Review

5.1 T&F

54. If a nonzero vector $\mathbf{v}$ is in the null space of a linear operator T, then $\mathbf{v}$ is an eigenvector of T.
55. If $\mathbf{v}$ is an eigenvector of a matrix A, then $c\mathbf{v}$ is also an eigenvector for any scalar c.
56. If $\mathbf{v}$ is an eigenvector of a matrix A, then $c\mathbf{v}$ is also an eigenvector for any nonzero scalar c.
57. If A and B are $n \times n$ matrices and λ is an eigenvalue of both A and B, then λ is an eigenvalue of $A + B$.
58. If A and B are $n \times n$ matrices and $\mathbf{v}$ is an eigenvector of both A and B, then $\mathbf{v}$ is an eigenvector of $A + B$.
59. If A and B are $n \times n$ matrices and λ is an eigenvalue of both A and B, then λ is an eigenvalue of AB.

60. If A and B are $n \times n$ matrices and $\mathbf{v}$ is an eigenvector of both A and B, then $\mathbf{v}$ is an eigenvector of AB.

5.1

72. An $n \times n$ matrix A is called *nilpotent* if, for some positive integer k, $A^k = O$, where O is the $n \times n$ zero matrix. Prove that 0 is the only eigenvalue of a nilpotent matrix.

72. Suppose λ is an eigenvalue of a nilpotent matrix A. Then $A\mathbf{v} = \lambda\mathbf{v}$ for some $\mathbf{v} \neq \mathbf{0}$. Multiplying both sides by A^{k-1} and using the result of Exercise 46, we obtain $\mathbf{0} = O\mathbf{v} = A^k\mathbf{v} = \lambda^k\mathbf{v}$. Since $\mathbf{v} \neq \mathbf{0}$, we must have $\lambda = 0$.

5.2

75. Suppose that the characteristic polynomial of an $n \times n$ matrix A is

$$a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0.$$

Determine the characteristic polynomial of $-A$.

(b) Establish a relationship between the characteristic polynomial of any square matrix B and that of B^T.

(c) What does (b) imply about the relationship between the eigenvalues of a square matrix B and those of B^T?

(d) Is there a relationship between the eigenvectors of a square matrix B and those of B^T?

5.2

84. Let A and B be $n \times n$ matrices such that $B = P^{-1}AP$, and let λ be an eigenvalue of A (and hence of B). Prove the following results:

(a) A vector $\mathbf{v}$ in $\mathcal{R}^n$ is in the eigenspace of A corresponding to λ if and only if $P^{-1}\mathbf{v}$ is in the eigenspace of B corresponding to λ.

(b) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis for the eigenspace of A corresponding to λ, then $\{P^{-1}\mathbf{v}_1, P^{-1}\mathbf{v}_2, \ldots, P^{-1}\mathbf{v}_k\}$ is a basis for the eigenspace of B corresponding to λ.

(c) The eigenspaces of A and B that correspond to the same eigenvalue have the same dimension.

5.3 T&F

38. If, for each eigenvalue λ of A, the multiplicity of λ equals the dimension of the corresponding eigenspace, then A is diagonalizable.

45. If S is a set of distinct eigenvectors of a matrix, then S is linearly independent.

46. If S is a set of eigenvectors of a matrix A that correspond to distinct eigenvalues of A, then S is linearly independent.

41. A diagonal $n \times n$ matrix has n distinct eigenvalues.

47. If the characteristic polynomial of a matrix A factors into a product of linear factors, then A is diagonalizable.

5.3

$$\mathrm{tr}(A) = \sum_i \lambda_i . \qquad \det(A) = \prod_i \lambda_i$$

88. Let A be a diagonalizable $n \times n$ matrix. Prove that if the characteristic polynomial of A is $f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$, then $f(A) = O$, where $f(A) = a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 I_n$. (This result is called the *Cayley-Hamilton theorem.*[7]) *Hint:* If $A = PDP^{-1}$, show that $f(A) = Pf(D)P^{-1}$.

89. The **trace** of a square matrix is the sum of its diagonal entries.

    (a) Prove that if A is a diagonalizable matrix, then the trace of A equals the sum of the eigenvalues of A. *Hint:* For all $n \times n$ matrices A and B, show that the trace of AB equals the trace of BA.

Chapter 5 review

13. If P is an invertible $n \times n$ matrix and D is a diagonal $n \times n$ matrix such that $A = P^{-1}DP$, then the columns of P form a basis for $\mathcal{R}^n$ consisting of eigenvectors of A.

13. False, if $A = PDP^{-1}$, where P is an invertible matrix and D is a diagonal matrix, then the columns of P are a basis for $\mathcal{R}^n$ consisting of eigenvectors of A.

