1.3 Sampling Theorem ––Replacing Continuous-time Signals by Sequences of Data

Question: Under What Kind of Conditions Can Signal Waveforms Uniquely Recovered from Its Samples?

$$x(t) \quad \rightarrow \quad x[n] = x(nT_s) \xrightarrow{?} x(t)$$

recovered

continuous-time signal

discrete-time signal processed by computers

T_s : sampling period

$\omega_s = \dfrac{2\pi}{T_s}$: sampling frequency

- In general there can be infinite numbers of continuous-time signals having the same samples

See Fig. 7.1, p. 515 of Oppenheim

<u>Considerations from Single Frequency Components</u>

- **$x(t) = A\,\mathit{cos}\,(\omega_0 t)$**

  $y(t) = b\,\mathit{cos}\,[(\omega_0+\omega_s)\,t]$, ω_s : sampling frequency

  $z(t) = x(t) + y(t)$

  - $x(nT_s) = A\,\mathit{cos}\,(\omega_0\, nT_s)$

    $y(nT_s) = b\,\mathit{cos}\,[(\omega_0 + \omega_s)\, nT_s] = b\,\mathit{cos}\,(\omega_0\, nT_s)$

    $\omega_s = \dfrac{2\pi}{T_s}$

    $z(nT_s) = (A + b)\,\mathit{cos}\,(\omega_0\, nT_s)$

  - Any two frequency components ω_1 , ω_2 become indistinguishable if $|\omega_1 - \omega_2| = m \bullet \omega_s$, m: integer,

    when sampled at frequency ω_s

  - Aliasing effect

  - For discrete-time signal x[n] at sampling frequency ω_s , only the frequency range [0, ω_s] (or an equivalent) makes sense. Other frequency ranges are simply repetitive

Considerations for a Continuous Spectrum

- **$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$**

  assume $X(\omega)$ has a bandwidth W,

  $| X(\omega) | = 0 , \quad | \omega | > W$

  $x(t)$ sampled at ω_s , $x[n] = x(nT_s)$

- if $\omega_s < 2W$

  aliasing effect occurs
  frequency components mixed, can’t be recovered

- if $\omega_s > 2W$

  No aliasing effect
  Spectra do not overlap
  Original signal can be recovered by taking only those frequencies below W, or low pass filtering

Sampling Theorem

- if $\omega_s > 2W$
  the original signal can be uniquely recovered by low pass filtering

- if $\omega_s < 2W$
  the original signal can't be recovered

  See Fig. 7.3, 7.4, pp. 518-519 of Oppenheim

- **Mathematical formulation**

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - nT_s)$$

$$x_p(t) = x(t)p(t) = \sum_{k=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

See Fig. 7.2, p. 516 of Oppenheim

- It can be shown the Fourier transform of $x_p(t)$ is

$$X_p(\omega) = \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

See Fig. 7.3, p.518 of Oppenheim

- **An Example for Aliasing Effect**

  See Fig. 7.15, 7.16, pp. 529-531 of Oppenheim

Practical Considerations

- Over-sampling
- Pre-filtering

  Ref. 7.0, 7.1.1, 7.3 of Oppenheim

1.4 Pulse-coded Modulation (PCM) – Digital Representation of Continuous-time Signals

$$x(t) \rightarrow x[n] = x(\,nT_s\,) \rightarrow 1010110110$$

binary representation of a real number

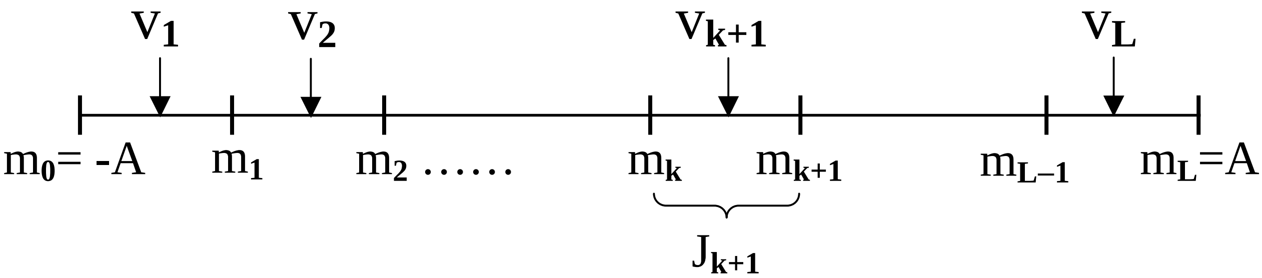
$$\begin{pmatrix}\text{number of bits}\\ \text{per sample}\end{pmatrix} \longleftrightarrow \begin{pmatrix}\text{truncation}\\ \text{error}\end{pmatrix}$$

tradeoff

transmission/storage condition

perceptual acceptability condition

Quantization (Scalar Quantization)



