Basis

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Outline

- What is a basis for a subspace?
- Confirming that a set is a basis for a subspace

- Reference: Textbook 4.2

What is Basis?

Basis

Why nonzero?

- Let V be a nonzero subspace of R^n. A basis B for V is a linearly independent generation set of V.

$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis for $\mathcal{R}^n$.

1. $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is independent
2. $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ generates $\mathcal{R}^n$.

$\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is a basis for $\mathcal{R}^2$

$\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$ $\left\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}-3\\1\end{bmatrix}\right\}$ $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right\}$ …… any two independent vectors form a basis for $\mathcal{R}^2$

Basis

- The pivot columns of a matrix form a basis for its columns space.

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

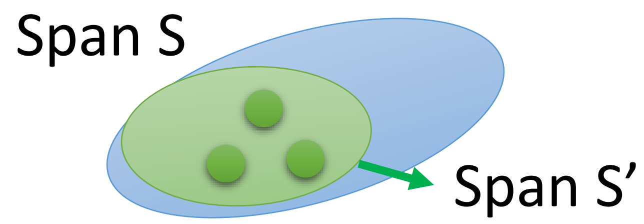
$$\text{Col } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} \right\}$$

Property

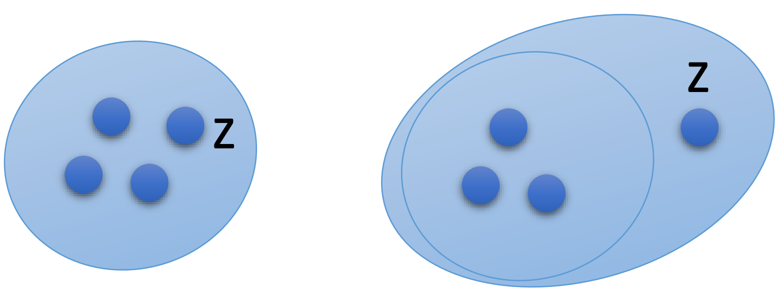
- (a) S is contained in Span S

Basis is always in its subspace

- (b) If a finite set S’ is contained in Span S, then Span S’ is also contained in Span S
  - Because Span S is a subspace



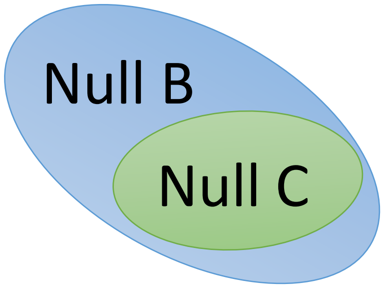
- (c) For any vector z, Span S = Span SU{z} if and only if z belongs to the Span S



Theorem

- 1. A basis is the smallest generation set.
- 2. A basis is the largest independent vector set in the subspace.
- 3. Any two bases for a subspace contain the same number of vectors.
  - The number of vectors in a basis for a nonzero subspace V is called dimension of V (dim V).

Theorem 3



- Any two bases of a subspace V contain the same number of vectors

Suppose $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_p\}$ are two bases of V.

Let $A = [\mathbf{u}_1\ \mathbf{u}_2 \cdots \mathbf{u}_k]$ and $B = [\mathbf{w}_1\ \mathbf{w}_2 \cdots \mathbf{w}_p]$.

Since $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ spans V, $\exists\ \mathbf{c}_i \in \mathcal{R}^k$ s.t. $A\mathbf{c}_i = \mathbf{w}_i$ for all i

$\Rightarrow A[\mathbf{c}_1\ \mathbf{c}_2 \cdots \mathbf{c}_p] = [\mathbf{w}_1\ \mathbf{w}_2 \cdots \mathbf{w}_p] \Rightarrow AC = B$

Now $C\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \in \mathcal{R}^p \Rightarrow AC\mathbf{x} = B\mathbf{x} = \mathbf{0}$

B is independent vector set $\Rightarrow x = \mathbf{0} \Rightarrow \mathbf{c}_1\ \mathbf{c}_2 \cdots \mathbf{c}_p$ are independent

