Matrix Multiplication

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Reference
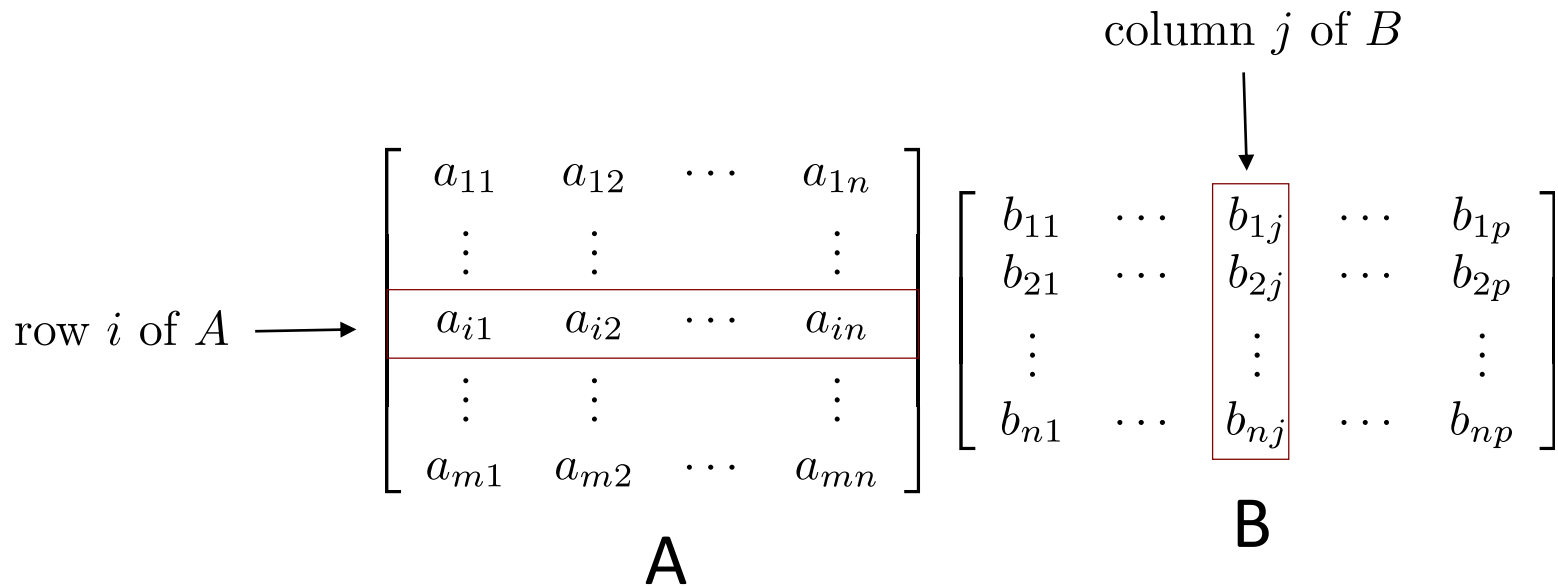
- Textbook: Chapter 2.1

Inner Product

(What you have learned
in high school)

Matrix Multiplication

- Given two matrices A and B, the (i, j)-entry of AB is the inner product of row i of A and column j of B

$$\text{column } j \text{ of } B$$

$$\text{row } i \text{ of } A \longrightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix}$$

A

B

$$C = AB \qquad c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$
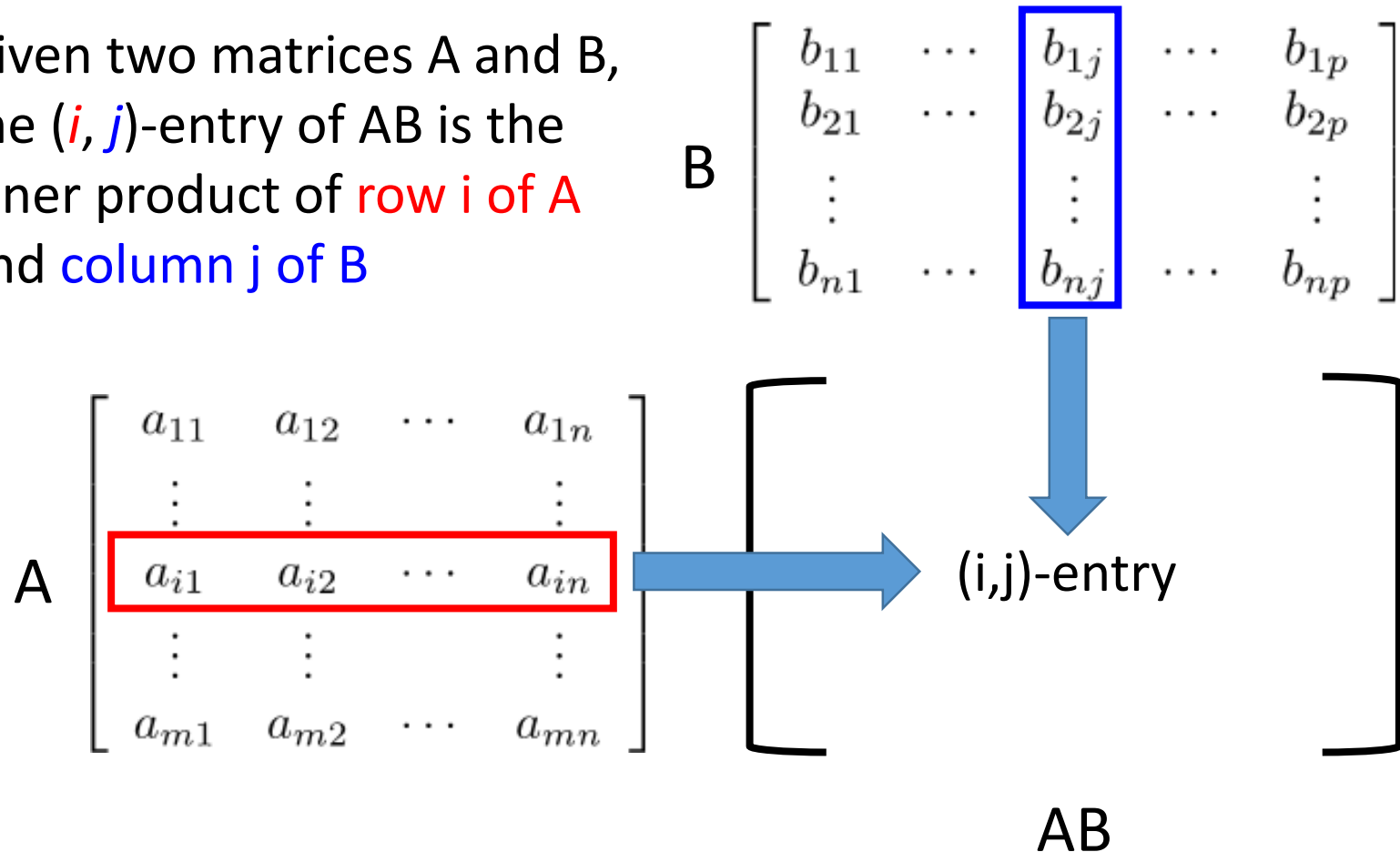
Matrix Multiplication

- Given two matrices A and B, the (i, j)-entry of AB is the inner product of row i of A and column j of B