- **Assume | x[n] | ≤ A**

  divide the range [–A , A] into L quantization levels

  $\{ J_1 , J_2 , \ldots\ldots J_k , \ldots\ldots J_L \}$

  $J_k : [m_{k-1} , m_k]$

  $L = 2^R$

  each quantization level J_k is represented by a value v_k

  $S = \bigcup_{k=1}^{L} J_k$, $V = \{ v_1 , v_2 , \ldots\ldots v_k , \ldots\ldots v_L \}$

- **Quantization is a mapping relation**

  $Q : S \rightarrow V$

  $Q [x[n]] = v_k \quad \text{if} \quad x[n] \in J_k$

  each v_k is represented by an R-bit pattern

  after transmission/storage the sample x[n] has only L values, $\{ v_1 , v_2 , \ldots\ldots v_L \}$

  \- Question: how to design the quantizer characteristic (codebook) represented by $\{ J_1 , J_2 , \ldots\ldots J_L \}$and $\{ v_1 , v_2 , \ldots v_L \}$ so as to achieve a good tradeoff between the transmission/storage condition and the perceptual acceptability condition?

Quantization Error

$$x_Q[n] = x[n] + e[n]$$

- **Assume simplest case –uniform quantization**

$$m_k - m_{k-1} = \Delta_k = \text{step size for k-th level}$$
$$= \Delta \quad , \quad \text{same for all k}$$
$$\Delta = \frac{2A}{L}$$

See Fig. 3.11, p. 195 of Haykin

$$|\, e[n] \,| < \frac{\Delta}{2}$$

assume e[n] is uniformly distributed (this is reasonable if L is large enough or Δ is small enough)

$$f_e(e) = \begin{cases} \frac{1}{\Delta} \;, & |\, e[n] \,| < \frac{\Delta}{2} \\ 0 \;, & \text{else} \end{cases}$$

- **Mean square error**

$$\sigma_e^2 = E[\,(e[n])^2\,] = \int_{-\Delta/2}^{\Delta/2} e^2 f_e(e)\, de = \frac{1}{12}\,\Delta^2$$

for a quantization level

- for all possible values x[n]

$$\sigma_e^2 = \sum_{k=1}^{L} P_k \left(\frac{1}{12}\,\Delta_k^2 \right) = \frac{1}{12}\,\Delta^2$$
$$P_k = \text{Prob}[\, x[n] \in J_k \,]$$

$$\Delta = \frac{2A}{L} = \frac{2A}{2^R}$$

$$\sigma_e^2 = \frac{1}{3} A^2\, 2^{-2R} \;(\alpha\; 2^{-2R})$$

Quantization Error

- **Signal-to-noise Ratio**

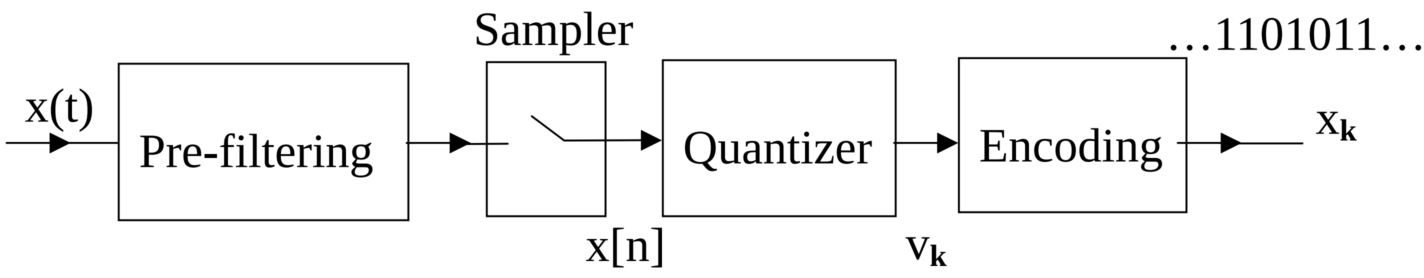
$$\sigma_{\mathbf{x}}^{\mathbf{2}} = E[\,(x[n])^{\mathbf{2}}\,]$$

$$SNR_{\mathbf{Q}} = \frac{\sigma_{\mathbf{x}}^{2}}{\sigma_{\mathbf{e}}^{2}} = \frac{3\sigma_{\mathbf{x}}^{2}}{A^{2}}\, 2^{\mathbf{2R}} \;(\alpha\; 2^{\mathbf{2R}} = L^{2})$$

- 6 dB reduction of $\sigma_{\mathbf{e}}^{\mathbf{2}}$ every extra bit per sample

- Quantization error can be arbitrarily suppressed by using more bits per sample

PCM Processes



for digital transmission,
storage or processing

Ref : 3.6 of Hankin