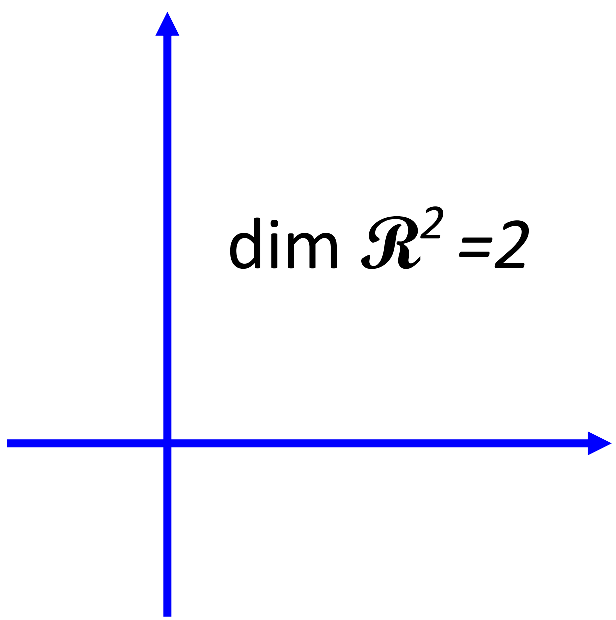
$\mathbf{c}_i \in \mathcal{R}^k \Rightarrow p \leq k$

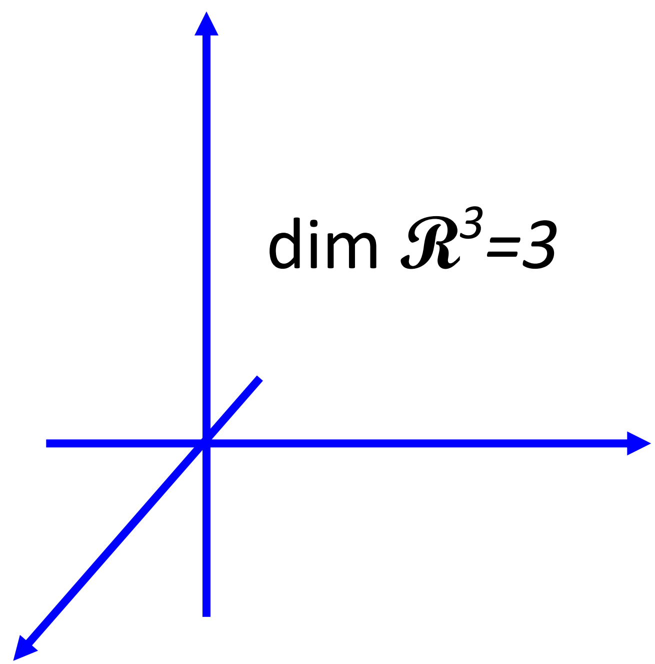
Reversing the roles of the two bases one has $k \leq p \Rightarrow p = k$.

Theorem 3

Every basis of $\mathscr{R}^n$ has n vectors.

- The number of vectors in a basis for a subspace V is called the dimension of V, and is denoted dim V
  - The dimension of zero subspace is 0





Example

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathcal{R}^4 : \begin{matrix} x_1 - 3x_2 + 5x_3 - 6x_4 = 0 \\ x_1 = 3x_2 - 5x_3 + 6x_4 \end{matrix} \right\}$$

Find dim V

dim V = 3

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_2 - 5x_3 + 6x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Basis? Independent vector set that generates V

Theorem 1

A basis is the smallest generation set.

If there is a generation set S for subspace V,

The size of basis for V is smaller than or equal to S.

Reduction Theorem

There is a basis containing in any generation set S.

S can be reduced to a basis for V by removing some vectors.

Theorem 1 – Reduction Theorem

所有的 generation set 心中都有一個 basis

S can be reduced to a basis for V by removing some vectors.

Suppose $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is a generation set of subspace V

Subspace $V = Span\ S$
$= Col\ A$

Let $A = [\ \mathbf{u}_1\ \ \mathbf{u}_2\ \ \cdots\ \ \mathbf{u}_k\ ]$.

⇒ The basis of Col A is the pivot columns of A   Subset of $\mathcal{S}$

Theorem 1 – Reduction Theorem

所有的 generation set 心中都有一個 basis

Subspace $V = Span\, S = Col\, A$ = Span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} \right\}$

Smallest generation set

$$S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 3 \\ 9 \end{bmatrix} \right\}$$

Generation set

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem 2

A basis is the largest independent set in the subspace.

If the size of basis is k, then you cannot find more than k ***independent*** vectors in the subspace.