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix}$$

$$C = AB = \begin{bmatrix} (-1) \times 1 + 3 \times 2 & 1 \times 1 + 2 \times 2 \\ (-1) \times 3 + 3 \times 4 & 1 \times 3 + 2 \times 4 \\ (-1) \times 5 + 3 \times 6 & 1 \times 5 + 2 \times 6 \end{bmatrix}$$
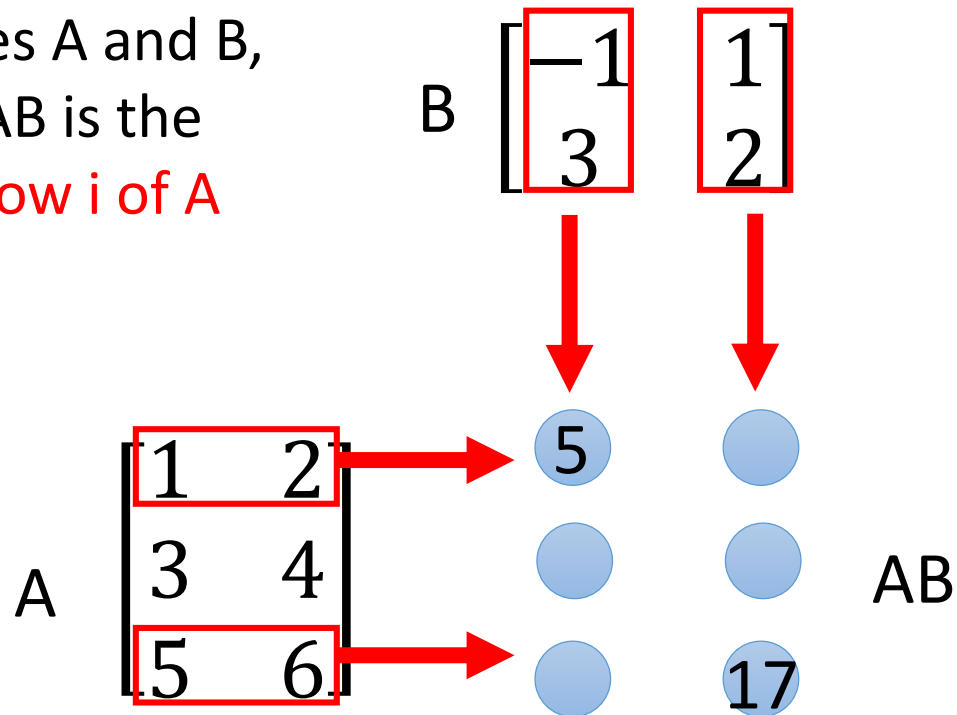
Matrix Multiplication

Given two matrices A and B, the (i, j)-entry of AB is the inner product of row i of A and column j of B

$$B \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix}$$

$$A \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

(i,j)-entry

AB

Matrix Multiplication − 4 ways

Given two matrices A and B, the (i, j)-entry of AB is the inner product of row i of A and column j of B

$$B \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix}$$

$$A \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$
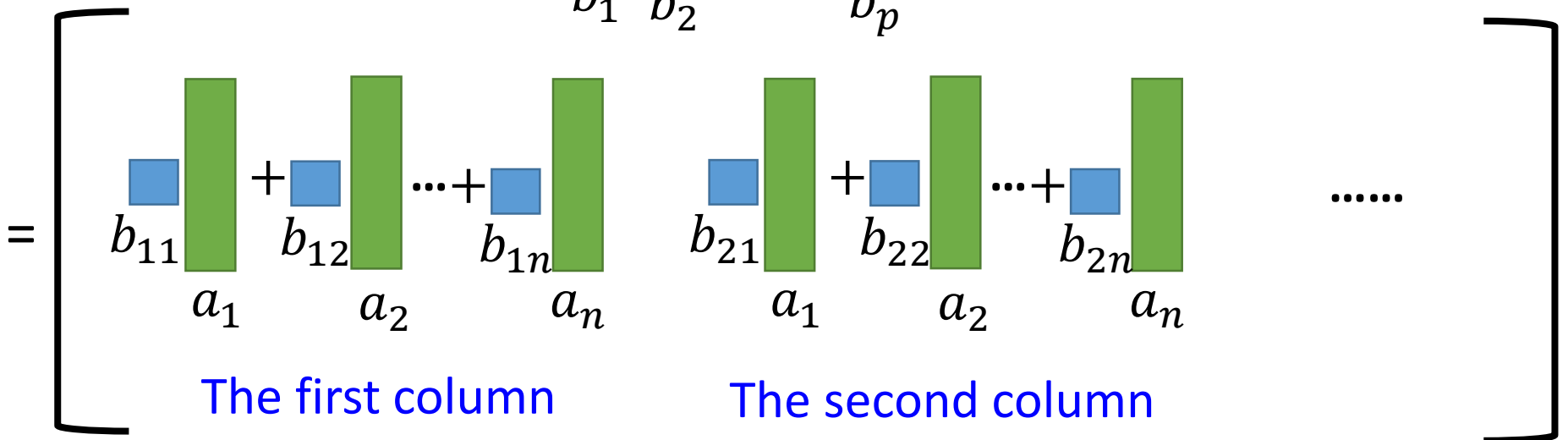
AB

5

17

Linear Combination of Columns and Rows

Matrix Multiplication

- Linear combination of columns

$$AB = A[b_1 \quad b_2 \quad \cdots \quad b_p]$$

$$= [Ab_1 \quad Ab_2 \quad \cdots \quad Ab_p]$$

$a_1 \quad a_2 \quad a_n$

$b_1 \quad b_2 \quad b_p$

$$= \begin{bmatrix} b_{11} \, a_1 + b_{12} \, a_2 \cdots + b_{1n} \, a_n & b_{21} \, a_1 + b_{22} \, a_2 \cdots + b_{2n} \, a_n & \cdots\cdots \end{bmatrix}$$

The first column   The second column

Matrix Multiplication $-$ 4 ways

- Linear combination of columns
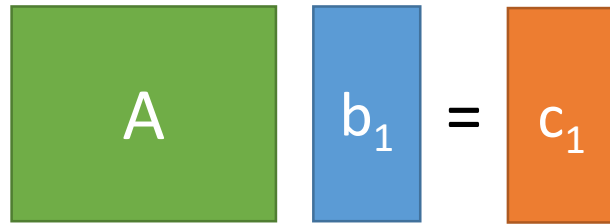
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix}$$

$$AB = A[b_1 \quad b_2 \quad \cdots \quad b_p]$$

$$= [Ab_1 \quad Ab_2 \quad \cdots \quad Ab_p]$$
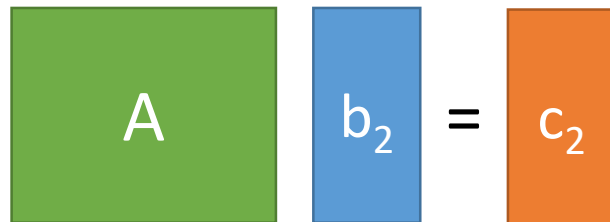
$$= \begin{bmatrix} -1\begin{bmatrix}1\\3\\5\end{bmatrix} + 3\begin{bmatrix}2\\4\\6\end{bmatrix} & 1\begin{bmatrix}1\\3\\5\end{bmatrix} + 2\begin{bmatrix}2\\4\\6\end{bmatrix} \end{bmatrix}$$

