Determinant

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Reference

- MIT OCW Linear Algebra:
  - Lecture 18: Properties of determinants
    - http://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/video-lectures/lecture-18-properties-of-determinants/
  - Lecture 19: Determinant formulas and cofactors
    - http://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/video-lectures/lecture-19-determinant-formulas-and-cofactors/
  - Lecture 20: Cramer's rule, inverse matrix, and volume
    - http://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/video-lectures/lecture-20-cramers-rule-inverse-matrix-and-volume/
- Textbook: Chapter 3

Determinant

- The determinant of a **square matrix** is a **scalar** that provides information about the matrix.
  - E.g. **Invertibility** of the matrix.
- Learning Target
  - The formula of Determinants
  - The properties of Determinants
  - Cramer’s Rule

Formula for Determinants

Determinants in High School

- 2 X 2

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$det(A) = \ ad - bc$$

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

- 3 x 3

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$$

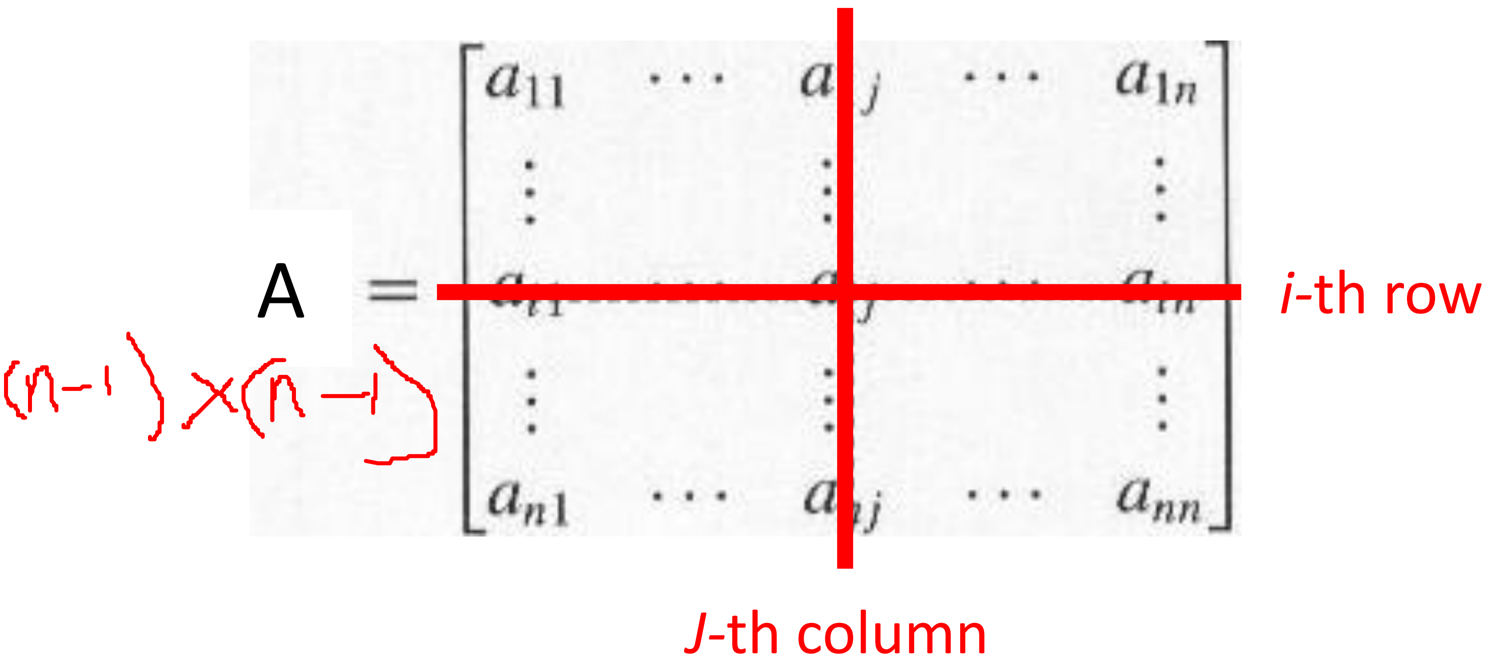
$$det(A) =$$

$$a_1a_5a_9 + a_2a_6a_7 + a_3a_4a_8$$

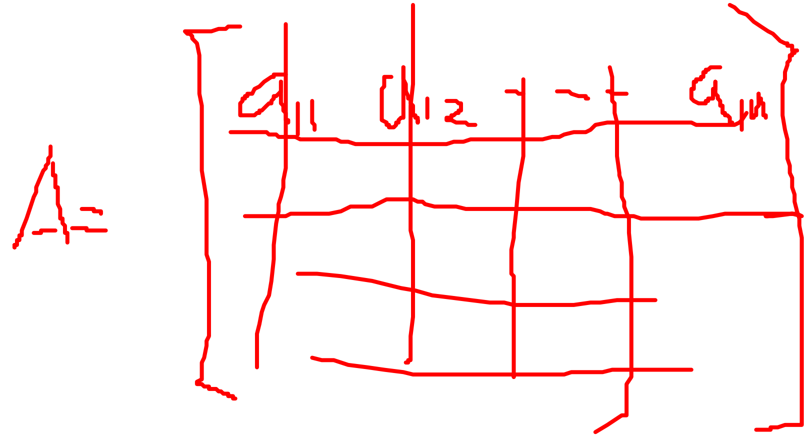
$$-a_3a_5a_7 - a_2a_4a_9 - a_1a_6a_8$$

Cofactor Expansion

- Suppose A is an n x n matrix. A_{ij} is defined as the submatrix of A obtained by removing the i-th row and the j-th column.



Cofactor Expansion



- Pick row 1

$$detA = a_{11}c_{11} + a_{12}c_{12} + \cdots + a_{1n}c_{1n}$$

c_{ij}: (i,j)-cofactor

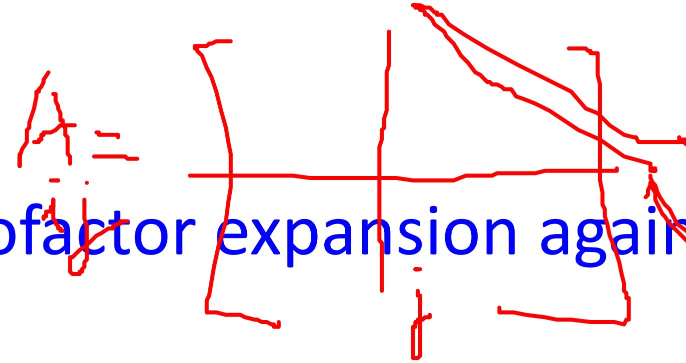
- Or pick row i

$$detA = a_{i1}c_{i1} + a_{i2}c_{i2} + \cdots + a_{in}c_{in}$$

