1.0 Fundamentals

A Signal

· **A signal is a function of one or more variables, which conveys information on the nature of some physical phenomena.**

· **Examples**

$f(t)$	: a voice signal, a music signal
$f(x, y)$	: an image signal, a picture
$f(x, y, t)$	: a video signal
x_n	: a sequence of data (n: integer)
b_n	: a bit stream (b:1 or 0)

Continuous/Discrete-time Signals x(t), x[n] Exponential/Sinusoidal Signals

· **Basic Building Blocks from which one can construct many different signals**

$x(t) = e^{jw_0t}$, w_0 : rad/sec $x[n] = e^{jw_0n}$, w_0 : rad

frequency: w_0

· **Sinusoidal signal**

$x(t) = A\,cos\,(w_0t + f) = Re\{ Ae^{j(w_0t+f)} \}$ $x[n] = A\,cos\,(w_0n + f) = Re\{ Ae^{j(w_0n+f)} \}$ A :

amplitude (envelope)

f : phase

· **Harmonically related signal sets**

$\{ f_k(t) = e^{jkw_0t}, k = 0, \pm1, \pm2, ... \}$ $\{ f_k[n] = e^{jkw_0n}, k = 0, \pm1, \pm2, ... \}$

frequency: kw_0

1.1 Fourier Analysis of Signals

Fourier Series Representation of Continuous-time Periodic Signals

$x(t) = x(t+T)$, T : fundamental period

- **Harmonically Related complex exponentials**

$\{ e^{jk\omega_o t}, k = 0, \pm1, \pm2, \cdots\cdots \}$, $\omega_0 = \frac{2\pi}{T}$ (rad/sec)

$e^{jk\omega_o t}$ with period $\frac{T}{|k|}$

all with period T

- **Fourier Series**

∞

$k=-\infty$

$$x(t) = \Sigma \ a_k e^{jk\omega_0 t}$$

$a_k e^{jk\omega_0 t}$: k-th harmonic components

- $x(t)$ real

  $a_k^* = a_{-k}$

$$x(t) = a_0 + 2 \sum_{k=1}^{\infty} A_k \mathit{cos}\,(k\omega_0 t + \theta_k)\ ,\ a_k = A_k\, e^{j\theta_k}$$

Fourier Series Representation of Continuous-time Periodic Signals

- **Determination of a_k**

$$\int_T x(t)e^{-jn\omega_0 t}dt = \int_T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}dt$$

$$\int_T e^{j(k-n)\omega_0 t}dt \begin{array}{l} = T \ , \quad k = n \\ = 0 \ , \quad k \neq n \end{array}$$

$a_n = \dfrac{1}{T} \int_T x(t)e^{-jn\omega_0 t}dt$, Fourier series coefficients

$a_0 = \dfrac{1}{T} \int_T x(t)dt$, dc component

- **Vector Space Interpretation**

- vector space concept

$$\mathbf{V = \{ v \mid v \text{ is a vector} \}}$$

$v_1 + v_2$, av_1

inner product : inner product space

$v_1 \cdot v_2 : V \times V \rightarrow R$

- vector space for periodic signals

{ x(t) : x(t) is periodic with period T }
could be a vector space

$x_1(t) + x_2(t)$, $ax(t)$

$[x_1(t)].\ [x_2(t)] = \int_T x_1(t)\, x_2^*(t)dt$

Fourier Series Representation of Continuous-time Periodic Signals

- **Vector Space Interpretation**

  - orthogonal basis

    $$[e^{jk\omega_0 t}] \cdot [e^{jn\omega_0 t}] = 0 \ , \quad k \neq n$$

    $$= T \ , \quad k = n$$

    $\{ e^{jk\omega_0 t}, k = 0, \pm 1, \pm 2, \ldots\}$

    is a set of orthogonal (not normalized) basis expanding a vector space of periodic signals with period T

  - Fourier Series

    $$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

    $$a_n = \frac{1}{T}\int_T x(t)\, e^{-jn\omega_0 t} dt = \frac{1}{T}\,[x(t)] \cdot [e^{jn\omega_0 t}]$$

  - 3-dim Vector Space

    $$\hat{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

    $$\hat{a}_1 = \hat{A} \cdot \hat{i} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot \hat{i}$$