The first column   The second column

Matrix Multiplication - Meaning

- **_Multiple Input_** $C = AB$
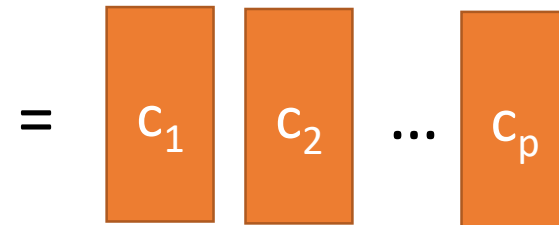
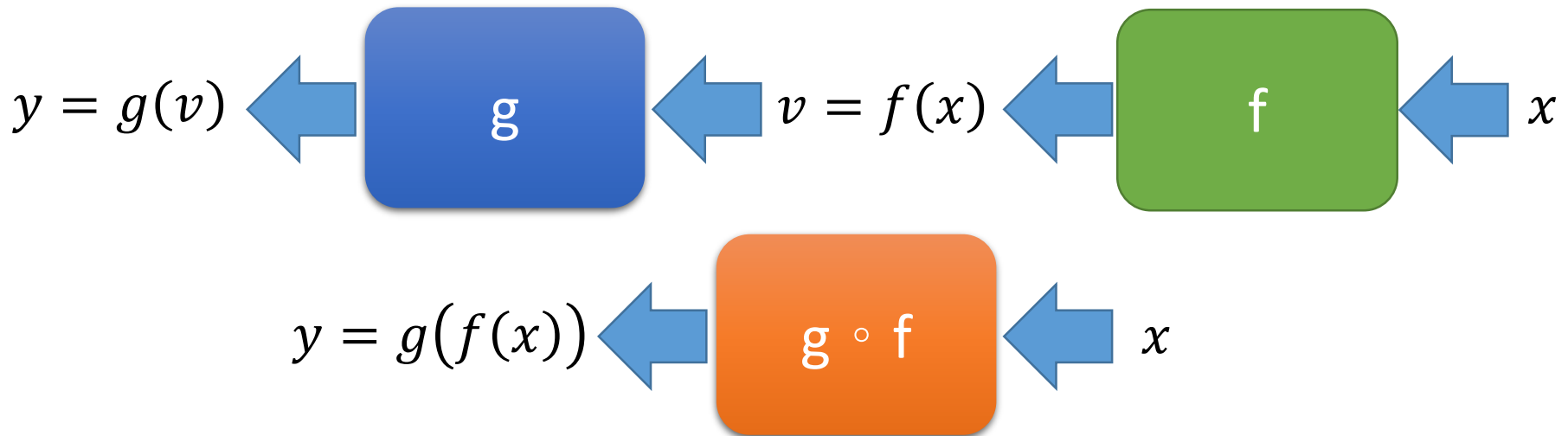$$AB = A[b_1 \quad b_2 \quad \cdots \quad b_p]$$

$$= [Ab_1 \quad Ab_2 \quad \cdots \quad Ab_p]$$

Matrix Multiplication - Meaning
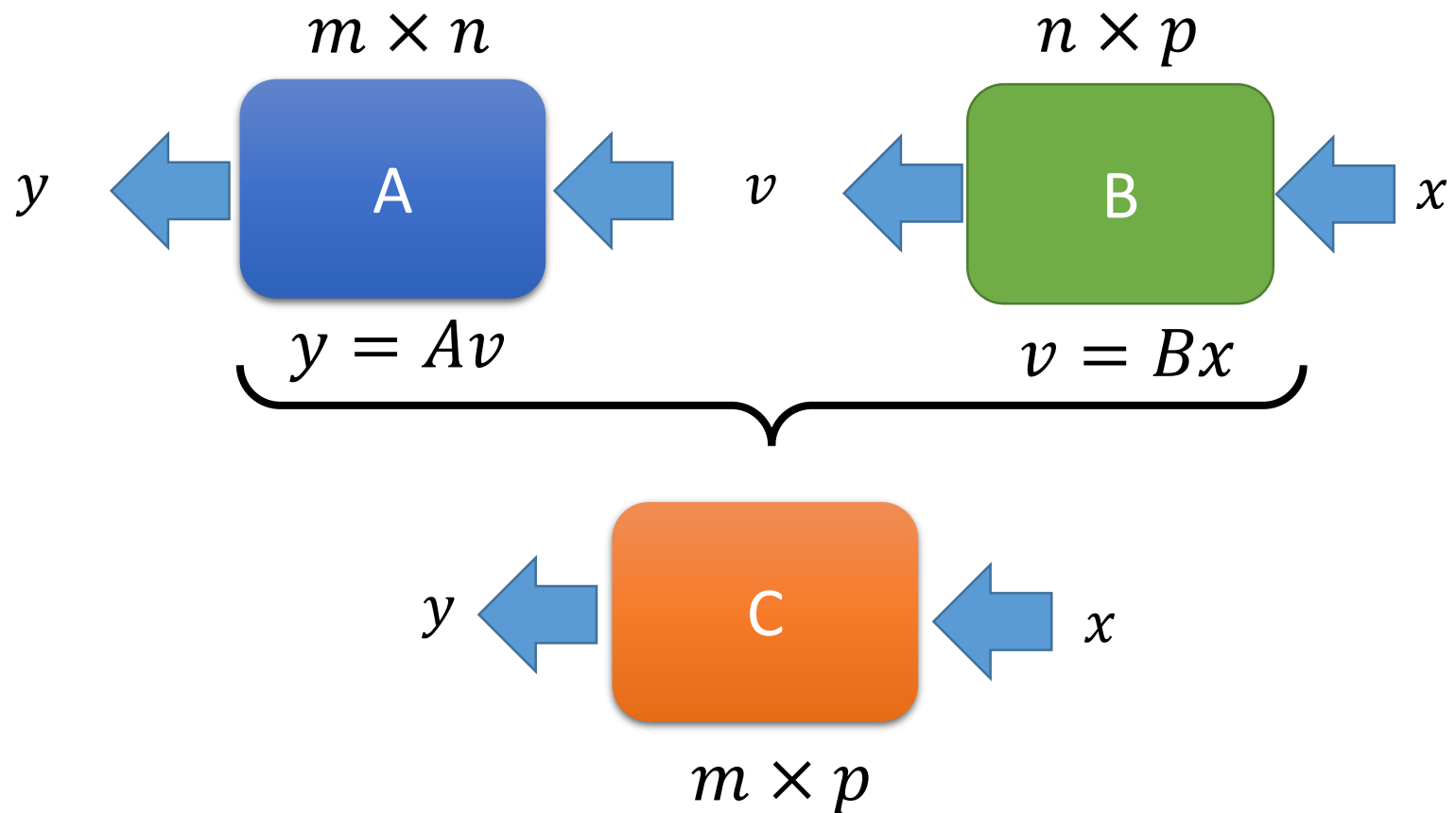
- ***Composition***
    - Given two functions f and g, the function $g\big(f(.)\big)$ is the composition g°f.
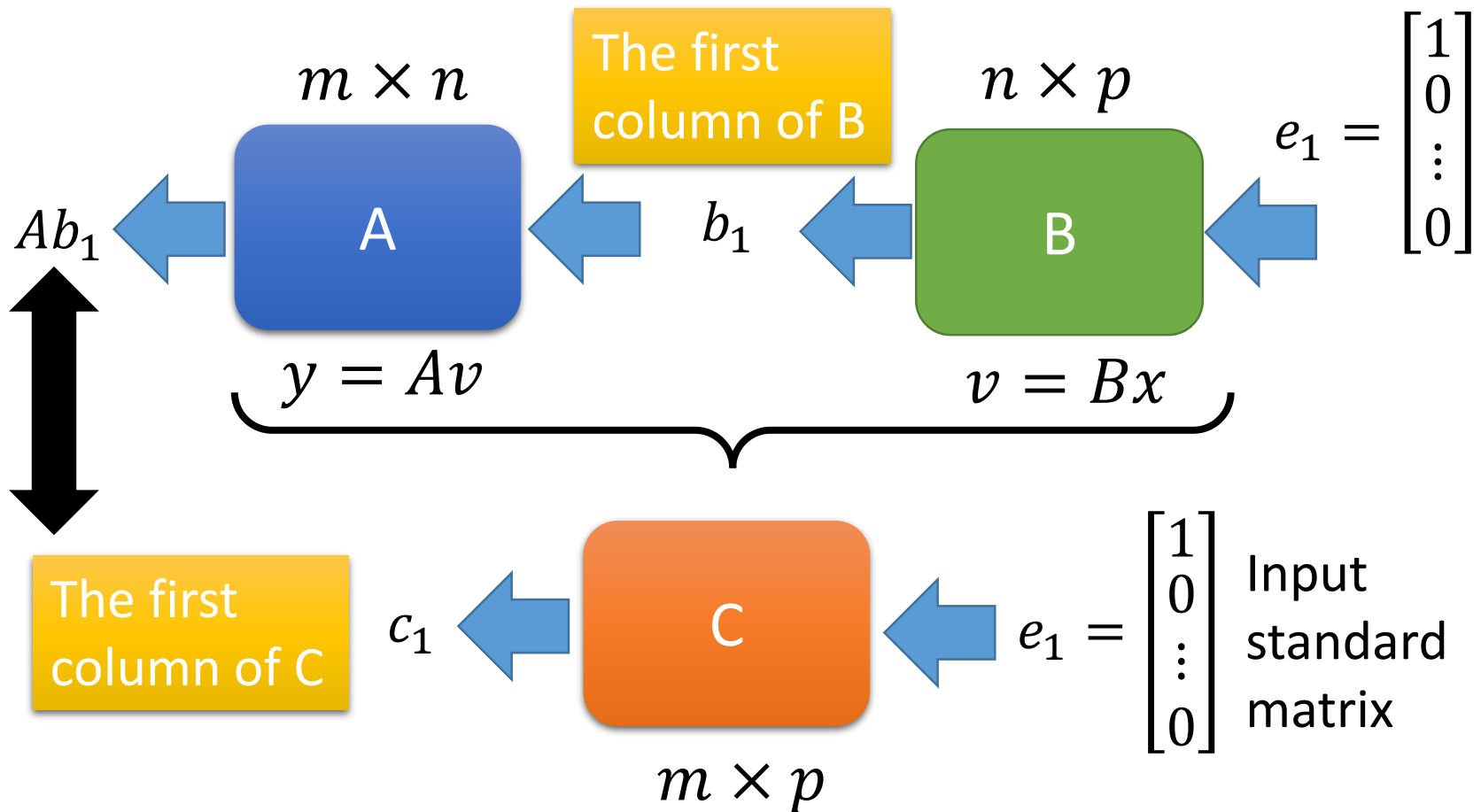
$$y = g(v) \leftarrow \boxed{g} \leftarrow v = f(x) \leftarrow \boxed{f} \leftarrow x$$

