

Vectors, Matrices and their Products

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Learning Target

- A system of linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Describe a system of linear equations
by Matrix-Vector Products

Vectors, Matrices and their Products

Vector

Vectors

- A vector v is a set of numbers

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Column vector

$$v = [1 \quad 2 \quad 3]$$

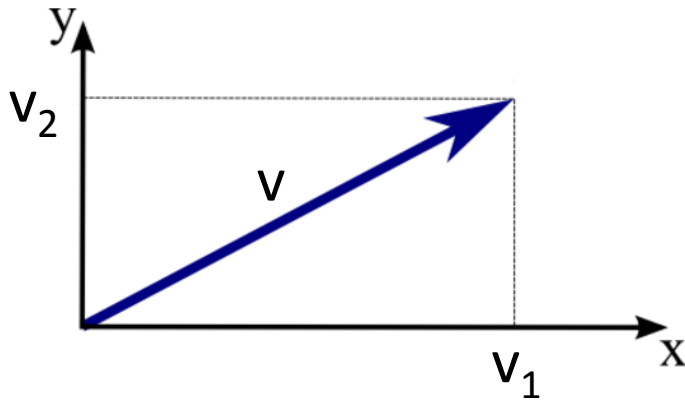
Row vector

In this course, the term **vector** refers to a **column vector** unless being explicitly mentioned otherwise.

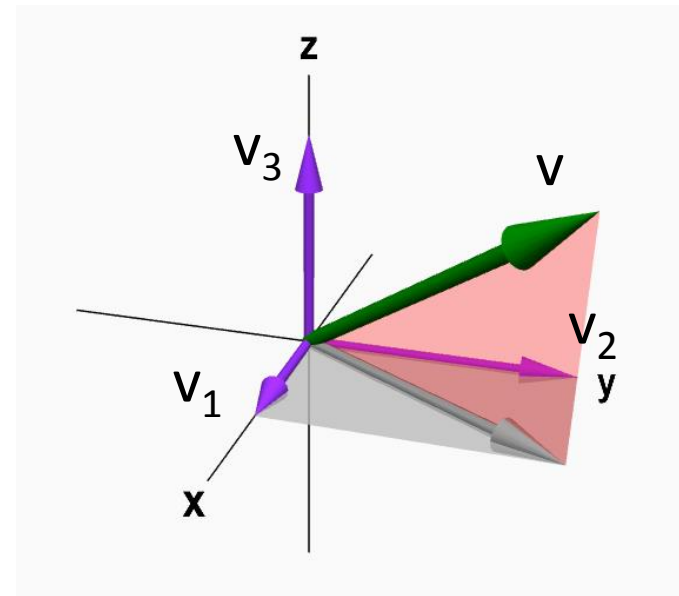
Vectors

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

- **components**: the entries of a vector.
 - The i -th component of vector v refers to v_i
 - $v_1=1, v_2=2, v_3=3$
- If a vector only has less than three components, you can visualize it.