- Or pick column j

$$detA = a_{1j}c_{1j} + a_{2j}c_{2j} + \cdots + a_{nj}c_{nj}$$

$$c_{ij} = (-1)^{i+j} detA_{ij}$$



Cofactor expansion again ......

2 x 2 matrix

$$c_{ij} = (-1)^{i+j} det A_{ij}$$

- Define det([a]) = a

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$det(A) = ad - bc$$

Pick the first row

$$det(A) = ac_{11} + bc_{12}$$

$$c_{11} = (-1)^{1+1} det([d]) = d$$

$$c_{12} = (-1)^{1+2} det([c]) = -c$$

3 x 3 matrix

$$c_{ij} = (-1)^{i+j} detA_{ij}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Pick row 2

$$detA = a_{21}c_{21} + a_{22}c_{22} + a_{23}c_{23}$$

4 5 6

$$(-1)^{2+1} detA_{21} \qquad (-1)^{2+2} detA_{22} \qquad (-1)^{2+3} detA_{23}$$

$$A_{21} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad A_{22} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad A_{23} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Example

- Given tridiagonal n x n matrix A

$$A = \begin{bmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix}$$

Find detA when n = 999

$$A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\det A_4$$

$$= a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} + a_{14}c_{14}$$

1 1 0 0

A_n

$$c_{11} = (-1)^2 \det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

A_{n-1}

$$= \det(A_3)$$

$$c_{12} = (-1)^3 \det \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= -\det(A_2)$$

$$= a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13}$$

1 1 0 0

$$= \det(A_2)$$

Example

$$det(A_4) = det(A_3) - det(A_2)$$

$$det(A_n) = det(A_{n-1}) - det(A_{n-2})$$

$det(A_1) = 1$ $\quad det(A_2) = 0$ $\quad det(A_3) = -1$

$det(A_4) = -1$ $\quad det(A_5) = 0$ $\quad det(A_6) = 1$

$det(A_7) = 1$ $\quad det(A_8) = 0$ $\quad$ ... ...