$$y = g\big(f(x)\big) \leftarrow \boxed{g \circ f} \leftarrow x$$

Matrix multiplication is the
composition of two linear functions.

Matrix Multiplication - Meaning

- ***Composition***

$$m \times n \qquad\qquad n \times p$$

$$y \xleftarrow{} \boxed{A} \xleftarrow{} v \qquad v \xleftarrow{} \boxed{B} \xleftarrow{} x$$

$$y = Av \qquad\qquad v = Bx$$

$$y \xleftarrow{} \boxed{C} \xleftarrow{} x$$

$$m \times p$$

Matrix Multiplication - Meaning

$m \times n$

The first column of B

$n \times p$

$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Ab_1

A

b_1

B

$y = Av$

$v = Bx$

The first column of C

c_1

C

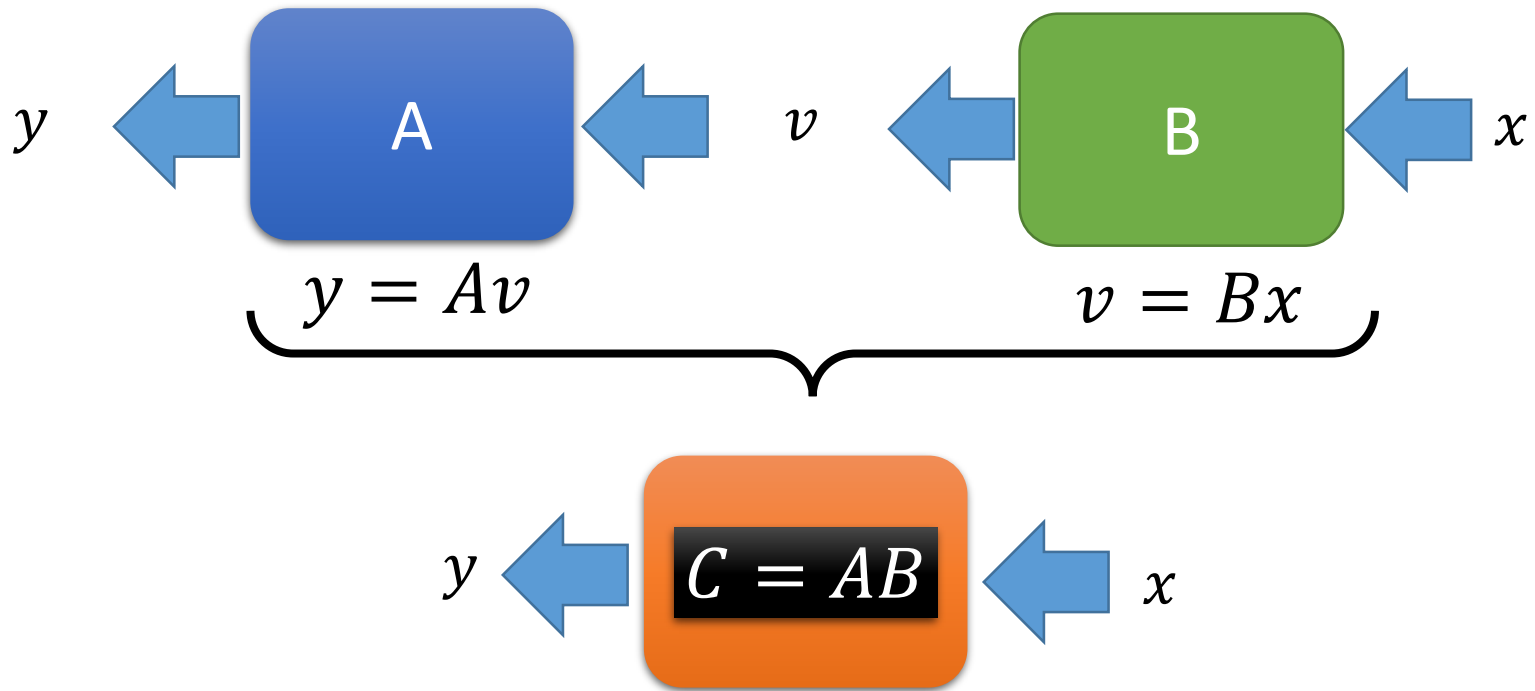$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ Input standard matrix