http://mathinsight.org/vectors_cartesian_coordinates_2d_3d#vector3D

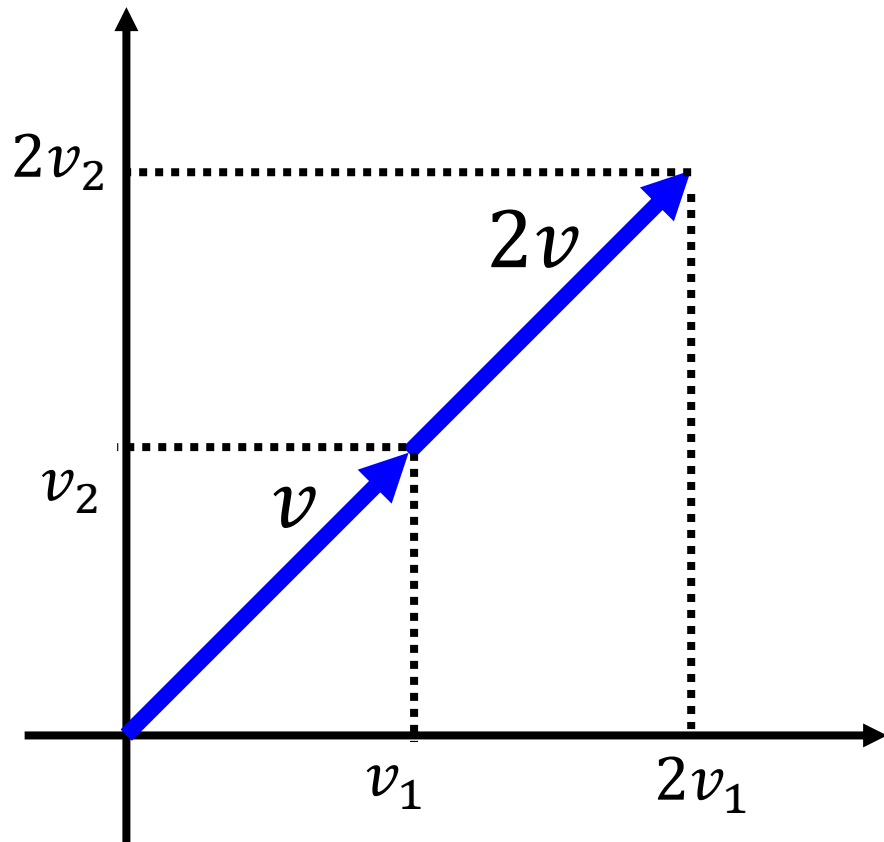


Scalar Multiplication

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

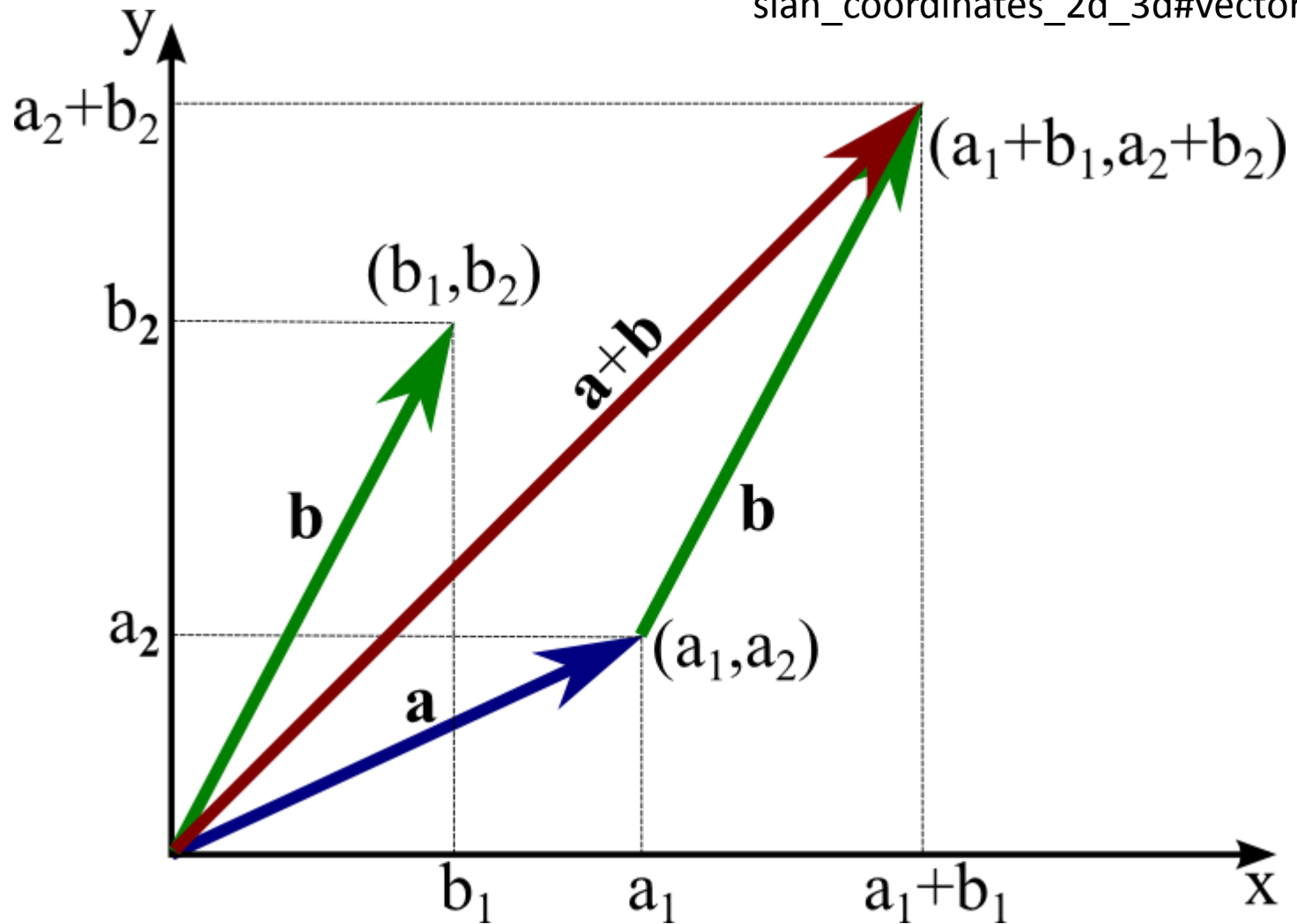


cv



Vector Addition

http://mathinsight.org/vectors_cartesian_coordinates_2d_3d#vector3D



Special Vectors

- zero vector $\mathbf{0}$