Properties of Determinants

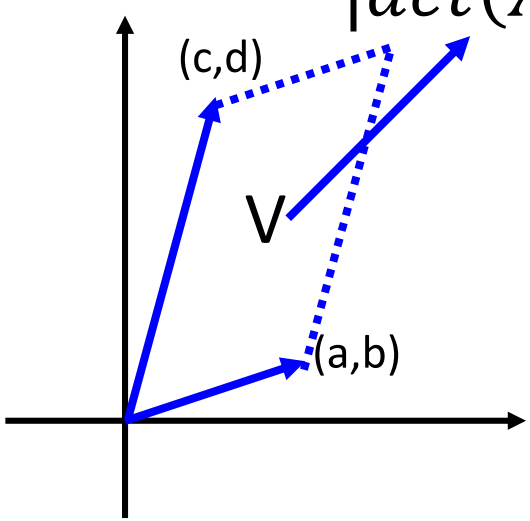
“Volume” in high dimensions (?)

Determinants in High School

- 2 X 2

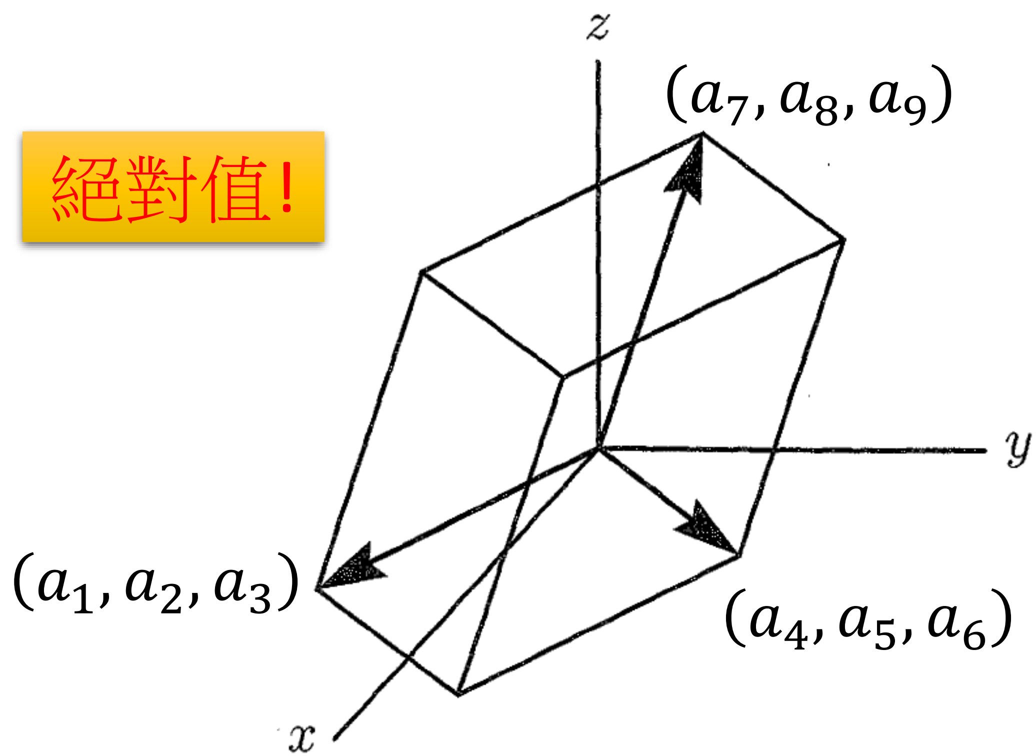
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$|det(A)|$ 絕對值!



- 3 x 3

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$$



Three Basic Properties

- Basic Property 1: $det(I) = 1$
- Basic Property 2: Exchange rows only reverses the sign of det (do not change absolute value)
- Basic Property 3: Determinant is "linear" for each row

Area in 2d and Volume in 3d have the above properties

Can we say determinant is the "Volume" also in high dimension?

