2.2 Differential PCM (DPCM)

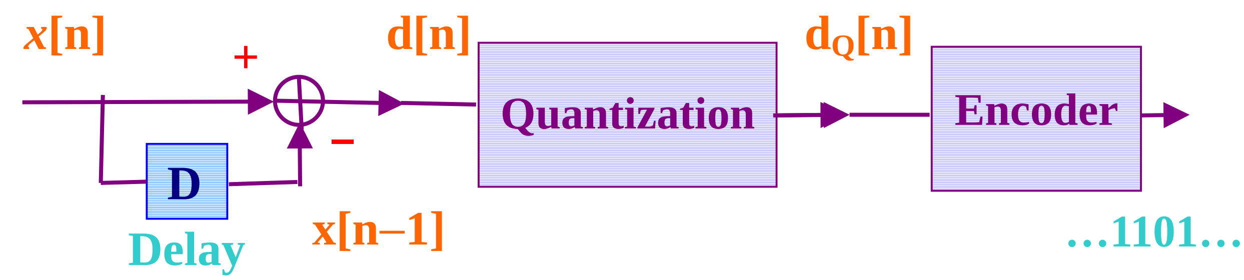
- Quantize and transmit only the differences in samples

$$d[n] = x[n] - x[n-1]$$

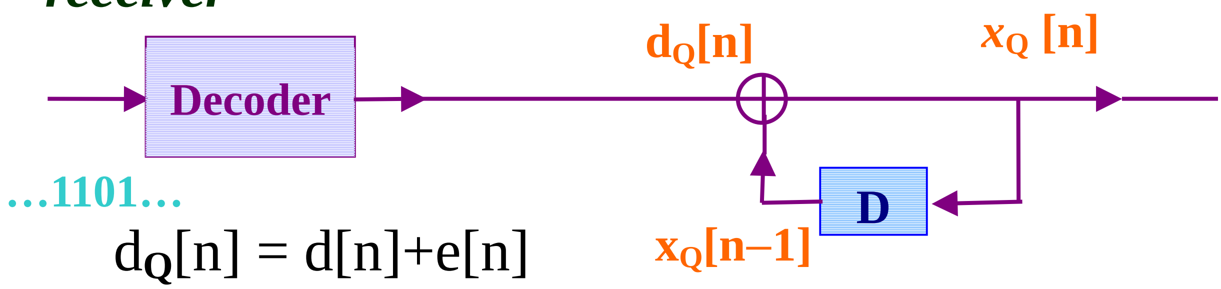
much smaller values of d[n] give much better performance

Error Accumulation Problem

transmitter



receiver



$$d_Q[n] = d[n]+e[n]$$

$$x_Q[n] = x_Q[n-1]+d_Q[n]$$

$$= x_Q[n-1]+d[n]+e[n]$$

$$x_Q[n+1] = x_Q[n]+d_Q[n+1]$$

$$= x_Q[n-1]+d[n]+e[n] +d[n+1]+e[n+1]$$

quantization error accumulated

Error Accumulation Problem

- **Solution**

redefine $d[n] = x[n] - x_Q[n-1]$

$$= x[n] - (x_Q[n-2] + d[n-1] + e[n-1])$$

previous errors absorbed

See Fig. 3.28 , p. 228 of Haykin

use of very limited memory and computation

Processing Gain

$$SNR'_Q = \frac{\sigma_x^2}{\sigma_e^2} = \frac{\sigma_x^2}{\sigma_d^2} \bullet \underbrace{\frac{\sigma_d^2}{\sigma_e^2}}_{\parallel \; SNR_Q}$$

$$G = \frac{\sigma_x^2}{\sigma_d^2}$$

$$\sigma_d^2 = E[(d[n])^2] = E[(x[n] - x[n-1])^2]$$

$$= E[(x[n])^2] + E[(x[n-1])^2] - 2E[(x[n]x[n-1])]$$

define $r_1 = \dfrac{E[(x[n]x[n-1])]}{E\,[(x[n])^2]}$

$$= E[(x[n])^2]\,(2-2r_1)$$

$$= \sigma_x^2 \bullet 2(1-r_1)$$

$$G = \frac{1}{2(1-r_1)}$$

r_1 :varying for speech, audio signals, etc.
averaged over a short period only

note : e[n] neglected here

Linear Prediction of Signals

- signal samples are highly correlated locally
- previous samples can be used to predict the next sample