$$\mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{Can be any size}$$

$$\mathbf{0} + v = v$$

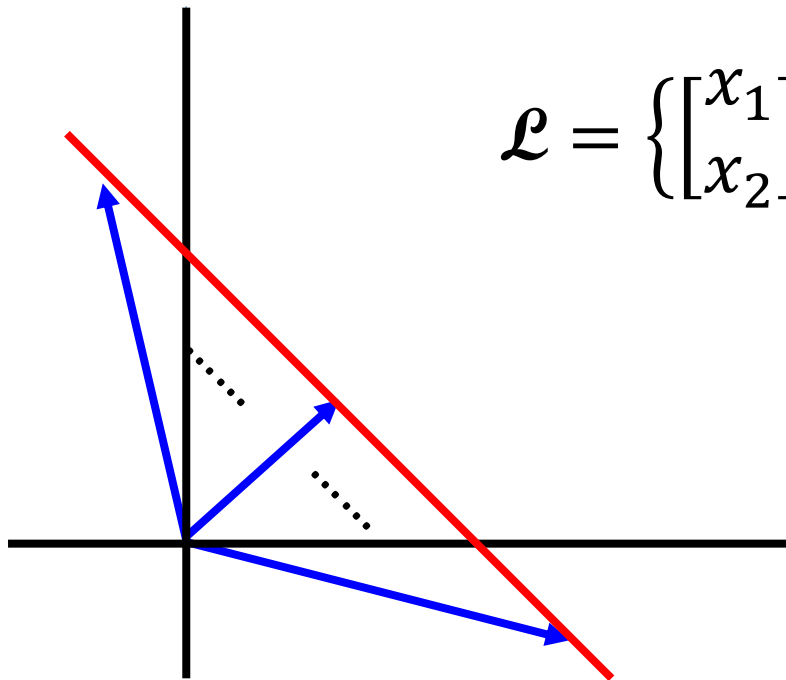
$$0v = \mathbf{0}$$

- Standard vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Vector Set $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 6 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 9 \\ 0 \\ 2 \end{bmatrix} \right\}$ A vector set with 4 elements