  - multi-dim Vector Space

    $$\hat{A} = \sum_k a_k \hat{v}_k$$

    $$a_k = \hat{A} \cdot \hat{v}_k = (\sum_j a_j \hat{v}_j) \cdot \hat{v}_k$$

- **Examples**

See Figs. 3.6, 3.7, pp. 193, 195 of Oppenheim

Fourier Series Representation of Discrete-time Periodic Signals

$x[n] = x[n+N]$, periodic with period N

- **Harmonically related signal sets**

$\{ e^{jk(\frac{2\pi}{N})n}, \quad k = 0, \pm 1, \pm 2, \cdots \}$

all with period N, $\omega_0 = \frac{2\pi}{N}$ (rad)

$e^{jk(\frac{2\pi}{N})n} = e^{j(k+rN)(\frac{2\pi}{N})n}$, r: integer

only N distinct signals

- **Fourier Series**

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_o n} = \sum_{k=<N>} a_k e^{jk(\frac{2\pi}{N})n}$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_o n} = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk(\frac{2\pi}{N})n}$$

- **Fast Fourier Transform (FFT)**

fast algorithms to compute the transform

- **Vector Space Interpretation**

$\{ x[n] : x[n]$ is periodic with period N $\}$

inner product

$$(x_1[n]) \cdot (x_2[n]) = \sum_{n=<N>} x_1[n] \, x_2^*[n]$$

$\{ e^{jk(\frac{2\pi}{N})n}, k = <N> \}$ is a set of orthogonal basis

Fourier Transform for Continuous-time Aperiodic Signals

- **Fourier Series : for periodic signal**

$x(t) = x(t+T)$, T : fundamental period

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T}$$

as T increases , $\omega_0 = 2\pi/T$ decreases

the envelope Ta_k is sampled at closer and closer spacing

See Figs. 3.6, 3.7, pp. 193, 195,
Fig. 4.2, p.286 of Oppenheim

- aperiodic : $T \to \infty$, $\omega_0 \to 0$, $a_k \to X(\omega)$

- **Fourier Transform**

$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$: spectrum, frequency domain
Fourier Transform

$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$: signal, time domain
Inverse Fourier Transform

Fourier Transform pair

$$x(t) \stackrel{F}{\longleftrightarrow} X(\omega)$$

$$X(\omega) = F\{x(t)\}\ , \quad x(t) = F^{-1}\{X(\omega)\}$$

Fourier Transform for Continuous-time Aperiodic Signals

- **Vector Space Interpretation**

$\{x(t)$: $x(t)$ is a continuous-time aperiodic signal$\}$

inner product

$$[\ x_1(t)\ ] \cdot [\ x_2(t)] = \int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)dt$$

$$[\ e^{j\omega_1 t}\ ] \cdot [\ e^{j\omega_2 t}\ ] = 0 \quad \text{if} \quad \omega_1 \neq \omega_2$$

$\{\ e^{j\omega t}$, all $\omega\}$ is a set of orthogonal (not normalized) basis

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt = [\ x(t)\ ] \cdot [\ e^{j\omega t}\ ]$$

Discrete-time Fourier Transform of Discrete-time Aperiodic Signals

- **Fourier Series for Discrete-time Periodic Signals**

$x[n] = x[n+N]$, periodic with period N

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_o n} = \sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk\omega_o n} = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$

- **As $N \to \infty$, $\omega_0 \to 0$, $a_k \to X(\omega)$**

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

signal, time-domain,
Inverse Discrete-time
Fourier Transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

spectrum, frequency domain
Discrete-time Fourier Transform

- Similar format to all Fourier analysis representation previously discussed

- **Note:** $X(\omega)$ is continuous and periodic with period 2π
  Integration over 2π only

