Eigenvalues and Eigenvectors

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Chapter 5

- In chapter 4, we already know how to consider a function from different aspects (coordinate system)
- Learn how to find a “good” coordinate system for a function
- Scope: Chapter 5.1 – 5.4
  - Chapter 5.4 has *

Outline

- What is Eigenvalue and Eigenvector?
  - Eigen (German word): "unique to” or "belonging to"
- How to find eigenvectors (given eigenvalues)?
- Check whether a scalar is an eigenvalue

- Reference: Textbook Chapter 5.1

What is Eigenvalue and Eigenvector?

Eigenvalues and Eigenvectors

- If $Av = \lambda v$ (v is a vector, λ is a scalar)
  - v is an eigenvector of A excluding zero vector
  - λ is an eigenvalue of A that corresponds to v

A must be square

Eigen value

$$\begin{bmatrix} 5 & 2 & 1 \\ -2 & 1 & -1 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

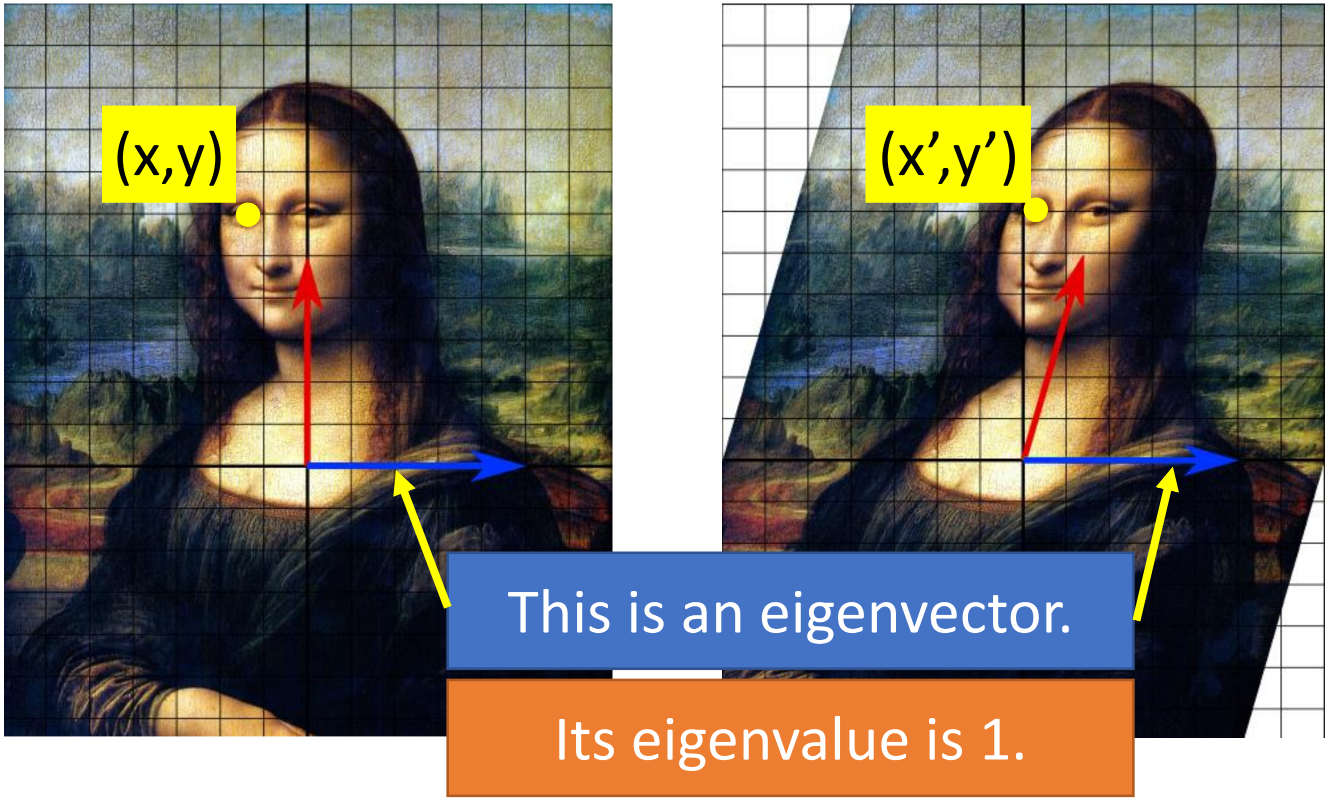
Eigen vector

Eigenvalues and Eigenvectors

- If $Av = \lambda v$ (v is a vector, λ is a scalar)
  - v is an eigenvector of A excluding zero vector
  - λ is an eigenvalue of A that corresponds to v
- T is a ***<u>linear operator.</u>*** If $\mathrm{T}(v) = \lambda v$ (v is a vector, λ is a scalar)
  - v is an eigenvector of T excluding zero vector
  - λ is an eigenvalue of T that corresponds to v

Eigenvalues and Eigenvectors

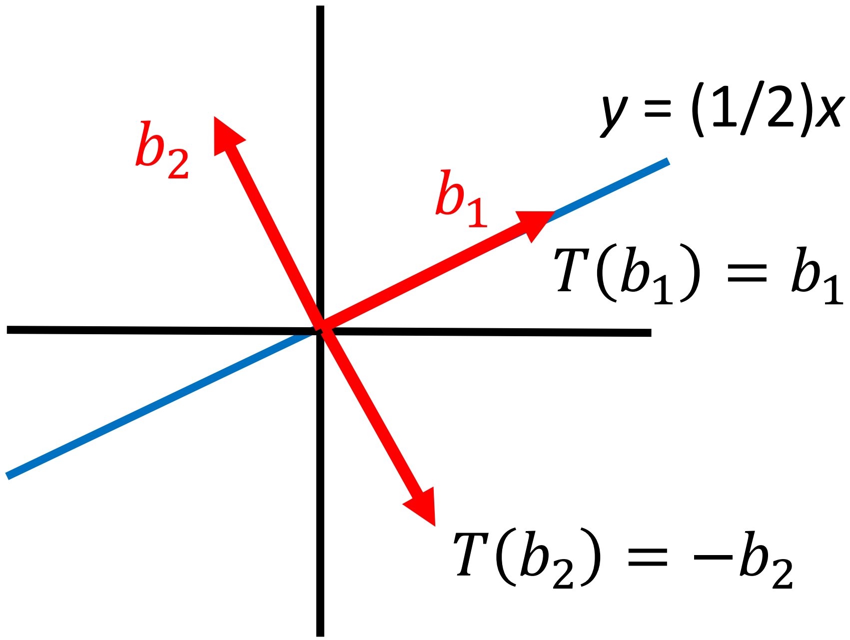
- Example: Shear Transform $\begin{bmatrix} x' \\ y' \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$



Eigenvalues and Eigenvectors

- Example: Reflection

reflection operator *T* about the line $y = (1/2)x$



$\mathbf{b}_1$ is an eigenvector of *T*

Its eigenvalue is 1.

$\mathbf{b}_2$ is an eigenvector of *T*

Its eigenvalue is -1.

