

Diagonalization

Hung-yi Lee

Review

- If $A\mathbf{v} = \lambda\mathbf{v}$ (\mathbf{v} is a vector, λ is a scalar)
 - \mathbf{v} is an eigenvector of A **excluding zero vector**
 - λ is an eigenvalue of A that corresponds to \mathbf{v}
- Eigenvectors corresponding to λ are **nonzero** solution of $(A - \lambda I_n)\mathbf{v} = \mathbf{0}$

Eigenvectors
corresponding to λ

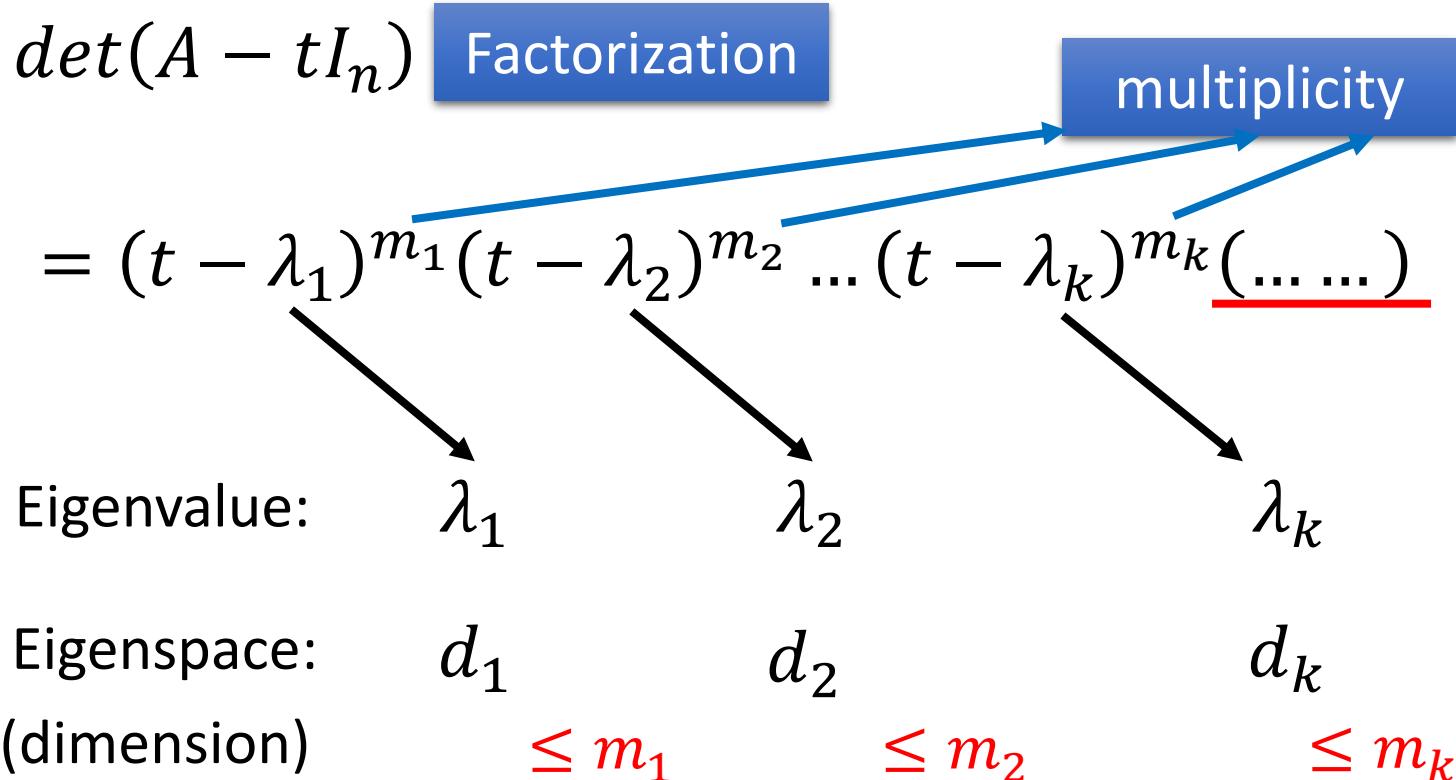
$$= \frac{\text{Null}(A - \lambda I_n)}{\text{eigenspace}} - \{\mathbf{0}\}$$