- **Vector Space Interpretation**

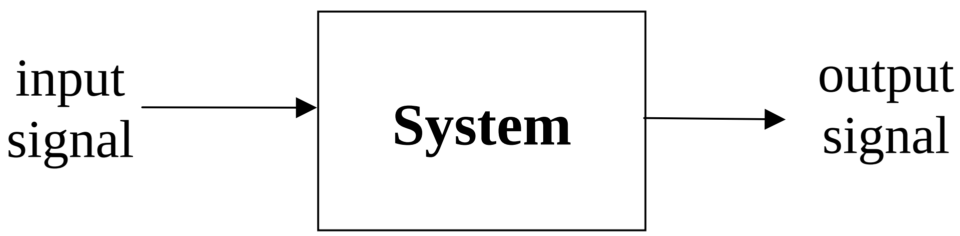
$$X(\omega) = (x[n]) \cdot (e^{j\omega n})$$

Ref : Oppenheim 3.3, 3.6, 4.1.1, 5.1.1

1.2 Systems and Input/Output Relationships

<u>A System</u>

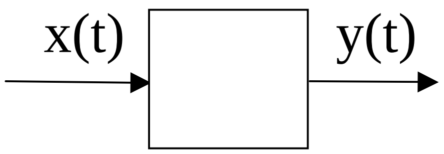
- **An entity that manipulates one or more signals to accomplish some function, including yielding some new signals.**

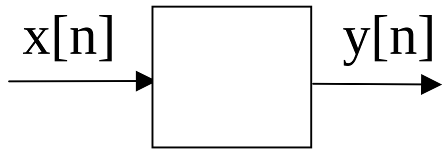


- **Examples**

  - an electric circuit
  - a telephone handset
  - a PC software receiving pictures from Internet
  - a TV set
  - a computer with some software handling some data

Continuous/Discrete-time Systems





- **Linearity**
  - linear: superposition property

    $x_k[n] \rightarrow y_k[n]$

    $\sum_k a_k x_k[n] \rightarrow \sum_k a_k y_k[n]$

  - scaling or homogeneity property

    $x[n] \rightarrow y[n]$

    $ax[n] \rightarrow ay[n]$

  - additive property

    $x_i[n] \rightarrow y_i[n]$

    $x_1[n] + x_2[n] \rightarrow y_1[n] + y_2[n]$

- **Time Invariance**
  - system characteristics fixed over time

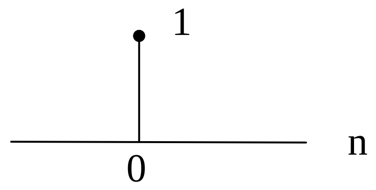
    $x[n] \rightarrow y[n]$

    $x[n-k] \rightarrow y[n-k]$

Unit Impulse

- **Discrete-time**

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$



– Unit Impulse Representation of A Discrete-time Signal

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

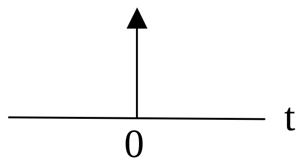
a unit impulse located at n = k on index n

See Fig. 2.1 , P. 76 of Oppenheim

<u>Unit Impulse</u>

- **Continuous-time**

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty \text{ or undefined}, & t = 0 \end{cases}$$



– First Derivative of Unit Step Function

$$u(t) = \begin{cases} 1\ , & t \geq 0 \\ 0\ , & t < 0 \end{cases}$$

$$\delta(t) = \frac{du(t)}{dt}$$

See Fig. 1.33 , Fig. 1.34, P. 33 of Oppenheim

– Unit Area Definition

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

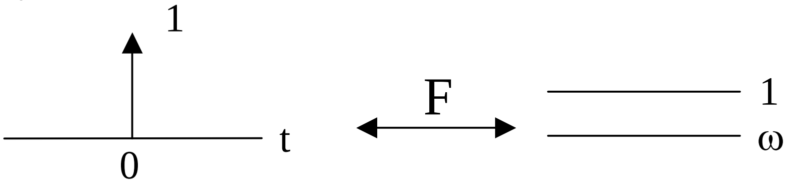
– Operational Definition, Unit Impulse Representation of A Continuous-time Signal