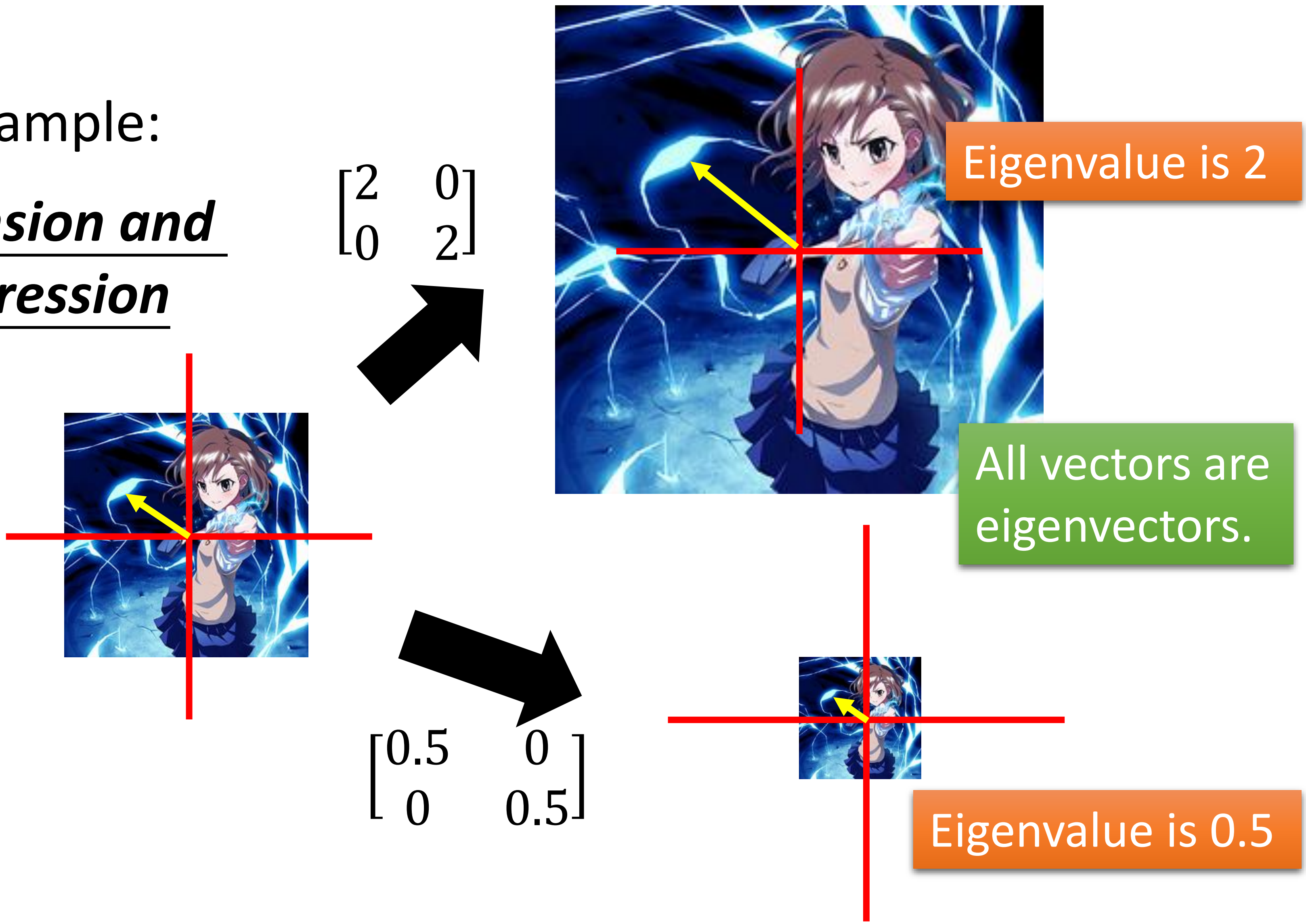
Eigenvalues and Eigenvectors

- Example:

Expansion and Compression

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$



Eigenvalues and Eigenvectors

- Example: Rotation

Do any n x n matrix or linear operator have eigenvalues?

How to find eigenvectors (given eigenvalues)

Eigenvalues and Eigenvectors

- An eigenvector of A corresponds to a unique eigenvalue.
- An eigenvalue of A has infinitely many eigenvectors.

Example:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \quad v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad u = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

Eigenvalue= -1

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Eigenvalue= -1

Do the eigenvectors correspond to the same eigenvalue form a subspace?

