1.5 Modeling of Stochastic Processes

Stochastic Process

- **A sample space**
  - large number of sample points
  - each sample point corresponds to a random experiment
  - the outcome of a random experiment is a time function
  - the time functions are called sample functions
  - the sample functions share some probabilistic low
  - the sample space or ensemble of sample functions is called a stochastic (or random) process

  See Fig1.1,P 32 of Haykin

- **X(t) ： a stochastic process**

  $x_1(t)$, $x_2(t)$, ……$x_n(t)$,……sample functions

  $X(t_k)$: a random variable for any t_k

  $x_1(t_k)$, $x_2(t_k)$,……, $x_n(t_k)$,……

  - the outcome of a random experiment at $t = t_k$ is a number

- **A stochastic process is an indexed family of random variables**

  $X(t_1)$, $X(t_2)$, ……, $X(t_k)$, …

- **Signals and noise can be considered as stochastic processes**

Stationarity and Moments

- **Stationary**

- the statistical characteristics of a process is independent of time

- **First-order stationary**

- $f_{\mathbf{X}}(x)$: probability density function of a random variable X

- $f_{\mathbf{X}(t_k)}(x) = f_{\mathbf{X}(t_k + h)}(x)$, all h

- mean or first-order moment

$$M_{\mathbf{X}} = E[\, X(t)\,] = \int_{-\infty}^{\infty} x f_{\mathbf{X}(t_k)}(x)\,dx$$

constant mean

- zero-mean process

$$M_{\mathbf{X}} = 0$$

Stationarity and Moments

- **Second-order stationary**

  - $f_{X_1X_2}(x_1, x_2)$: joint density function of the two random variables X_1, X_2

  - $f_{X(t_1)X(t_2)}(x_1, x_2) = f_{X(t_1+h)X(t_2+h)}(x_1, x_2)$, all h

  - autocorrelation function or second-order moment

    $R_X(t_1, t_2) = E[\,X(t_1)X(t_2)\,]$

    $$= \int\int_{-\infty}^{\infty} x_1x_2\, f_{X(t_1)X(t_2)}(x_1, x_2)dx_1dx_2$$

  - Set $h = -t_1$

    $R_X(t_1, t_2) = E[\,X(0)X(t_2 - t_1)\,]$

    depends on $t_2 - t_1$ only

  - Set $\tau = t_2 - t_1$

    $R_X(\tau) = E[\,X(t)X(t + \tau)\,]$

    a deterministic function

- **Wide-sense Stationary**

  - autocorrelation function depends on $t_2 - t_1$ only

  - constant mean

Stationarity and Moments

- **Power Spectral Density**

$$S_{\mathbf{x}}(\omega) = F\{ R_{\mathbf{x}}(\tau) \} = \int_{-\infty}^{\infty} R_{\mathbf{x}}(\tau)\, e^{-j\omega\tau} d\tau$$

$$= \lim_{T \to \infty} E\left[\frac{1}{2T} \left| \int_{-T}^{T} X(t) e^{-j\omega t} dt \right|^2 \right]$$

- averaged power distribution on frequency domain

- a deterministic function

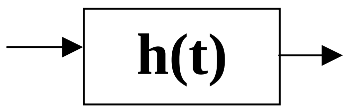
Ref: 1.2, 1.3, 1.4 of Haykin

Linear Time-invariant Systems

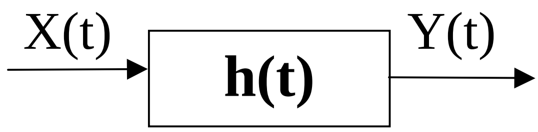
・ **wide-sense stationary processes**

X(t) , Y(t) : stochastic processes

$x_1(t)$, $x_2(t)$……, $y_1(t)$, $y_2(t)$……sample functions



$$X(t)\ \left\{\begin{array}{l} x_1(t) \longrightarrow y_1(t) = \int_{-\infty}^{\infty} h(\tau)x_1(t-\tau)d\tau \\ x_2(t) \longrightarrow y_2(t) = \int_{-\infty}^{\infty} h(\tau)x_2(t-\tau)d\tau \\ \quad\vdots \qquad\qquad\qquad\qquad \vdots \end{array}\right\} Y(t)$$



$$Y(t) = \int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau = h(t) * X(t)$$