Eigenspace of λ :

Eigenvectors
corresponding to λ + $\{\mathbf{0}\}$

Review

- Characteristic polynomial of A is



Outline

- An $n \times n$ matrix A is called **diagonalizable** if $A = PDP^{-1}$
 - D : $n \times n$ diagonal matrix
 - P : $n \times n$ invertible matrix
- Is a matrix A **diagonalizable**?
 - If yes, find D and P
- An application of diagonalization
- Reference: Textbook 5.3

Diagonalizable

- An $n \times n$ matrix A is called **diagonalizable** if $A = PDP^{-1}$

- D : $n \times n$ diagonal matrix
- P : $n \times n$ invertible matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\rightarrow A^2 = 0 \quad (?)$$

- Not all matrices are diagonalizable

If $A = PDP^{-1}$ for some invertible P and diagonal D

$$\rightarrow A^2 = PD^2P^{-1} = 0 \rightarrow D^2 = 0 \rightarrow D = 0$$

$$\rightarrow A = 0 \quad \text{X}$$

D is diagonal

Diagonalizable

- If A is diagonalizable

$$P = [p_1 \quad \cdots \quad p_n]$$

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

$$A = PDP^{-1} \longrightarrow AP = PD$$

$$\longrightarrow AP = [Ap_1 \quad \cdots \quad Ap_n]$$

$$\longrightarrow PD = P[d_1e_1 \quad \cdots \quad d_ne_n]$$

$$= [Pd_1e_1 \quad \cdots \quad Pd_ne_n]$$

$$= [d_1Pe_1 \quad \cdots \quad d_nPe_n]$$

$$= [\underline{d_1p_1} \quad \cdots \quad \underline{d_np_n}] \longrightarrow Ap_i = d_ip_i$$

p_i is an eigenvector of A corresponding to eigenvalue d_i

Diagonalizable

- If A is diagonalizable

$$A = PDP^{-1}$$

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p_i is an eigenvector of A
corresponding to eigenvalue d_i

There are n eigenvector that forms an invertible matrix

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There are n independent eigenvectors

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The eigenvectors of A can form a basis for \mathbb{R}^n .

$$P = [p_1 \quad \cdots \quad p_n]$$

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

Diagonalizable

- If A is diagonalizable

$$A = PDP^{-1}$$

p_i is an eigenvector of A
corresponding to eigenvalue d_i

How to diagonalize a matrix A?

Step 1: Find n L.I. eigenvectors corresponding if possible,
and form an invertible P

Step 2: The eigenvalues corresponding to the
eigenvectors in P form the diagonal matrix D .

$$P = [p_1 \quad \cdots \quad p_n]$$

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

Diagonalizable

A set of eigenvectors that correspond to distinct eigenvalues is linear independent.

$$\det(A - tI_n) \quad \text{Factorization}$$

$$= (t - \lambda_1)^{m_1} (t - \lambda_2)^{m_2} \dots (t - \lambda_k)^{m_k} (\dots \dots)$$

Eigenvalue:

$$\lambda_1$$

$$\lambda_2$$

.....

$$\lambda_k$$

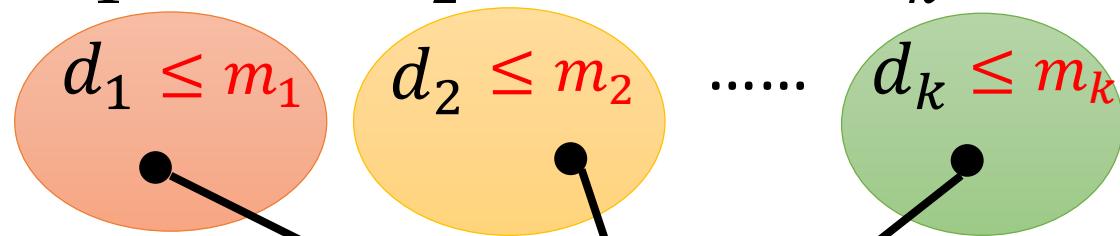
Eigenspace:
(dimension)

$$d_1 \leq m_1$$

$$d_2 \leq m_2$$

.....

$$d_k \leq m_k$$



Independent

Diagonalizable

A set of eigenvectors that correspond to distinct eigenvalues is linear independent.