$m \times p$
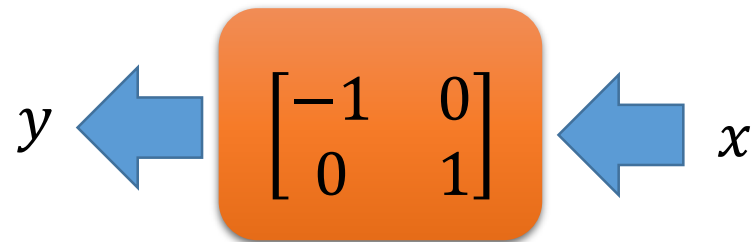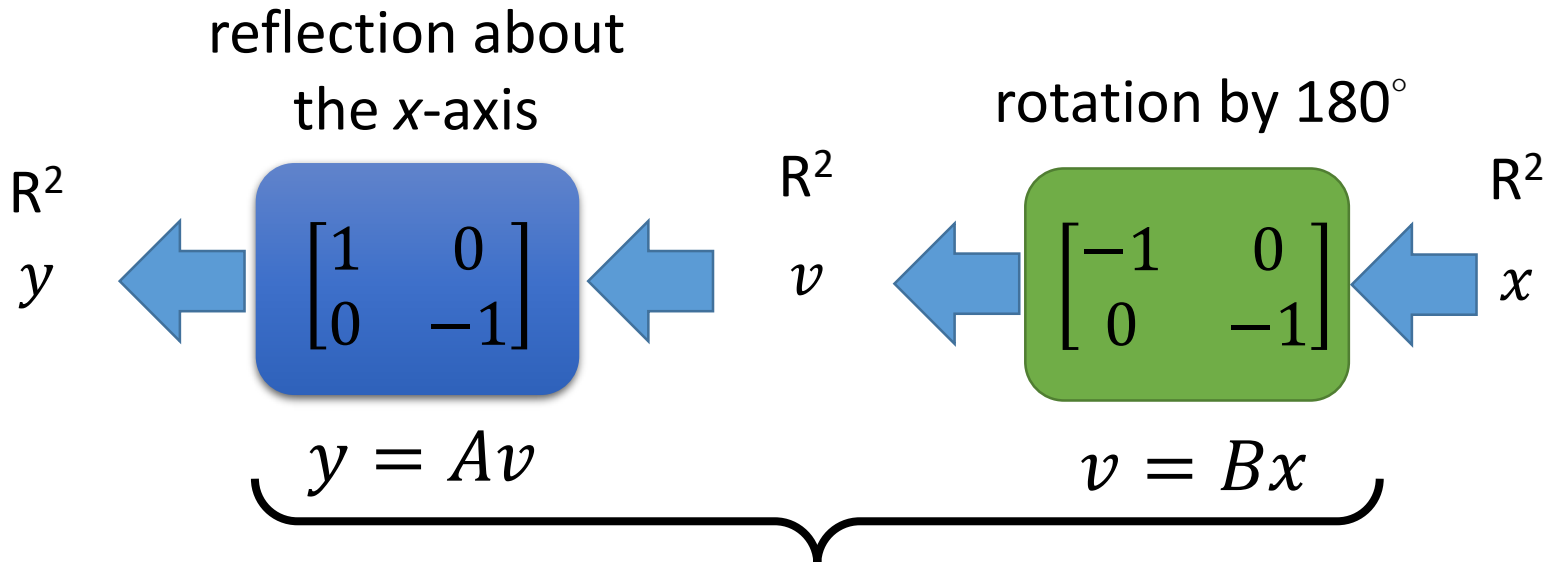
Matrix Multiplication - Meaning
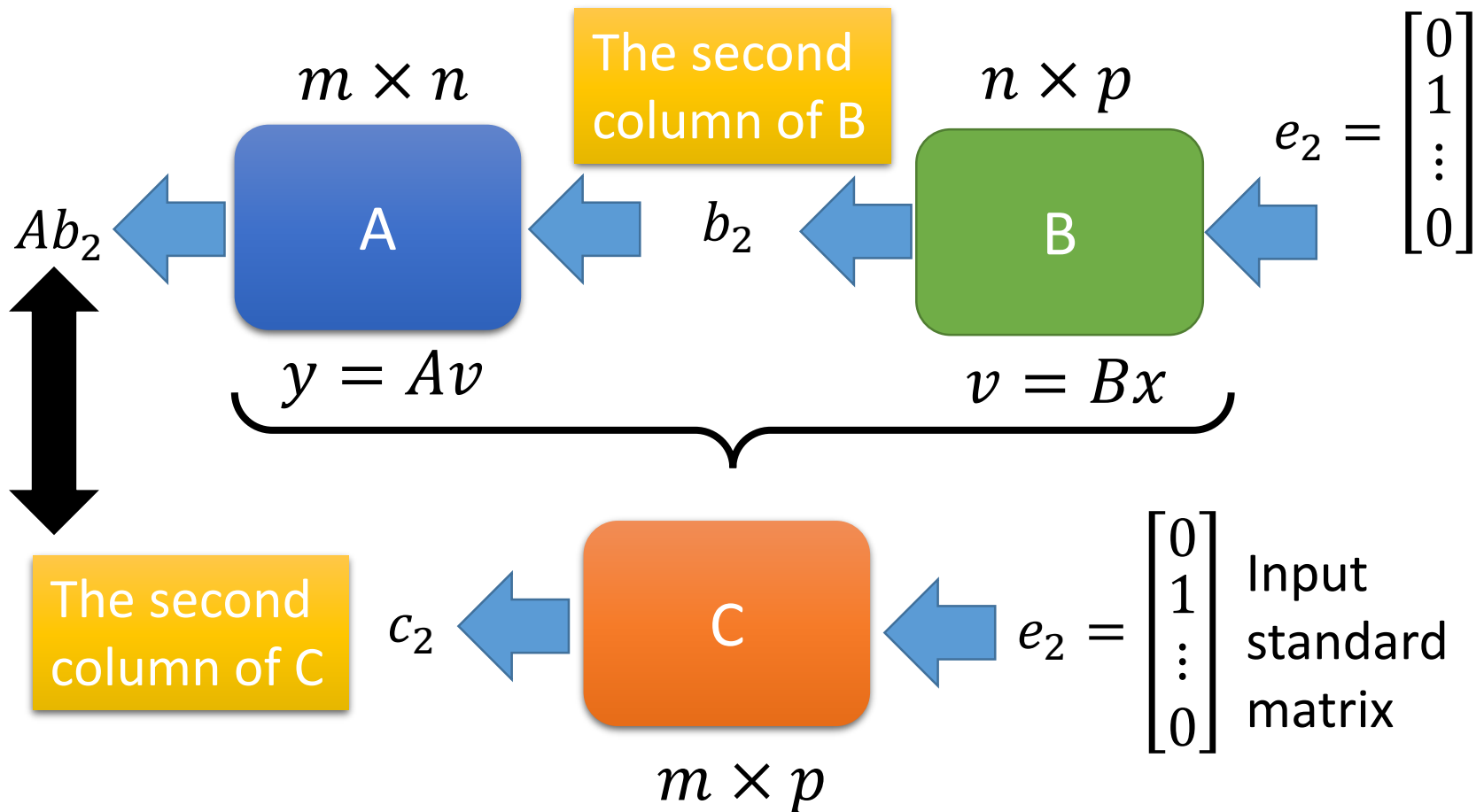
The composition of A and B is

$$C = [Ab_1 \quad Ab_2 \quad \cdots \quad Ab_p]$$

Matrix Multiplication

Example

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

reflection about the x-axis

rotation by 180°

R^2
y

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

R^2
v

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

R^2
x

$y = Av$

$v = Bx$

y

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

x

reflection about the y-axis

Matrix Multiplication

- Linear combination of rows

$$a_1^T$$ [green bar] $$b_1^T$$ [blue bar]

$$a_2^T$$ [green bar] $$b_2^T$$ [blue bar]

$$\vdots$$ $$\vdots$$

$$a_m^T$$ [green bar] $$b_n^T$$ [blue bar]

$$
= \begin{bmatrix} a_{11}b_1^T + a_{12}b_2^T \cdots + a_{1n}b_n^T \\ a_{21}b_1^T + a_{22}b_2^T \cdots + a_{2n}b_n^T \\ \vdots \\ a_{m1}b_1^T + a_{m2}b_2^T \cdots + a_{mn}b_n^T \end{bmatrix}
$$

