Subspace

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In Chapter 4 ……

- 民濕寢則腰疾偏死，鰌然乎哉？木處則惴慄恂懼，猨猴然乎哉？三者孰知正處？民食芻豢，麋鹿食薦，蝍蛆甘帶，鴟鴉耆鼠，四者孰知正味？猨猵狙以為雌，麋與鹿交，鰌與魚游。毛嬙、西施，人之所美也；魚見之深入，鳥見之高飛，麋鹿見之決驟，四者孰知天下之正色哉？《莊子·齊物論》

The same vector or operation is represented differently when they are in different coordinate systems.

Subspace

Reference

- Textbook: chapter 4.1

Subspace

- A vector set V is called a subspace if it has the following three properties:
- 1. The zero vector **0** belongs to V
- 2. If **u** and **w** belong to V, then **u+w** belongs to V

Closed under (vector) addition

- 3. If **u** belongs to V, and c is a scalar, then c**u** belongs to V

Closed under scalar multiplication

2+3 is linear combination

Examples

$$W = \left\{ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \in \mathcal{R}^3 : 6w_1 - 5w_2 + 4w_3 = 0 \right\}$$ Subspace?

Property 1. $\mathbf{0} \in W$ $\Longrightarrow$ $6(0) - 5(0) + 4(0) = 0$

Property 2. $\mathbf{u}, \mathbf{v} \in W \Rightarrow \mathbf{u}+\mathbf{v} \in W$

$\mathbf{u} = [\ u_1\ \ u_2\ \ u_3\ ]^T$, $\mathbf{v} = [\ v_1\ \ v_2\ \ v_3\ ]^T$ $\mathbf{u}+\mathbf{v}=[\ u_1+v_1\ \ u_2+v_1\ \ u_3+v_1\ ]^T$

$6(u_1+v_1) - 5(u_2+v_2) + 4(u_3+v_3)$

$= (6u_1 - 5u_2 + 4u_3\ ) + (6v_1 - 5v_2 + 4v_3\ ) = 0 + 0 = 0$

Property 3. $\mathbf{u} \in W \Rightarrow c\mathbf{u} \in W$

$6(cu_1) - 5(cu_2) + 4(cu_3) = c(6u_1 - 5u_2 + 4u_3) = c0 = 0$

Examples

$V = \{c\mathbf{w} \mid c \in \mathcal{R}\}$ Subspace?

$$\mathcal{S}_1 = \left\{ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathcal{R}^2 : w_1 \geq 0 \text{ and } w_2 \geq 0 \right\}$$

Subspace? $\mathbf{u} \in \mathcal{S}_1, \mathbf{u} \neq \mathbf{0} \Rightarrow -\mathbf{u} \notin \mathcal{S}_1$

$$\mathcal{S}_2 = \left\{ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathcal{R}^2 : w_1^2 = w_2^2 \right\}$$

Subspace? $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \in S_2 \text{ but } \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \notin S_2$

$\mathcal{R}^n$ Subspace? $\{\mathbf{0}\}$ Subspace? zero subspace

Null Space

- The null space of a matrix A is the solution set of Ax=0. It is denoted as Null A.

  Null $A = \{\, \mathbf{v} \in \mathcal{R}^n : \; A\mathbf{v} = \mathbf{0} \,\}$

  The solution set of the homogeneous system of linear equations $A\mathbf{v} = \mathbf{0}$.

- Null A is a subspace

A linear function is one-to-one $\rightleftarrows$ Null space only contain **0**

Null Space - Example

$$T : \mathcal{R}^3 \to \mathcal{R}^2 \text{ with } T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + 2x_3 \\ -x_1 + x_2 - 3x_3 \end{bmatrix}$$

Find a generating set for the null space of *T*.

The null space of *T* is the set of solutions to $A\mathbf{x} = \mathbf{0}$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -3 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x_1 = x_2$$

$$\cancel{x_1 - x_2 = 0}$$

$$x_3 = 0$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

a generating set for the null space

Subspace v.s. Span

- The span of a vector set is a subspace

  Let $S = \{w_1, w_2, \cdots, w_k\}$ $\quad V = Span\ S$

  Property 1. $\mathbf{0} \in V$

  Property 2. $\boldsymbol{u}, \boldsymbol{v} \in V, \boldsymbol{u} + \boldsymbol{v} \in$V

  Property 3. $\boldsymbol{u} \in V$, c$\boldsymbol{u} \in V$

Column Space and Row Space

- Column space of a matrix A is the span of its columns. It is denoted as Col A.

$$A \in \mathcal{R}^{m \times n} \Rightarrow \text{Col } A = \{A\mathbf{v} : \mathbf{v} \in \mathcal{R}^n\}$$

If matrix A represents a function

Col A is the range of the function

- Row space of a matrix A is the span of its rows. It is denoted as Row A.

$$\text{Row } A = \text{Col } A^T$$

RREF

- Original Matrix A v.s. its RREF R
  - Columns:
    - The relations between the columns are the same.
    - The span of the columns are different.

$$Col\ \mathrm{A} \neq Col\ R$$

  - Rows:
    - The relations between the rows are changed.
    - The span of the rows are the same.

$$Row\ \mathrm{A} = Row\ R$$

Column Space = Range

- The range of a linear transformation is the same as the column space of its standard matrix.

Linear Transformation

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + x_3 - x_4 \\ 2x_1 + 4x_2 - 8x_4 \\ 2x_3 + 6x_4 \end{bmatrix}$$

Standard matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 0 & -8 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$ $\Rightarrow$ Range of *T* =

$$\text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -8 \\ 6 \end{bmatrix}\right\}$$

Consistent

Ax = b have solution (consistent)

b is the linear combination of columns of A

b is in the span of the columns of A

b is in Col A

$$A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 0 & -8 \\ 0 & 0 & 2 & 6 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \in \mathrm{Col}\, A? \quad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \in \mathrm{Col}\, A?$$

Solving Ax = u

$$\text{RREF([A u])} = \begin{bmatrix} 1 & 2 & 0 & -4 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Solving Ax = v

$$\text{RREF([A v])} = \begin{bmatrix} 1 & 2 & 0 & -4 & 0.5 \\ 0 & 0 & 1 & 3 & 1.5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Conclusion: Subspace is Closed under addition and multiplication