Eigenvalue: $\lambda_1 \quad \lambda_2 \quad \dots \quad \lambda_m$ Assume dependent
Eigenvector: $v_1 \quad v_2 \quad \dots \quad v_m$  a contradiction

$$v_k = c_1 v_1 + c_2 v_2 + \dots + c_{k-1} v_{k-1}$$

$$A v_k = c_1 A v_1 + c_2 A v_2 + \dots + c_{k-1} A v_{k-1}$$

$$\lambda_k v_k = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \dots + c_{k-1} \lambda_{k-1} v_{k-1}$$

$$-\lambda_k v_k = c_1 \lambda_k v_1 + c_2 \lambda_k v_2 + \dots + c_{k-1} \lambda_k v_{k-1}$$

$$0 = c_1 (\lambda_1 - \lambda_k) v_1 + c_2 (\lambda_2 - \lambda_k) v_2 + \dots + c_{k-1} (\lambda_{k-1} - \lambda_k) v_{k-1}$$

Not $c_1 = c_2 = \dots = c_{k-1} = 0$  Same eigenvalue  a contradiction

Diagonalizable

- If A is diagonalizable

$$A = PDP^{-1}$$

p_i is an eigenvector of A corresponding to eigenvalue d_i

$$\det(A - tI_n)$$

$$= (t - \lambda_1)^{m_1} (t - \lambda_2)^{m_2} \dots (t - \lambda_k)^{m_k} (\dots \dots)$$

Eigenvalue:

$$\lambda_1$$

$$\lambda_2$$

.....

$$\lambda_k$$

Eigenspace:

{ Basis for λ_1 Basis for λ_2 Basis for λ_3 }

You can't find more!

Independent Eigenvectors

Diagonalizable - Example

- Diagonalize a given matrix $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

characteristic polynomial is $-(t + 1)^2(t - 3) \rightarrow$ eigenvalues: 3, -1

eigenvalue 3

$$B_1 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

eigenvalue -1

$$B_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

$$A = PDP^{-1},$$

where

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Test for a Diagonalizable Matrix

- An $n \times n$ matrix A is diagonalizable if and only if both the following conditions are met.

The characteristic polynomial of A factors into a product of linear factors.

$$\det(A - tI_n) \quad \text{Factorization}$$
$$= (t - \lambda_1)^{m_1} (t - \lambda_2)^{m_2} \dots (t - \lambda_k)^{m_k} (\underline{\hspace{2cm}})$$

For each eigenvalue λ of A , the multiplicity of λ equals the dimension of the corresponding eigenspace.

Independent Eigenvectors

An $n \times n$ matrix A is diagonalizable

||

The eigenvectors of A can form a basis for \mathbb{R}^n .

||

$$\det(A - tI_n)$$

$$= (t - \lambda_1)^{m_1} (t - \lambda_2)^{m_2} \dots (t - \lambda_k)^{m_k} (\dots)$$