Matrix Multiplication − 4 ways

- Way 3: Linear combination of rows

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1[-1 \quad 1] + 2[3 \quad 2] \\ \\ 3[-1 \quad 1] + 4[3 \quad 2] \\ \\ 5[-1 \quad 1] + 6[3 \quad 2] \end{bmatrix}$$
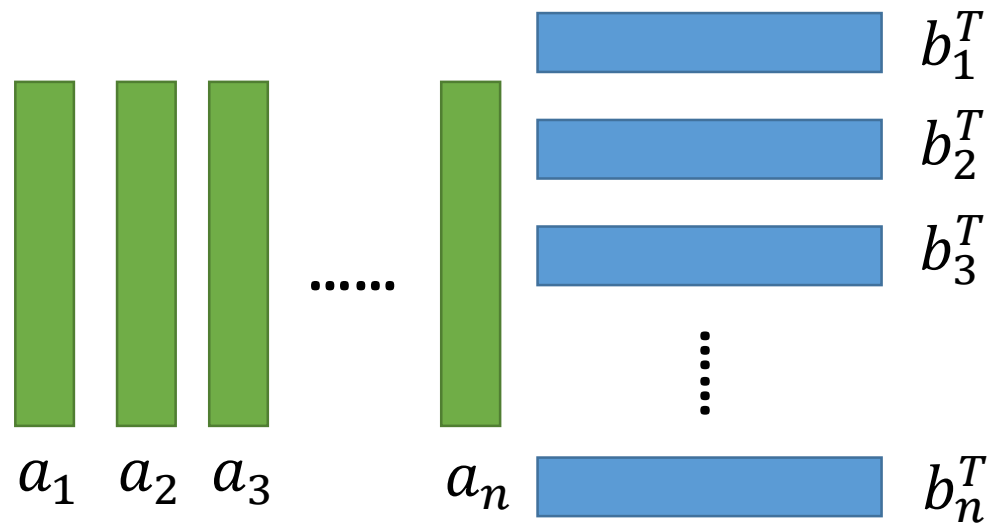
The first row

The second row

The third row

Summation of Matrices

Matrix Multiplication

- Summation of matrices

$$= a_1 b_1^T + a_2 b_2^T + \cdots + a_n b_n^T$$

matrices

Matrix Multiplication

- Summation of matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \begin{bmatrix} 3 & 2 \end{bmatrix} \end{bmatrix}$$

"1 x 2"

"2 x 1"

"1 x 1"

$$= \begin{bmatrix} -1 & 1 \\ -3 & 3 \\ -5 & 5 \end{bmatrix} + \begin{bmatrix} 6 & 4 \\ 12 & 8 \\ 18 & 12 \end{bmatrix}$$

Rank = ?   Rank = ?

Block Multiplication
(Section 2.5)

Augmentation and Partition

- Augment: the augment of A and B is [A B]
- Partition:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

Block Multiplication

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 5 & -1 & 6 \\ 1 & 0 & 3 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Multiply as the small matrices are scalar

$$= \begin{bmatrix} & \\ & \end{bmatrix}$$

Don't switch the order

Block Multiplication

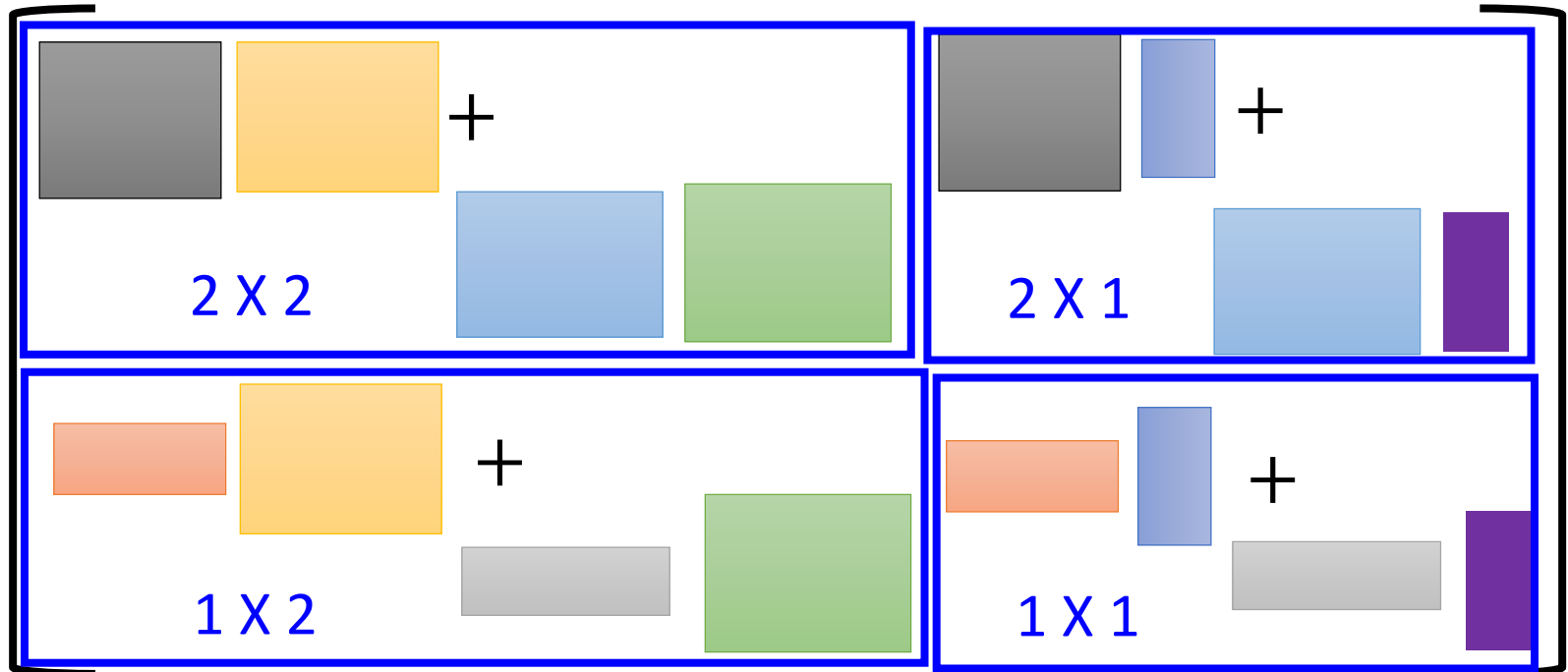$$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 5 & -1 & 6 \\ 1 & 0 & 3 & -1 \end{bmatrix}$$

"2 x 2"

$$B = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$$

"2 x 2"

$AB =$

"2 x 2"