Chapter 6-1

61. V is the set of all $n \times n$ matrices with determinant equal to 0.

62. V is the set of all $n \times n$ matrices A such that $A^2 = A$.

67. V is the subset of $\mathcal{P}$ consisting of the zero polynomial and all polynomials of the form $c_0 + c_1x + \cdots + c_mx^m$ with $c_k \neq 0$ if k is even.

68. V is the subset of $\mathcal{P}$ consisting of the zero polynomial and all polynomials of the form $c_0 + c_1x + \cdots + c_mx^m$ with $c_i \geq 0$ for all i.

72. Let s_1 and s_2 be elements of S, and let V be the set of all functions f in $\mathcal{F}(S)$ such that $f(s_1) \cdot f(s_2) = 0$.

Chapter 6-2

- T&F

46. The definite integral is a linear operator on $C([a,b])$, the vector space of continuous real-valued functions defined on $[a,b]$.

48. The solution set of the differential equation $y'' + 4y = \sin 2t$ is a subspace of C^∞.

59. Recall the set $C([a,b])$ in Example 3.

(b) Let $T: C([a,b]) \to C([a,b])$ be defined by

$$T(f)(x) = \int_a^x f(t)dt \qquad \text{for } a \le x \le b.$$

Prove that T is linear and one-to-one.

How to check one-to-one

Chapter 6-3

- T&F

31. If a set is infinite, it cannot be linearly independent.
32. Every vector space has a finite basis.
33. The dimension of the vector space $\mathcal{P}_n$ equals n.

35. It is possible for a vector space to have both an infinite basis and a finite basis.
36. If every finite subset of S is linearly independent, then S is linearly independent.
37. Every nonzero finite-dimensional vector space is isomorphic to $\mathcal{R}^n$ for some n.

Chapter 6-3

- Find Basis

52. Let W be the subspace of skew-symmetric 3×3 matrices and $V = \mathcal{M}_{3\times3}$.

55. Let W be the subspace of $V = \mathcal{P}_n$ consisting of polynomials $p(x)$ for which $p(1) = 0$.

56. Let $W = \{f \in \mathcal{D}(\mathcal{R}) : f' = f\}$, where f' is the derivative of f and $\mathcal{D}(\mathcal{R})$ is the set of functions in $\mathcal{F}(\mathcal{R})$ that are differentiable, and let $V = \mathcal{F}(\mathcal{R})$.

Chapter 6-4

- T&F

29. Every linear operator can be represented by a matrix.

30. Every linear operator on a nonzero finite-dimensional vector space can be represented by a matrix.

41. Let D be the derivative operator on $\mathcal{P}_2$.
    (a) Find the eigenvalues of D.
    (b) Find a basis for each of the corresponding eigenspaces.

For $\mathcal{B} = \{1, x, x^2\}$, which is a basis for $\mathcal{P}_2$, we see that

$$[D]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) Since the characteristic polynomial of $[D]_{\mathcal{B}}$ is $-t^3$, D has only one eigenvalue, $\lambda = 0$.

(b) Since

Chapter 6-5

1. $\langle u, u \rangle > 0$ if $u \neq 0$
2. $\langle u, v \rangle = \langle v, u \rangle$
3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
4. $\langle au, v \rangle = a\langle u, v \rangle$

47. Let V be a finite-dimensional vector space and $\mathcal{B}$ be a basis for V. For $\mathbf{u}$ and $\mathbf{v}$ in V, define

$$\langle \mathbf{u}, \mathbf{v} \rangle = [\mathbf{u}]_{\mathcal{B}} \cdot [\mathbf{v}]_{\mathcal{B}}.$$

Prove that this rule defines an inner product on V.

48. Let A be an $n \times n$ invertible matrix. For $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^n$, define

$$\langle \mathbf{u}, \mathbf{v} \rangle = (A\mathbf{u}) \cdot (A\mathbf{v}).$$

Prove that this rule defines an inner product on $\mathcal{R}^n$.

49. Let A be an $n \times n$ positive definite matrix (as defined in the exercises of Section 6.6). For $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^n$, define

$$\langle \mathbf{u}, \mathbf{v} \rangle = (A\mathbf{u}) \cdot \mathbf{v}.$$

Prove that this rule defines an inner product on $\mathcal{R}^n$.

Chapter 6-5

1. $\langle \boldsymbol{u}, \boldsymbol{u} \rangle > 0$ if $\boldsymbol{u} \neq \boldsymbol{0}$
2. $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$
3. $\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle$
4. $\langle a\boldsymbol{u}, \boldsymbol{v} \rangle = a\langle \boldsymbol{u}, \boldsymbol{v} \rangle$

53. Let $V = \mathsf{C}([0, 2])$, and

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt$$

for all f and g in V. (Note that the limits of integration are not 0 and 2.)

37. In an inner product space, $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$.