Eigenvalue: λ_1 λ_2 λ_k

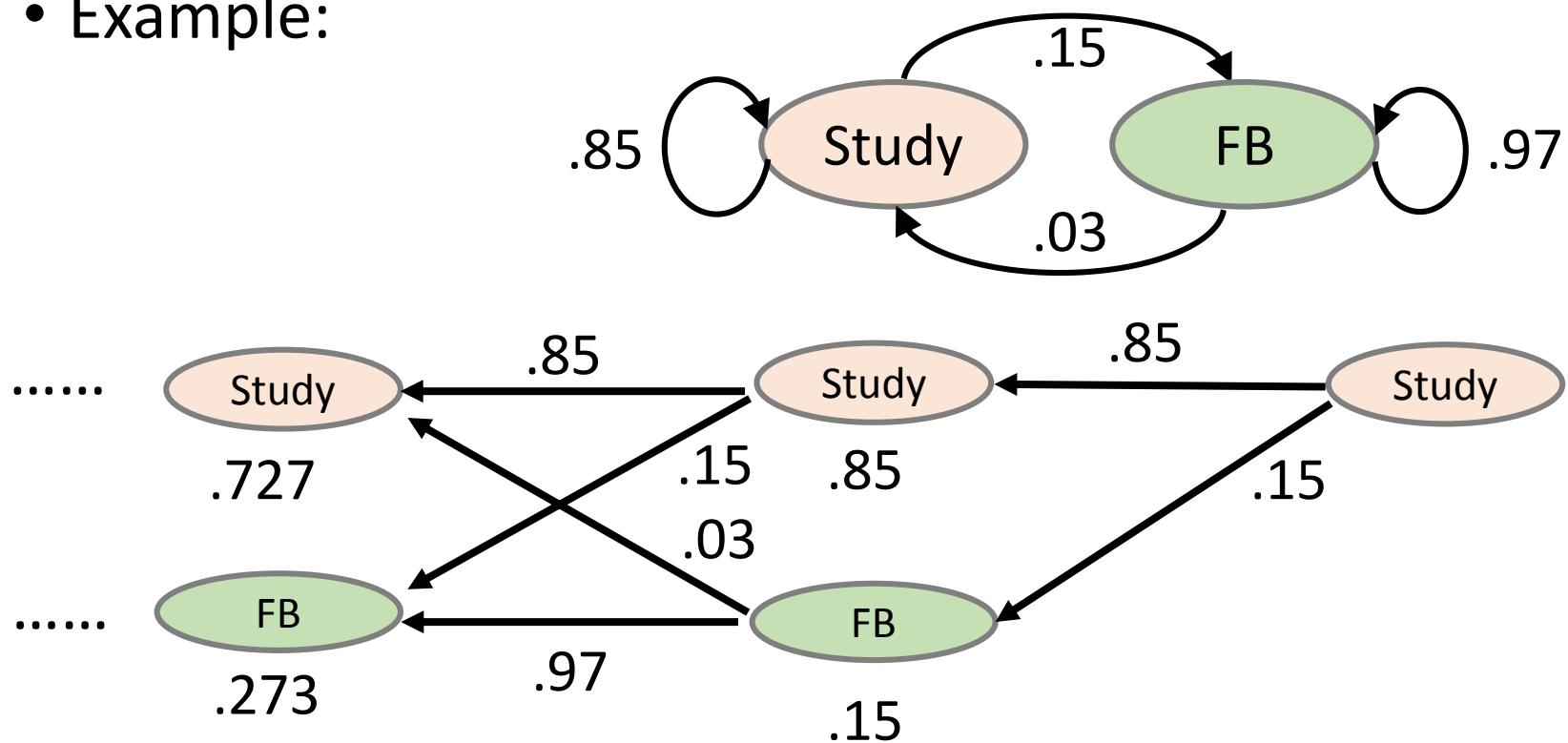
Eigenspace: $d_1 = m_1$ $d_2 = m_2$ $d_k = m_k$
(dimension)