Block Multiplication

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 6 & 8 & 5 & 0 \\ -7 & 9 & 0 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} I_2 & O \\ B & 5I_2 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & 8 \\ -7 & 9 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} I_2 & O \\ B & 5I_2 \end{bmatrix} \begin{bmatrix} I_2 & O \\ B & 5I_2 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$A^3 = AA^2 = \begin{bmatrix} I_2 & O \\ B & 5I_2 \end{bmatrix} \begin{bmatrix} I_2 & O \\ 6B & 25I_2 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

Properties
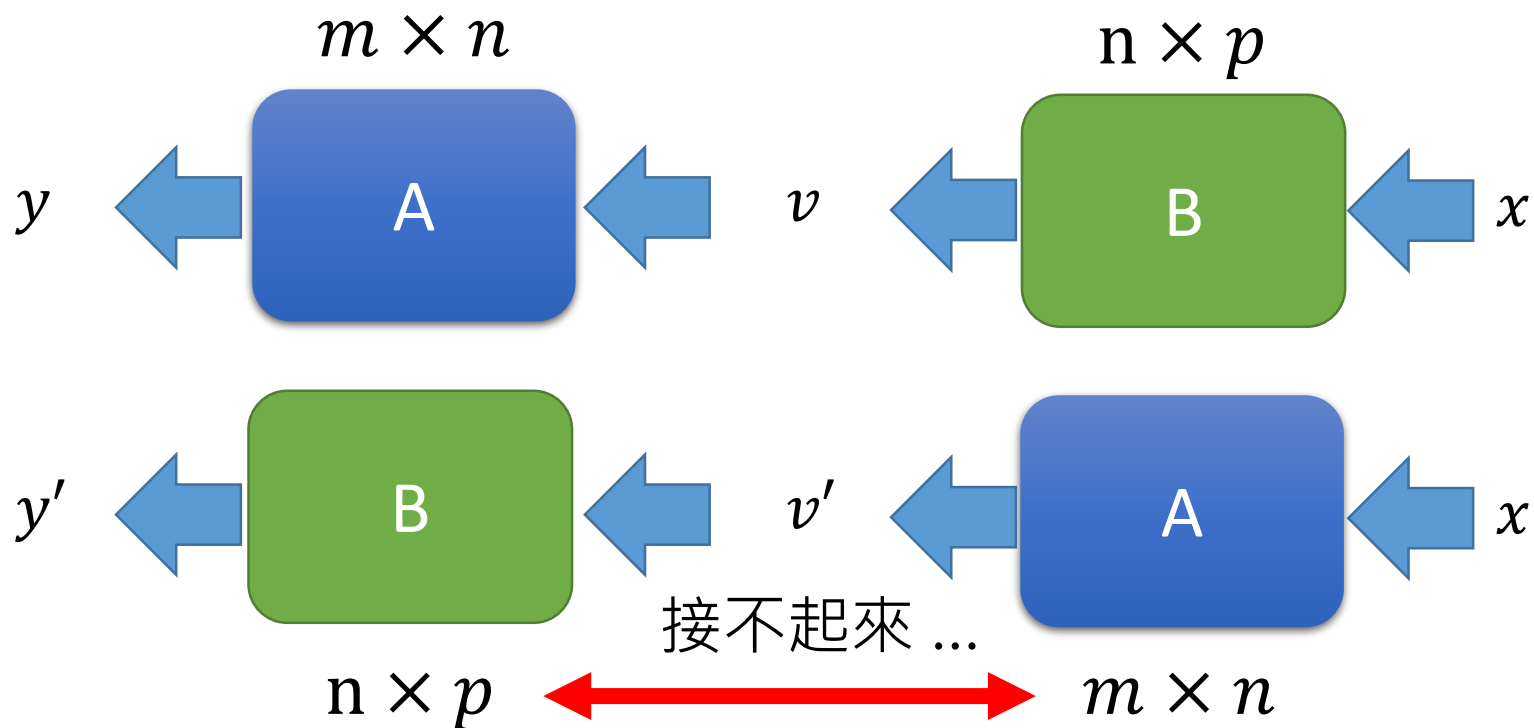
Not Communicative

- $AB \neq BA$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$\neq BA = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

Not Communicative

$m \times n$

$n \times p$

y A v B x

y' B v' A x

接不起來 ...

$n \times p$ $m \times n$

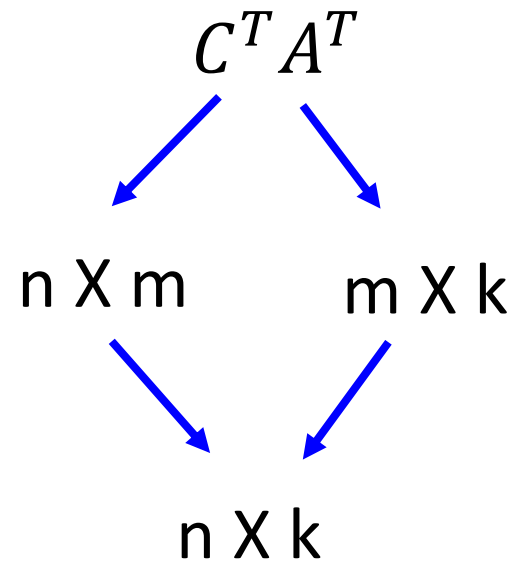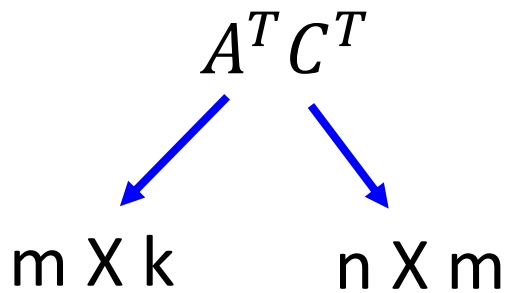If A and B are matrices, then both AB and BA are defined if and only if A and B are square matrices?

Properties

- Let A and B be k x m matrices, C be an m x n matrix, and P and Q be n x p matrices
  - For any scalar s, s(AC) = (sA)C = A(sC)
  - (A + B)C = AC + BC
  - C(P+Q)=CP+CQ
  - $I_k A = A = A I_m$
  - The product of any matrix and a zero matrix is a zero matrix
- Power of square matrices: $A \in \mathcal{M}_{n \times n}$, $A^k = A\,A \cdots A$ (k times), and by convention, $A^1 = A$, $A^0 = I_n$.