Three Basic Properties

- Basic Property 1:
  - $det(I) = 1$

正方形　　面積為 1

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$det(I_2) = 1$$

正立方體　　體積為 1

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$det(I_3) = 1$$

Three Basic Properties

- Basic Property 2:
  - Exchanging rows only reverses the sign of det

$$det\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right) = 1$$

$$det\left(\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right) = -1$$

$$det\left(\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right) = 1$$

$$det\left(\begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}\right) = -1$$

$$det\left(\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\1 & 0 & 0\end{bmatrix}\right) = 1$$

Three Basic Properties

- Basic Property 2:
  - Exchanging rows only reverses the sign of det

If a matrix A has ***<u>2 equal rows</u>***

⇒ $det(A) = 0$

$A \xrightarrow{\text{exchange the two rows}} A'$

$det(A) = K$ = $det(A') = -K$

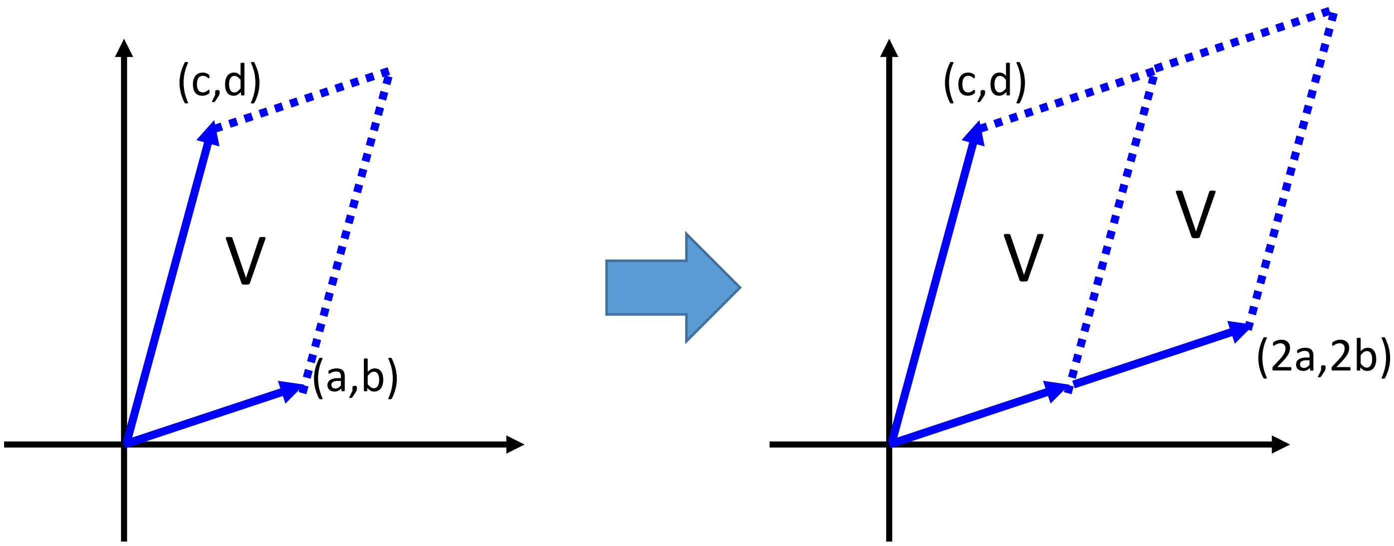
Exchanging the two equal rows yields the same matrix

Three Basic Properties

- Basic Property 3:
  - Determinant is "linear" for each row

3-a

$$det\left(\begin{bmatrix} ta & tb \\ c & d \end{bmatrix}\right) = t\,det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$



Three Basic Properties

- Basic Property 3:
  - Determinant is “linear” for each row

3-a

$$det\left(\begin{bmatrix} ta & tb \\ c & d \end{bmatrix}\right) = t\,det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

Q: find $det(2A)$

If A is n x n ……

A: $det(2A) = 2^n det(A)$

Three Basic Properties

- Basic Property 3:
  - Determinant is "linear" for each row

3-a

$$det\left(\begin{bmatrix} ta & tb \\ c & d \end{bmatrix}\right) = t\,det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