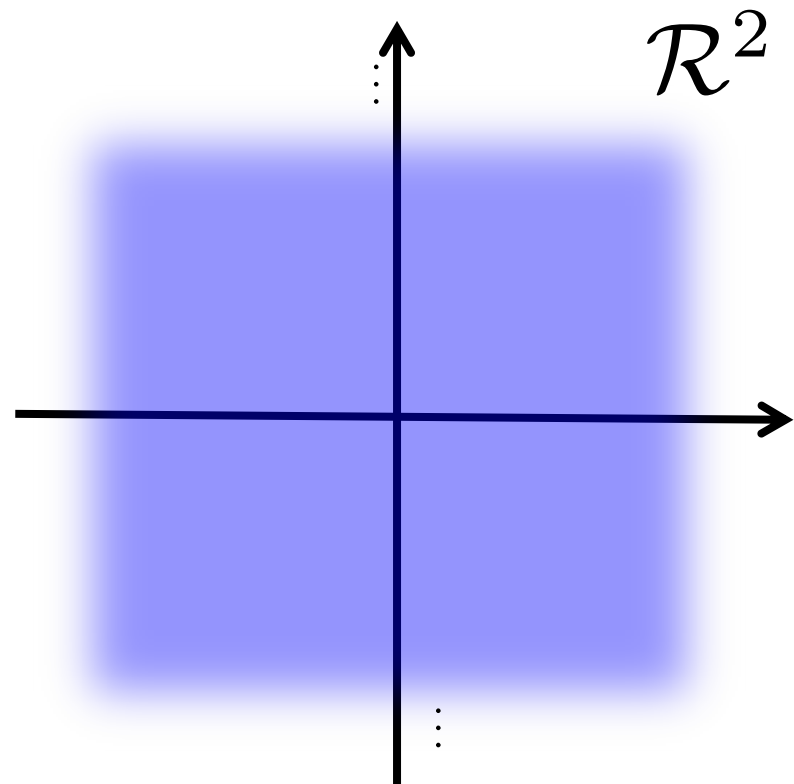
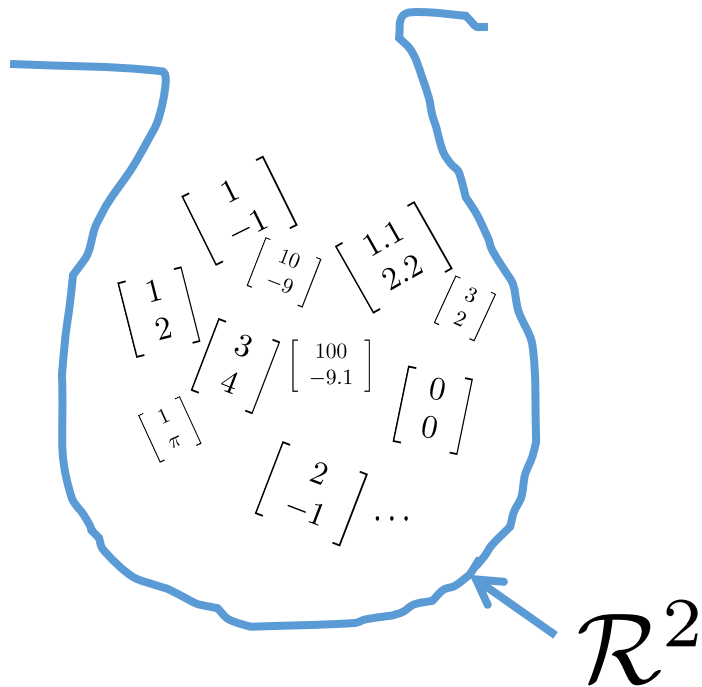
- A vector set can contain infinite elements



$$\mathcal{L} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 + x_2 = 1 \right\}$$

Vector Set

- \mathcal{R}^n : We denote the set of all **vectors** with n entries by \mathcal{R}^n .



Vectors, Matrices and their Products

Matrix

Matrix

- A matrix is a set of vectors

$$a_1 = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} \quad a_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \quad a_3 = \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix}$$

$$A = [a_1 \quad a_2 \quad a_3] = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 1 & -1 \\ -2 & 1 & 1 \end{bmatrix}$$

Matrix

- If the matrix has m rows and n columns, we say the size of the matrix is m by n , written $m \times n$
 - The matrix is called square if $m=n$
 - We use $\mathcal{M}_{m \times n}$ to denote the set that contains all matrices whose size is $m \times n$

$$\begin{array}{c} \text{3 columns} \\ \text{2 rows} \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in \mathcal{M}_{2 \times 3}$$

2×3

$$\begin{array}{c} \text{2 columns} \\ \text{3 rows} \end{array} \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \in \mathcal{M}_{3 \times 2}$$

3×2

先 Row 再 Column

Matrix

- **Index of component**: the scalar in the i-th row and j-th column is called (i,j)-entry of the matrix

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

先 Row 再 Column

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 1 & -1 \\ -2 & 1 & 1 \end{bmatrix}$$

(1,2)-entry

(3,1)-entry

(3,3)-entry

Matrix

- Two matrices with the same size can add or subtract.
- Matrix can multiply by a scalar

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 6 & 9 \\ 8 & 0 \\ 9 & 2 \end{bmatrix} \quad 9B$$

$$A + B$$

$$A - B$$

Zero Matrix

- **zero matrix**: matrix with all zero entries, denoted by O (any size) or $O_{m \times n}$.
 - For example, a 2-by-3 zero matrix can be denoted

$$O_{2 \times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A + O = A$$

$$OA = O$$

$$A - A = O$$

- Identity matrix: must be square
 - 對角線是 1, 其它都是 0

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Sometimes I_n is simply written as I (any size).