Properties

AC: k X n $(AC)^T$: n X k

- Let A be kxm matrices, C be an mxn matrix,
    - $(AC)^T =?$ $C^T A^T$

$A^T C^T$

m X k   n X m

$C^T A^T$

n X m   m X k

n X k

Special Matrix

- Diagonal Matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad AB = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$
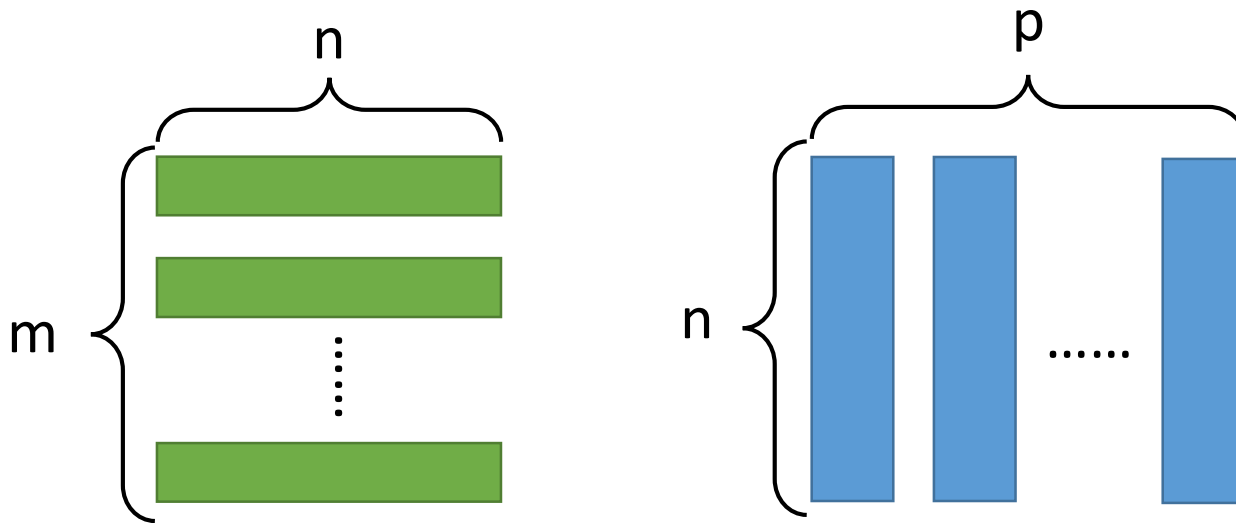
- Symmetric Matrix $\quad A^T = A$

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 5 \end{bmatrix} = A^T \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq B^T$$

AA^T and A^TA are square and symmetric

$$(AA^T)^T = A^{TT}A^T = AA^T \qquad (A^TA)^T = A^TA^{TT} = A^TA$$

Practical Issue

- Let A and B be k x m matrices, C be an m x n matrix, and P and Q be n x p matrices
  - A(CP) = (AC)P

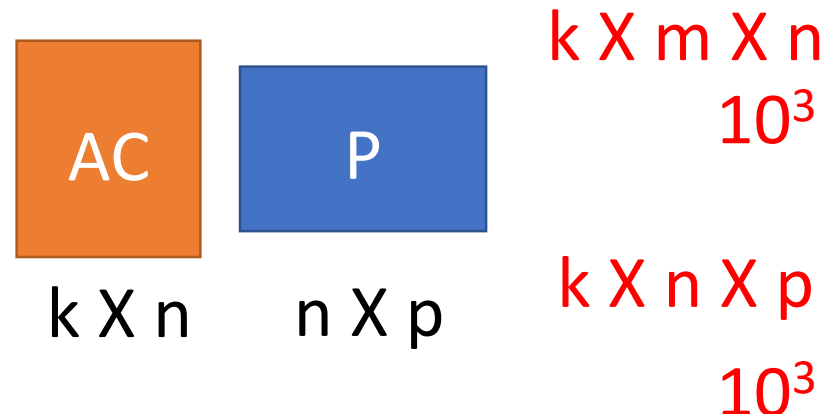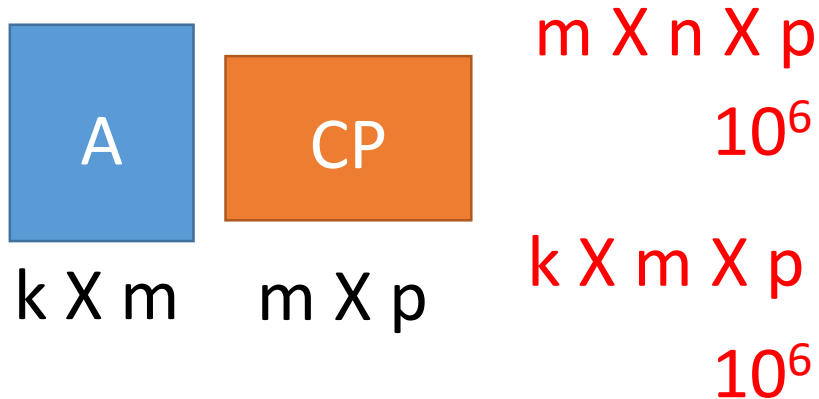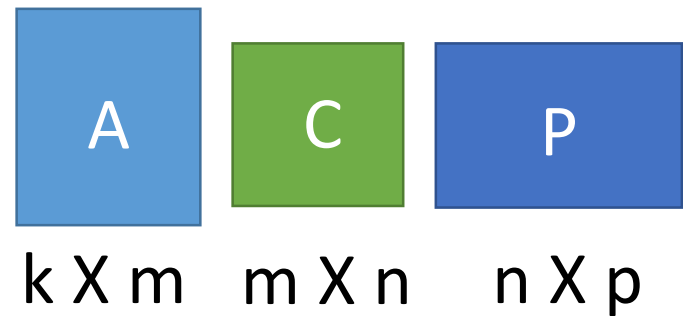Multiplication count: m X n X p

Practical Issue

k=1   m=1000

n=1   p=1000

- Let A and B be k x m matrices, C be an m x n matrix, and P and Q be n x p matrices
  - A(CP) = (AC)P

Practical Issue - GPU
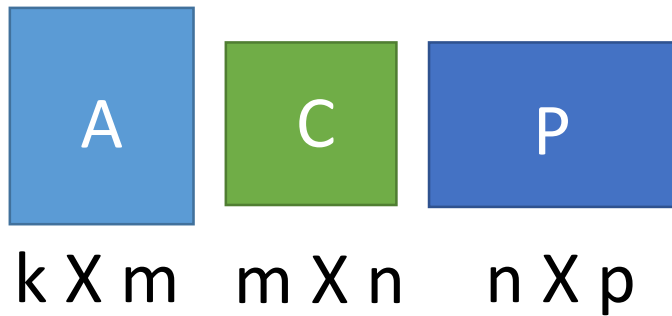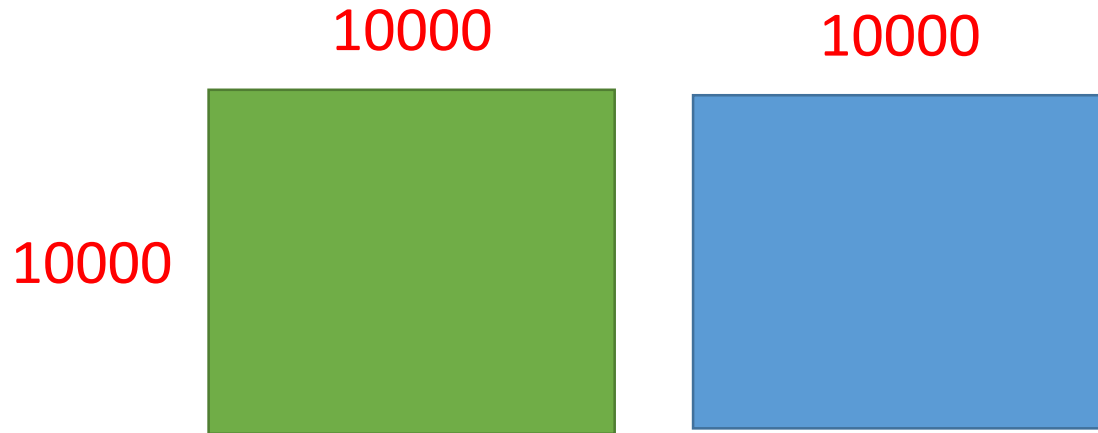

10000   10000

10000

Multiplying two 10000 X 10000 matrices

CPU: **21.2** 秒

GPU: **0.84** 秒 (GTX 980 Ti)

More than 20 times faster