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

a unit impulse located at $\tau = t$
on index τ

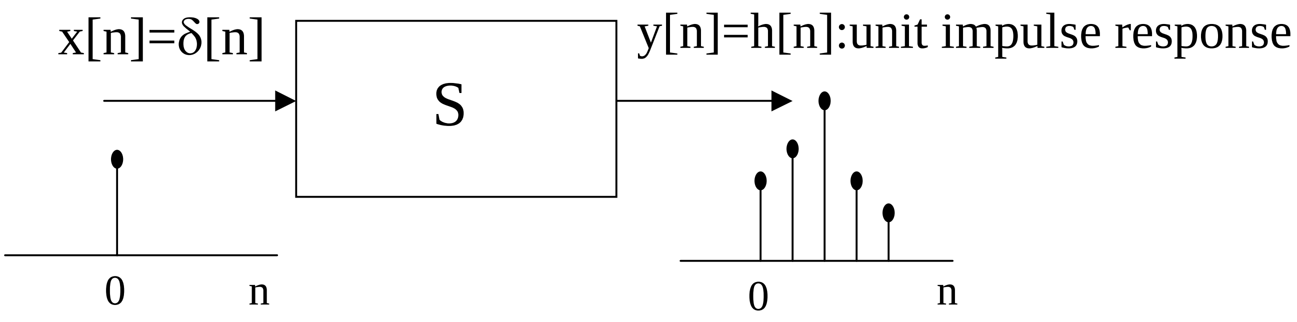
– Unit Step Function Has A Flat Spectrum

$$\int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = 1, \text{ all } \omega$$



Convolution Sum for Discrete-time Systems

- **Defining the Output for an Unit Impulse Input as the Unit Impulse Response**



- **By Linearity (Superposition Property) and Time Invariance**

  - The output for an arbitrary input signal is the superposition of a series of "shifted, scaled unit impulse response"

$$\sum_{k} \underset{\uparrow \atop x[k]}{a_k} \underset{\uparrow \atop \delta[n-k]}{x_k[n]} \rightarrow \sum_{k} \underset{\uparrow \atop x[k]}{a_k} \underset{\uparrow \atop h[n-k]}{y_k[n]}$$

$$x[n] = \sum_{k} x[k]\delta[n-k]$$

$$\boxed{\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \equiv h[n] * x[n] \\ &\qquad \text{Convolution Sum} \end{aligned}}$$

See Fig. 2.3 , P. 80 of Oppenheim

Convolution Sum for Discrete-time Systems

- **A Different Way to Visualize the Convolution Sum**

  - looked at on the index k

$$y[n] \;=\; \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

input signal

Contribution to
the output signal
at time n

reflected-over version of
h[k] located at k = n

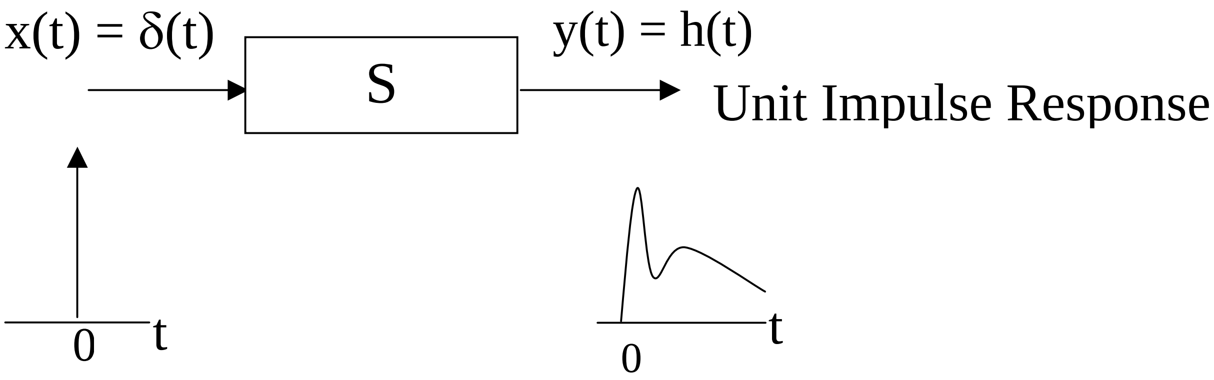
  – on the dummy index k ,h[k] is reflected over and shifted to k=n ,weighted by x[k] and summed to produce an output sample y[n] at time n

  See Figs 2.5, 2.6, 2.7, pp. 83-85 of Oppenheim

- **A linear time-invariant discrete-time system is completely characterized by its unit impulse response**