Properties

- A, B, C are $m \times n$ matrices, and s and t are scalars
 - $A + B = B + A$
 - $(A + B) + C = A + (B + C)$
 - $(st)A = s(tA)$
 - $s(A + B) = sA + sB$
 - $(s+t)A = sA + tA$

Transpose

- If A is an $m \times n$ matrix
- A^T (transpose of A) is an $n \times m$ matrix whose (i,j) -entry is the (j,i) -entry of A

$$A = \begin{bmatrix} 6 & 9 \\ 8 & 0 \\ 9 & 2 \end{bmatrix} \xrightarrow{\text{Transpose}} A^T = \begin{bmatrix} 6 & 8 & 9 \\ 9 & 0 & 2 \end{bmatrix}$$

The diagram illustrates the transpose operation. Matrix A is a 3×2 matrix with elements 6, 9, 8, 0, 9, 2. The element 9 at row 1, column 2 is highlighted with a blue box and labeled (1,2). The element 2 at row 3, column 2 is highlighted with a red box and labeled (3,2). A blue arrow labeled "Transpose" points to matrix A^T , which is a 2×3 matrix with elements 6, 8, 9, 9, 0, 2. The element 9 at row 2, column 1 is highlighted with a blue box and labeled (2,1). The element 2 at row 2, column 3 is highlighted with a red box and labeled (2,3).

以左上到右下的對角線為軸
進行翻轉

Transpose

$$A = \begin{bmatrix} 5 & 5 \\ 6 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 7 \\ 8 & 8 \end{bmatrix}$$

- A and B are mxn matrices, and s is a scalar

- $(A^T)^T = A$

- $(sA)^T = sA^T$

$$2A = \begin{bmatrix} 10 & 10 \\ 12 & 12 \end{bmatrix} \quad (2A)^T = \begin{bmatrix} 10 & 12 \\ 10 & 12 \end{bmatrix}$$

- $(A + B)^T = A^T + B^T$

$$A^T = \begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix} \quad 2A^T = \begin{bmatrix} 10 & 12 \\ 10 & 12 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 12 & 12 \\ 14 & 14 \end{bmatrix}$$

$$(A + B)^T = \begin{bmatrix} 12 & 14 \\ 12 & 14 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix} \quad B^T = \begin{bmatrix} 7 & 8 \\ 7 & 8 \end{bmatrix}$$

$$A^T + B^T = \begin{bmatrix} 12 & 14 \\ 12 & 14 \end{bmatrix}$$

Vectors, Matrices
and their Products
Matrix-Vector Products

Matrix-Vector Product

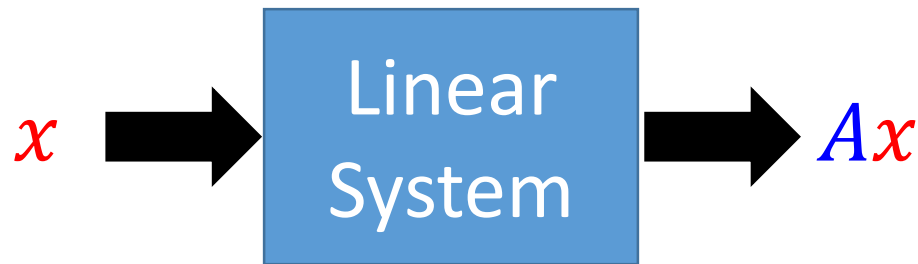
$$A = \begin{matrix} m \times n \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{matrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

Matrix-Vector Product

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{array} = \begin{array}{l} b_1 \\ b_2 \\ \vdots \\ b_m \end{array}$$