• **Formulation**

$$\tilde{x}[n] = \sum_{k=1}^{P} w_k \, x[n-k]$$

P : prediction order

$$d[n] = x[n] - \tilde{x}[n]$$

See Fig. 3.26 , p. 223 of Haykin

$$\sigma_d^2 = E[(d[n])^2] = \min$$

$$\frac{\partial \sigma_d^2}{\partial w_k} = 0$$

Linear Prediction of Signals

- **Solution**

$$\overline{w} = R^{-1}\overline{r}$$

$$\overline{w} = [w_1, w_2, \ldots, w_p]^t$$

$$\overline{r} = [r_1, r_2, \ldots, r_p]^t$$

$$R = \begin{bmatrix} r_0 & r_1 & r_2 & \ldots & r_{p-1} \\ r_1 & r_0 & r_1 & \ldots & r_{p-2} \\ r_2 & r_1 & r_0 & \ldots & r_{p-3} \\ \vdots & \vdots & \vdots & & \vdots \\ r_{p-1} & r_{p-2} & r_{p-3} & \ldots & r_0 \end{bmatrix}$$

$$r_k = \frac{E[(x[n]x[n-k])]}{E[(x[n])^2]}$$

Wiener-Hopf Equation

Vector Space Interpretation of Wiener-Hopf Equation

- **3-dim Vector Space –– an estimation problem**

  Given $\overline{y}_1$, $\overline{y}_2$, $\overline{x}$, find

  $\overline{x}^* = a_1\overline{y}_1 + a_2\overline{y}_2$, a best estimate of $\overline{x}$

  such that $\| \overline{x} - \overline{x}^* \|^2 = [(\overline{x} - \overline{x}^*) \bullet (\overline{x} - \overline{x}^*)] = \min$

  - Solution :

  $$\begin{cases} (\overline{x} - a_1\overline{y}_1 - a_2\overline{y}_2) \bullet \overline{y}_1 = 0 \\ (\overline{x} - a_1\overline{y}_1 - a_2\overline{y}_2) \bullet \overline{y}_2 = 0 \end{cases}$$

  - Orthogonality Principle : the difference vector should be orthogonal to the plane of (y_1, y_2) , or to both y_1 and y_2
  - This principle can be extended to all vector space with valid definition of inner product

- **Vector Space of Random Variables**

  V = { X ; X is a random variable }

  - $X \bullet Y \equiv E[XY]$

    $\|X\| = E[X^2]$

    X is orthogonal to Y if $E[XY] = 0$

Vector Space Interpretation of Wiener-Hope Equation

- **Orthogonality Principle for Linear Prediction**
  - Given x[n–1] , x[n–2], … x[n–p] and x[n] , find

$$\tilde{x}[n] = \sum_{k=1}^{P} w_k\, x[n-k]$$

such that $\| x[n] - \tilde{x}[n] \|^2 = \min$

  - Orthogonality Principle

$$\left(x[n] - \sum_{k=1}^{P} w_k\, x[n-k]\right) \bullet (x[n-j]) = 0\ ,$$

$$j = 1, 2, 3, \ldots, p$$

$$E[(x[n]x[n-j])] = \sum_{k=1}^{P} w_k\, E[(x[n-k])x[n-j])]$$

$$j = 1, 2, 3, \ldots, p$$

$$\bar{r} = R\,\bar{w}$$

$$\bar{w} = R^{-1}\,\bar{r} \qquad \text{Wiener-Hopf Equation}$$

Adaptive Linear Prediction

r_k in $\bar{r}$, R are time-varying

w_k are time-varying

$\hat{r}_k[n]$, $\hat{w}_k[n]$ estimated based on short-time statistics obtained with windowed x[n]

Improved DPCM with Linear Prediction

$$\tilde{x}[n] = \sum_{k=1}^{P} \hat{w}_k [n] x_Q[n-k]$$

$\hat{w}_k[n]$: estimate of w_k at time n

best estimate of x[n] by minimizing σ_d^2

$$d[n] = x[n] - \tilde{x}[n]$$

See Fig. 3.28 , p. 228 of Haykin

$$\sigma_d^2 = \min \text{ , } G = \max$$

- **Redundancy and Predictability**

$$x[n] = \tilde{x}[n] + d[n]$$

($\tilde{x}[n]$): predictable with previous samples, correlated with previous samples statistically, redundancy

($d[n]$): unpredictable, New Information, uncorrelated with previous samples

- DPCM is to remove the redundancy before quantization

- Using more computation and memory to improve performance