・ **mean of output process**

$$\mu_Y = E[\ Y(t)\ ] = E\ [\int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau\ ]$$

$$= \int_{-\infty}^{\infty} E[\ h(\tau)X(t-\tau)]d\tau = \int_{-\infty}^{\infty} h(\tau)E[X(t-\tau)]d\tau$$

$$= \mu_X \int_{-\infty}^{\infty} h(\tau)d\tau = \mu_X\ H(0)$$

- means of input/output processes related by H(0)
- zero-mean input gives zero-mean output

Linear Time-invariant Systems

- **Power Spectral Density**

$$X(t) \longrightarrow \boxed{\mathbf{h(t),\ H(\omega)}} \longrightarrow Y(t)$$

$$x_1(t) \overset{F}{\longleftrightarrow} X_1(\omega) \qquad y_1(t) \overset{F}{\longleftrightarrow} Y_1(\omega) = X_1(\omega)H(\omega)$$

$$x_2(t) \overset{F}{\longleftrightarrow} X_2(\omega) \qquad y_2(t) \overset{F}{\longleftrightarrow} Y_2(\omega) = X_2(\omega)H(\omega)$$

$$\vdots \qquad\qquad \vdots$$

$$S_Y(\omega) \qquad S_X(\omega)$$

- $S_Y(\omega) = S_X(\omega)\,|\,H(\omega)\,|^2$

Gaussian and White Processes

- **Gaussian Processes**

  - Gaussian random variable X

$$f_X(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-(x-m)^2/2\sigma^2}$$

  - Multivariate Gaussian distribution for n random variables

$$\overline{X} = [\, X_1 , X_2 , \ldots\ldots , X_n \,]^t$$

$$f_{\overline{X}}(\overline{x}) = \frac{1}{(2\pi)^{n/2}\Delta^{1/2}} e^{-\frac{1}{2}[\,(\overline{x}-\overline{\mu})^t \Sigma^{-1} (\overline{x}-\overline{\mu})\,]}$$

$$\overline{\mu} = [\, \mu_{X_1} , \mu_{X_2} , \ldots\ldots , \mu_{X_n} \,]^t$$

$\Sigma = [\, \sigma_{ij} \,]$, covariance matrix

$$\sigma_{ij} = E\,[\,(X_i - \mu_{X_i})(X_j - \mu_{X_j})\,]$$

Δ : determinant of Σ

  - X(t) is a Gaussian process if for any set $\{ t_1 , t_2 , \ldots\ldots , t_n \}$

$$\overline{X} = [\, X(t_1), X(t_2) , \ldots\ldots X(t_n)]^t$$

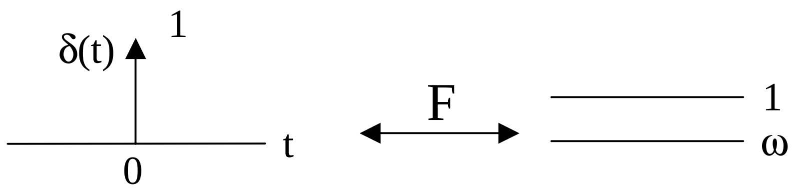
has a multivariate Gaussian distribution

  - When a Gaussian process operated by a linear time-invariant system, the output is another Gaussian process

Gaussian and White Processes

- **White Processes**

$$\delta(t) \xrightarrow{F} \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = 1, \text{ all } \omega$$



- A process X(t) is white if

$$R_X(\tau) = \frac{N_0}{2}\,\delta(\tau)\,, \quad S_X(\omega) = \frac{N_0}{2}\,, \text{ all } \omega$$

- N_0 : noise density

white noise

- $E[\,X(t_1)X(t_2)] = 0\,,\ t_1 \neq t_2$

Equal average power at all frequencies

Ref : 1.6, 1.7, 1.8, 1.9 of Haykin