Characteristic Polynomial

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Outline

- Last lecture:
  - Given eigenvalues, we know how to find eigenvectors or eigenspaces
  - Check eigenvalues
- This lecture: How to find eigenvalues?

- Reference: Textbook 5.2

Looking for Eigenvalues

A scalar t is an eigenvalue of A

⟺ Existing $v \neq 0$ such that $Av = tv$

⟺ Existing $v \neq 0$ such that $Av - tv = 0$

⟺ Existing $v \neq 0$ such that $(A - tI_n)v = 0$

⟺ $(A - tI_n)v = 0$ has multiple solution

⟺ The columns of $(A - tI_n)$ are Dependent

⟺ Rank $(A - tI_n)$ < n ⟺ $(A - tI_n)$ is not invertible

⟺ $det(A - tI_n) = 0$

Characteristic Polynomial

A scalar t is an eigenvalue of A $\Leftrightarrow$ $det(A - tI_n) = 0$

A is the standard matrix of linear operator T

$det(A - tI_n)$: Characteristic polynomial of A

linear operator T

$det(A - tI_n) = 0$: Characteristic equation of A

linear operator T

Eigenvalues are the roots of characteristic polynomial or solutions of characteristic equation.

Looking for Eigenvalues

- Example 1: Find the eigenvalues of $A = \begin{bmatrix} -4 & -3 \\ 3 & 6 \end{bmatrix}$

A scalar t is an eigenvalue of A ⟺ $det(A - tI_n) = 0$

$$A - tI_2 = \begin{bmatrix} -4-t & -3 \\ 3 & 6-t \end{bmatrix}$$

$\det(A - tI_2)$ =0

➡ t = -3 or 5

The eigenvalues of *A* are -3 or 5.

Looking for Eigenvalues

- Example 1: Find the eigenvalues of $A = \begin{bmatrix} -4 & -3 \\ 3 & 6 \end{bmatrix}$

The eigenvalues of *A* are -3 or 5.

<u>Eigenspace of -3</u>

$$Ax = -3x \quad \Rightarrow \quad (A + 3I)x = 0$$

find the solution

<u>Eigenspace of 5</u>

$$Ax = 5x \quad \Rightarrow \quad (A - 5I)x = 0$$

find the solution

Looking for Eigenvalues

- Example 2: find the eigenvalues of linear operator

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 \\ 2x_1 - x_2 - x_3 \\ -x_3 \end{bmatrix} \xrightarrow{\text{standard matrix}} A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

A scalar t is an eigenvalue of A $\Leftrightarrow$ $det(A - tI_n) = 0$

$$A - tI_n = \begin{bmatrix} -1-t & 0 & 0 \\ 2 & -1-t & -1 \\ 0 & 0 & -1-t \end{bmatrix}$$

$\Rightarrow$ $det(A - tI_n) = (-1 - t)^3$

Looking for Eigenvalues

- Example 3: linear operator on $\mathcal{R}^2$ that rotates a vector by 90°

A scalar t is an eigenvalue of A ⟺ $det(A - tI_n) = 0$

standard matrix of the 90°-rotation: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\det\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - tI_2\right)$$

No eigenvalues, no eigenvectors

Characteristic Polynomial

- In general, a matrix A and RREF of A have different characteristic polynomials. ➡ Different Eigenvalues
- Similar matrices have the same characteristic polynomials ➡ The same Eigenvalues

$B = P^{-1}AP$

$$det(B - tI) = det(P^{-1}AP - P^{-1}(tI)P)$$

$$= det(P^{-1}(A - tI)AP)$$

$$= det(P^{-1})det(A - tI)det(P)$$

$$= \left(\frac{1}{det(P)}\right)det(A - tI)det(P) = det(A - tI)$$

Characteristic Polynomial

- Question: What is the order of the characteristic polynomial of an $n \times n$ matrix A?
  - The characteristic polynomial of an $n \times n$ matrix is indeed a polynomial with degree n
  - Consider $\det(A - tI_n)$
- Question: What is the number of eigenvalues of an $n \times n$ matrix A?
  - Fact: An n x n matrix A have less than or equal to n eigenvalues
  - Consider complex roots and multiple roots

Characteristic Polynomial

- If nxn matrix A has n eigenvalues (including multiple roots)

Sum of n eigenvalues	=	Trace of A
Product of n eigenvalues	=	Determinant of A

Example

$$A = \begin{bmatrix} -4 & -3 \\ 3 & 6 \end{bmatrix}$$

Eigenvalues:
-3, 5

Characteristic Polynomial

- The eigenvalues of an upper triangular matrix are its diagonal entries.

Characteristic Polynomial:

$$\begin{bmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{bmatrix}$$

$$det\begin{bmatrix} a-t & * & * \\ 0 & b-t & * \\ 0 & 0 & c-t \end{bmatrix}$$

$$= (a-t)(b-t)(c-t)$$

The determinant of an upper triangular matrix is the product of its diagonal entries.

Characteristic Polynomial v.s. Eigenspace

- Characteristic polynomial of A is

$det(A - tI_n)$ Factorization

multiplicity

$$= (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2} \dots (t-\lambda_k)^{m_k}(\dots\dots)$$

Eigenvalue:	λ_1	λ_2	λ_k
Eigenspace: (dimension)	d_1 $\leq m_1$	d_2 $\leq m_2$	d_k $\leq m_k$

Characteristic Polynomial v.s. Eigenspace

- Example 1:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

characteristic polynomials:

$$-(t+1)^2(t-3)$$

Eigenvalue -1

Multiplicity of "-1" is 2 Dim of eigenspace is 1 or 2

Dim = 2

Eigenvalue 3

Multiplicity of "3" is 1 Dim of eigenspace must be 1

Characteristic Polynomial v.s. Eigenspace

- Example 2:

$$B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

characteristic polynomials:

$$-(t+1)(t-3)^2$$

Eigenvalue -1

Multiplicity of “-1” is 1

Dim of eigenspace must be 1

Eigenvalue 3

Multiplicity of “3” is 2

Dim of eigenspace is 1 or 2

Dim = 2

Characteristic Polynomial v.s. Eigenspace

- Example 3:

$$C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

characteristic polynomials:

$$-(t+1)(t-3)^2$$

Eigenvalue -1

Multiplicity of "-1" is 1 Dim of eigenspace must be 1

Eigenvalue 3

Multiplicity of "3" is 2 Dim of eigenspace is 1 or 2

Dim = 1

	Characteristic polynomial	Eigenvalues	Eigenspaces
$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$	$-(t+1)^2(t-3)$	-1 →	2
		3 →	1
$B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$	$-(t+1)(t-3)^2$	-1 →	1
		3 →	2
$C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$	$-(t+1)(t-3)^2$	-1 →	1
		3 →	1

Summary

- Characteristic polynomial of A is

$det(A - tI_n)$ Factorization

$= (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2} \dots (t-\lambda_k)^{m_k}(\dots\dots)$

multiplicity

Eigenvalue:	λ_1	λ_2	λ_k
Eigenspace: (dimension)	d_1	d_2	d_k
	$\leq m_1$	$\leq m_2$	$\leq m_k$

Homework