Eigenspace

- Assume we know λ is the eigenvalue of matrix A
- Eigenvectors corresponding to λ

$$A\mathbf{v} = \lambda\mathbf{v}$$

$$A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0}$$

$$A\mathbf{v} - \lambda I_n\mathbf{v} = \mathbf{0}$$

$$\underbrace{(A - \lambda I_n)}_{\text{matrix}}\mathbf{v} = \mathbf{0}$$

Eigenvectors corresponding to λ are nonzero solution of

$$(A - \lambda I_n)\mathbf{v} = \mathbf{0}$$

Eigenvectors corresponding to λ

$$= \underbrace{Null(A - \lambda I_n)}_{\text{eigenspace}} - \{\mathbf{0}\}$$

Eigenspace of λ:

Eigenvectors corresponding to λ + $\{\mathbf{0}\}$

Check whether a scalar is an eigenvalue

Check Eigenvalues

$Null(A - \lambda I_n)$:
eigenspace of λ

- How to know whether a scalar λ is the eigenvalue of A?

Check the dimension of eigenspace of λ

If the dimension is 0

→ Eigenspace only contains {0}

→ No eigenvector

→ λ is not eigenvalue

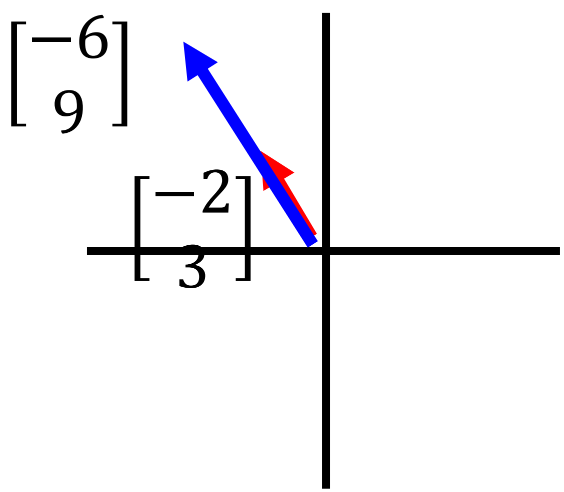
Check Eigenvalues

$Null(A-\lambda I_n)$: eigenspace of λ

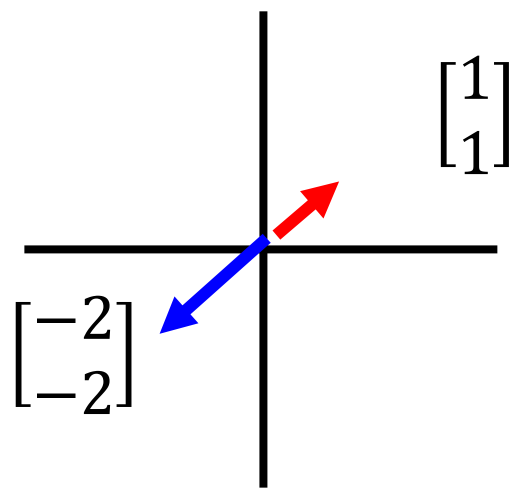
- Example: to check 3 and –2 are eigenvalues of the linear operator T

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -2x_2 \\ -3x_1 + x_2 \end{bmatrix} \quad A = \begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix}$$

$Null(A - 3I_n)$=?



$Null(A + 2I_n)$=?



Check Eigenvalues

$Null(A - \lambda I_n)$:
eigenspace of λ

- Example: check that 3 is an eigenvalue of B and find a basis for the corresponding eigenspace

$$B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

find the solution set of $(B - 3I_3)\mathbf{x} = \mathbf{0}$

find the RREF of

$B - 3I_3$

$$= \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_3 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Summary

- If $Av = \lambda v$ (v is a vector, λ is a scalar)
  - v is an eigenvector of A excluding zero vector
  - λ is an eigenvalue of A that corresponds to v
- Eigenvectors corresponding to λ are nonzero solution of $(A - \lambda I_n)\mathbf{v} = \mathbf{0}$

Eigenvectors corresponding to λ

$$= \underbrace{Null(A - \lambda I_n)}_{\text{eigenspace}} - \{\mathbf{0}\}$$

Eigenspace of λ:

Eigenvectors corresponding to λ $+ \{\mathbf{0}\}$

Homework