Convolution Integral for Continuous-time Systems

- **Defining the output for an unit impulse input as the unit impulse response**



- **By Linearity (Superposition Property)**
  - The output for an arbitrary input signal is the superposition of a series of "shifted, scaled unit impulse response"

$$\sum_{k} a_k x_k(t) \rightarrow \sum_{k} a_k y_k(t)$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$\boxed{\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \equiv x(t) * h(t) \\ &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \equiv h(t) * x(t) \\ &\quad \text{Convolution Integral} \end{aligned}}$$

<u>Convolution Integral for Continuous-time Systems</u>

- **A Different Way to visualize the convolution integral**

  - Looked at on the index τ

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau) d\tau$$

input signal

output signal at time t

reflected-over version of h(t) located at τ=t

  - On the dummy index τ, h(t) is reflected over and shifted to τ = t, weighted by x(t) and integrated to produce the output value at time t, y(t)

- **A linear time-invariant continuous-time system is completely characterized by its unit impulse response**

Response of A Linear Time-invariant System to An Exponential Signal

$$x(t),\ x[n] \longrightarrow \boxed{\begin{matrix} h(t) \\ h[n] \end{matrix}} \longrightarrow y(t),\ y[n]$$

- **Continuous-time**

$$x(t) = e^{j\omega_0 t}$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega_0(t-\tau)}\, d\tau$$

$$= (e^{j\omega_0 t}) \underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-j\omega_0\tau}\, d\tau}_{H(\omega_0)} \qquad h(t) \overset{F}{\longleftrightarrow} H(\omega)$$

  - if an input signal has a single frequency component, the output will be exactly the same, except scaled

  - $H(\omega)$: frequency response, or transfer function

- **Discrete-time**

$$x[n] = e^{j\omega_0 n}$$

$$y[n] = (\, e^{j\omega_0 n}\,).\ \underbrace{\sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega_0 k}}_{H(\omega_0)} \qquad h[n] \overset{F}{\longleftrightarrow} H(\omega)$$

System Characterization

- **Superposition Property for Frequency Domain**

  - periodic (Fourier Series)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t} \rightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k H(k\omega_0) e^{jk\omega_o t}$$

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_o n} \rightarrow y[n] = \sum_{k=\langle N\rangle} a_k H(k\omega_0) e^{jk\omega_o n}$$

  - aperiodic (Fourier Transform)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$\rightarrow y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)X(\omega)e^{j\omega t}\, d\omega$$

$$= F^{-1}[\, H(\omega)X(\omega)\,]$$

$$y(t) = x(t) * h(t) \xleftrightarrow{F} Y(\omega) = X(\omega)H(\omega)$$

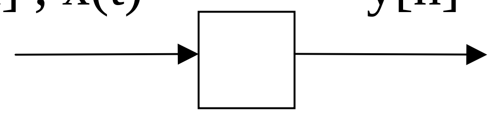
$$y[n] = x[n] * h[n] \xleftrightarrow{F} Y(\omega) = X(\omega)H(\omega)$$

- Convolution Property of Fourier Transform

- $H(\omega)$ frequency response,
  or transfer function

System Characterization

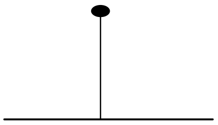
$x[n]$, $x(t)$ $\qquad$ $y[n]$, $y(t)$

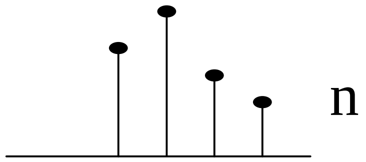


- **Unit impulse as signal components**

  - each component split to many other components, thus convolution required for computing the output

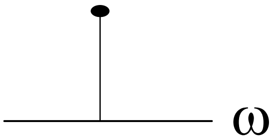
$\delta[n]$, $\delta(t)$ $\qquad$ $h[n]$, $h(t)$

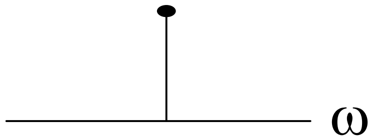




- **Single frequency as signal components**

$e^{\mathbf{j\omega n}}$, $e^{\mathbf{j\omega t}}$ $\qquad$ $H(\omega)e^{\mathbf{j\omega n}}$, $H(\omega)e^{\mathbf{j\omega t}}$