Extension Theorem

Given an independent vector set S in the space

S can be extended to a basis by adding more vectors

Theorem 2 – Extension Theorem

Independent set:
我不是一個 basis 就是正在成為一個 basis

There is a subspace V

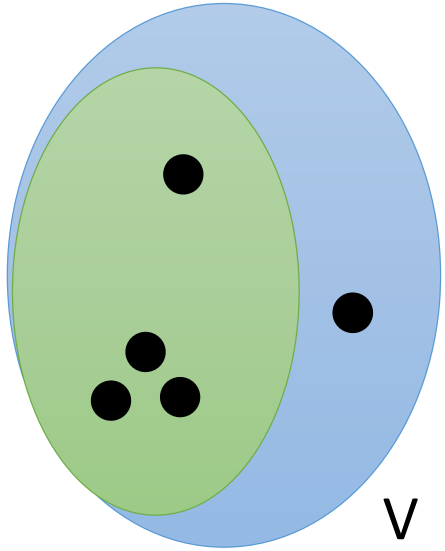
Given a independent vector set S (elements of S are in V)

- If Span S = V, then S is a basis
- If Span S ≠ V, find v_1 in V, but not in Span S

  S = S ∪ {v_1} is still an independent set

- If Span S = V, then S is a basis
- If Span S ≠ V, find v_2 in V, but not in Span S

  S = S ∪ {v_2} is still an independent set



…… You will find the basis in the end.

More from Theorems

A basis is the smallest generation set.

A vector set generates $\mathcal{R}^m$ must contain at least m vectors.

$\mathcal{R}^m$ have a basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$

Because a basis is the smallest generation set

Any other generation set has at least m vectors.

A basis is the largest independent set in the subspace.

Any independent vector set in $\mathcal{R}^m$ contain at most m vectors.

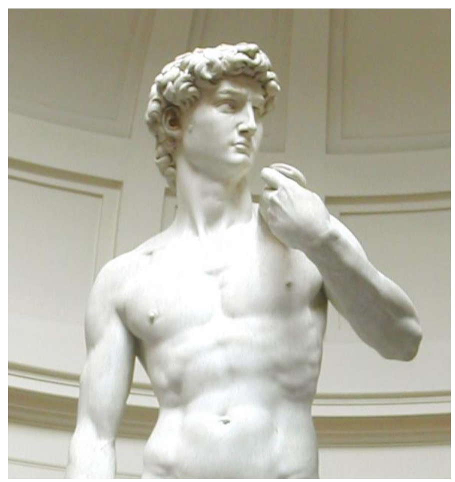
Summary

雕塑 ... 主要是使用雕（通過減除材料來造型）及塑（通過疊加材料來造型）的方式 ...... (from wiki)

Generation set

刪去

Same size



Basis

疊加

Independent vector set

Confirming that a set is a Basis

Intuitive Way

- Definition: A basis B for V is an independent generation set of V.

$$V = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathcal{R}^3 : v_1 - v_2 + 2v_3 = 0 \right\}$$

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Is $\mathcal{C}$ a basis of V?

Independent? yes

Generation set? difficult

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$ generates V

Another way

Find a basis for V

- Given a subspace V, assume that we already know that dim V = k. Suppose S is a subset of V with k vectors

If S is independent → S is basis

If S is a generation set → S is basis

$$V = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathcal{R}^3 : v_1 - v_2 + 2v_3 = 0 \right\} \quad \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Dim V = 2 (parametric representation)

Is $\mathcal{C}$ a basis of V?

$\mathcal{C}$ is a subset of V with 2 vectors

Independent? yes

→ $\mathcal{C}$ is a basis of V

Another way

Assume that dim V = k. Suppose S is a subset of V with k vectors

If S is independent ⟹ S is basis

By the extension theorem, we can add more vector into S to form a basis.

However, S already have k vectors, so it is already a basis.

If S is a generation set ⟹ S is basis

By the reduction theorem, we can remove some vector from S to form a basis.

However, S already have k vectors, so it is already a basis.

Example

- Is $\mathcal{B}$ a basis of V?

$$V = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \in \mathcal{R}^4 : v_1 + v_2 + v_4 = 0 \right\} \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

Independent set in V? yes

Dim V = ? 3 → $\mathcal{B}$ is a basis of V.

Example

- Is $\mathcal{B}$ a basis of V = Span $\mathcal{S}$?

$\mathcal{B}$ is a subset of V with 3 vectors

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\} \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$A = \begin{bmatrix} 1 & -1 & 3 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 2 & -1 & 1 & -1 \end{bmatrix} \Rightarrow R_A = \begin{bmatrix} 1 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad dim A = 3$$

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow R_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad Independent$$

⇒ $\mathcal{B}$ is a basis of V.