Application of Diagonalization

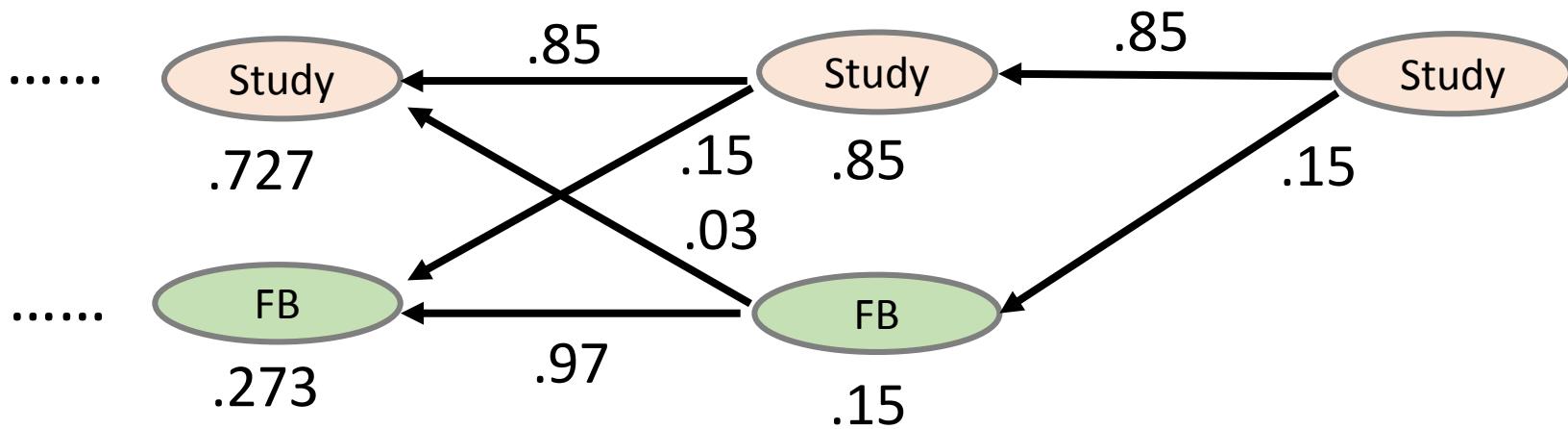
- If A is diagonalizable,

$$A = PDP^{-1} \longrightarrow A^m = PD^mP^{-1}$$

- Example:



$$\begin{array}{ccccc}
 & & \text{From} & & \\
 & & \text{Study} & & \text{FB} \\
 \text{To} & \text{Study} & \left[\begin{array}{cc} .85 & .03 \\ .15 & .97 \end{array} \right] & .85 \\
 & \text{FB} & & & \equiv A
 \end{array}$$



$$\begin{bmatrix} .727 \\ .273 \end{bmatrix} \begin{bmatrix} .85 & .03 \\ .15 & .97 \end{bmatrix} \begin{bmatrix} .85 \\ .15 \end{bmatrix} \begin{bmatrix} .85 & .03 \\ .15 & .97 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A = PDP^{-1} \longrightarrow A^m = PD^mP^{-1}$$

Diagonalizable

- Diagonalize a given matrix $A = \begin{bmatrix} .85 & .03 \\ .15 & .97 \end{bmatrix}$

$$\det(A - tI_2)$$

$$A - .82I_2 \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$A - I_2 \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -.2 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{p}_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$P = \begin{bmatrix} -1 & 1 \\ 1 & 5 \end{bmatrix}$ (invertible)

$D = \begin{bmatrix} .82 & 0 \\ 0 & 1 \end{bmatrix}$

Application of Diagonalization

$$A = PDP^{-1} \text{ where } P = \begin{bmatrix} -1 & 1 \\ 1 & 5 \end{bmatrix}, D = \begin{bmatrix} .82 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} A^m &= PD^mP^{-1} \\ &= \begin{bmatrix} -1 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} (.82)^m & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 5 \end{bmatrix}^{-1} \\ &= \frac{1}{6} \begin{bmatrix} 1 + 5(.82)^m & 1 - (.82)^m \\ 5 - 5(.82)^m & 5 + (.82)^m \end{bmatrix} \end{aligned}$$

When $m \rightarrow \infty$,

$$A^m = \begin{bmatrix} 1/6 & 1/6 \\ 5/6 & 5/6 \end{bmatrix}$$

The beginning condition does not influence.

工商時間

- [2016-04-29] Prof. 李宏毅 (Hung-yi Lee), "Towards Machine Comprehension of Spoken Content"
 - Date: 明天 2:20pm-3:30pm
 - Location: 資工系德田館 R103

Homework