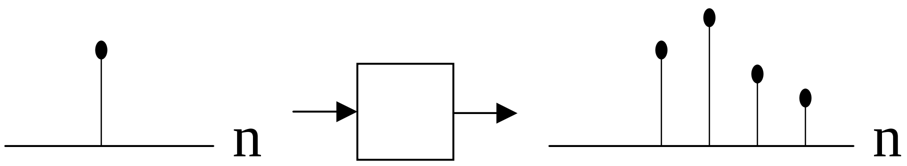
  - each frequency component never split to other frequency components, convolution reduced to multiplication

Vector Space Interpretation

- **Discrete-time**

  - $\{\, x[n],\ x[n] \text{ is a discrete-time signal} \,\} = V$
  - $\{\, \delta[n-k],\quad k = 0, \pm 1, \pm 2, \ldots \}$ is a set of orthonormal basis
  - $(\delta[n-k]) \cdot (\delta[n-j]) = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases}$

  $$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = (x[k]) \cdot (\delta[n-k])$$

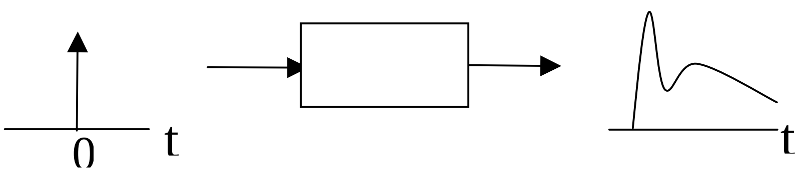


- **Continuous-time**

  - $\{\, x(t),\ x(t) \text{ is a continuous-time signal} \,\} = V$
  - $\{\, \delta(t-\tau),\ -\infty < \tau < \infty \,\}$ is a set of orthogonal (not normalized) basis

  $$[\delta(t-\tau_1)] \cdot [\delta(t-\tau_2)] = 0,\ \tau_1 \neq \tau_2$$

  $$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = [x(\tau)] \cdot [\delta(t-\tau)]$$



Vector Space Interpretation

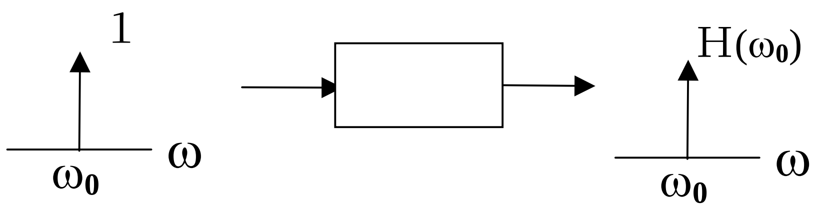
- **Frequency Domain**

  - Continuous-time

    $\{ e^{j\omega t}, -\infty < \omega < \infty \}$ is a set of orthogonal (not normalized) basis

    $[e^{j\omega_1 t}] \cdot [e^{j\omega_2 t}] = 0, \quad \omega_1 \neq \omega_2$

    $$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = [\, x(t)\,] \cdot [\, e^{j\omega t}\,]$$

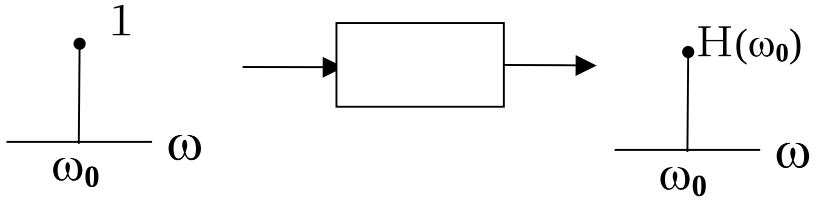


  - Discrete-time

    $\{ e^{j\omega n}, 0 < \omega < 2\pi \}$ is a set of orthogonal (not normalized) basis

    $[e^{j\omega_1 n}] \cdot [e^{j\omega_2 n}] = 0, \quad \omega_1 \neq \omega_2$

    $$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = (\, x[n]\,) \cdot (\, e^{j\omega n}\,)$$



Ref: Oppenheim 2.1, 2.2, 3.2, 4.4