$Ax = b$



Coefficients are A

Row Aspect

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

Column Aspect

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

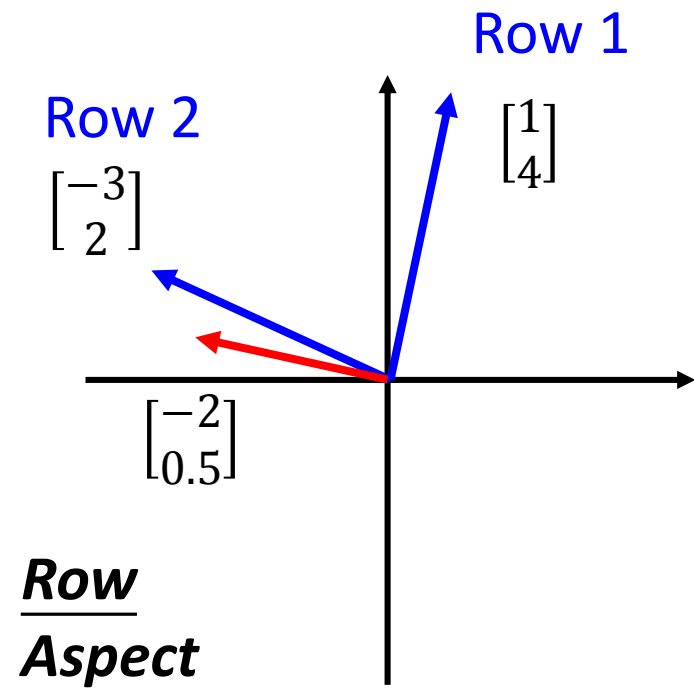
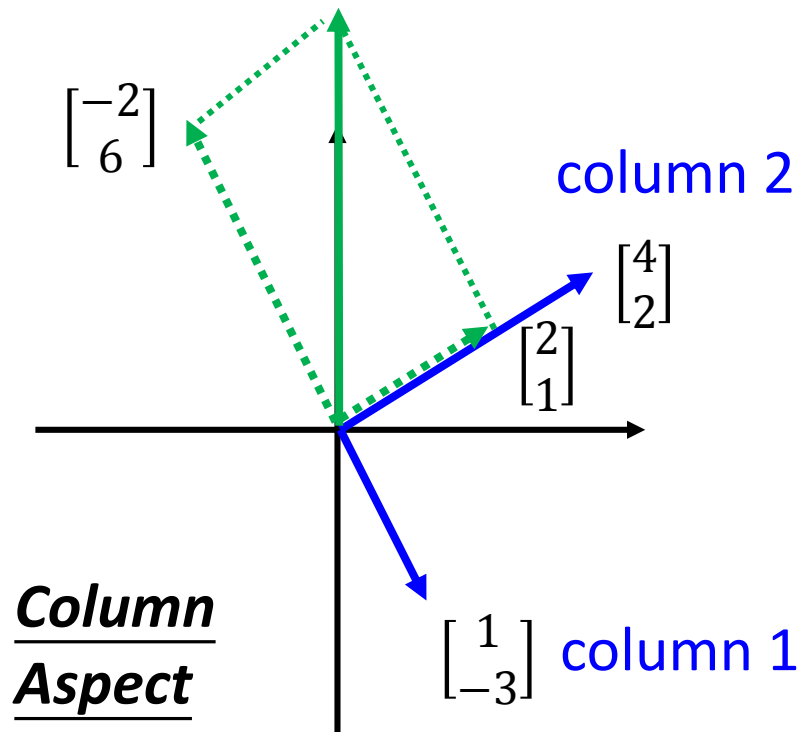
$$Ax =$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix}$$

$$\begin{aligned} x_1 + 4x_2 &= b_1 \\ -3x_1 + 2x_2 &= b_2 \end{aligned} \iff \begin{bmatrix} -2 \\ 0.5 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \end{matrix} \rightarrow \begin{bmatrix} A \end{bmatrix} \rightarrow \begin{matrix} b_1 \\ b_2 \end{matrix} = Ax$$



Matrix-vector Product

- The size of matrix and vector should be matched.

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 1 & -1 \\ -2 & 1 & 1 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A' = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$A'' = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 0 & -1 \\ 1 & -3 \end{bmatrix}$$

Properties of Matrix-vector Product

- A and B are $m \times n$ matrices, u and v are vectors in \mathbb{R}^n , and c is a scalar.
- $A(u + v) = Au + Av$
- $A(cu) = c(Au) = (cA)u$
- $(A + B)u = Au + Bu$
- $A\mathbf{0}$ is the $m \times 1$ zero vector
- $\mathbf{0}v$ is also the $m \times 1$ zero vector
- $I_n v = v$

Properties of Matrix-vector Product

- A and B are $m \times n$ matrices. If $Aw = Bw$ for all w in \mathbb{R}^n . Is it true that $A = B$?

$Ae_j = a_j$ for $j = 1, 2, \dots, n$, where e_j is the j -th standard vector in \mathbb{R}^n

$$e_1 = \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix} \quad Ae_1 = [a_1 \quad \cdots \quad a_n] \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix} = 1 \cdot a_1 + 0 \cdot a_2 + \cdots + 0 \cdot a_n = a_1$$

$$Ae_1 = Be_1$$



$$a_1 = b_1$$

$$Ae_2 = Be_2$$



$$a_2 = b_2$$

.....

$$Ae_n = Be_n$$



$$a_n = b_n$$



$$A = B$$

Concluding Remarks

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

Row
Aspect

$$= x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Column
Aspect