A row of zeros ⇒ $det(A) = 0$

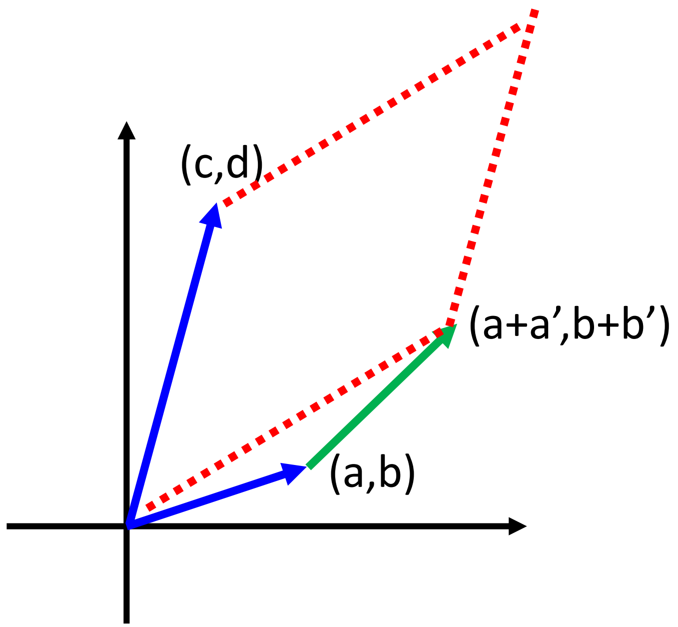
Set $t = 0$!

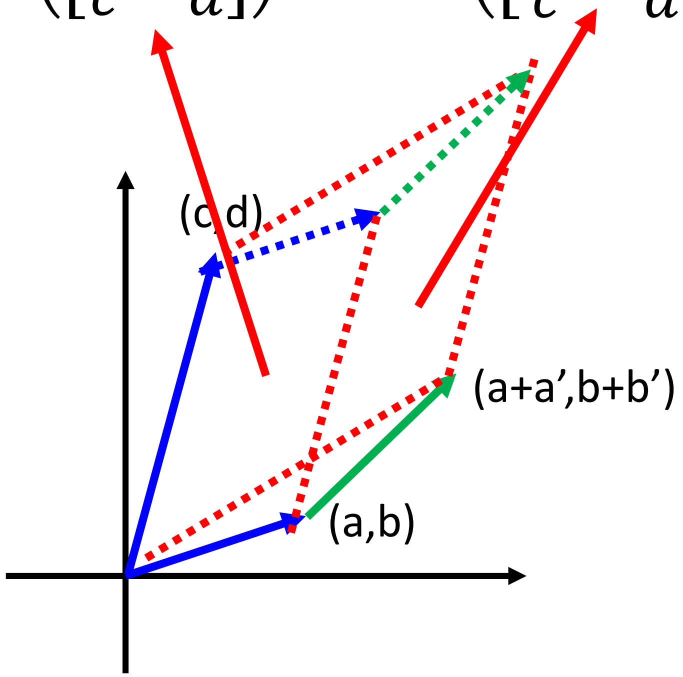
A row of zeros ⇒ "volume" is zero

Three Basic Properties

- Basic Property 3:
  - Determinant is “linear” for each row

3-b $$det\left(\begin{bmatrix} a+a' & b+b' \\ c & d \end{bmatrix}\right) = det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + det\left(\begin{bmatrix} a' & b' \\ c & d \end{bmatrix}\right)$$





Three Basic Properties

- Basic Property 3:
  - Determinant is “linear” for each row

Subtract k x row i from row j (elementary row operation)

Determinant doesn’t change

$$det\left(\begin{bmatrix} a & b \\ c-ka & d-kb \end{bmatrix}\right)$$

3-b $$= det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + det\left(\begin{bmatrix} a & b \\ -ka & -kb \end{bmatrix}\right)$$

3-a $$= det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) - k det\left(\begin{bmatrix} a & b \\ a & b \end{bmatrix}\right) = det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

Determinants for Upper Triangular Matrix

$$U = \begin{bmatrix} d_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

Killing everything above

Does not change the det

$$det(U) = det\left(\begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}\right)$$

Property 1

_3-a_ $= d_1 d_2 \cdots d_n det\left(\begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}\right)$

= 1

$$det(U) = d_1 d_2 \cdots d_n$$ (Products of diagonal)

Determinants v.s. Invertible

A is invertible ⟷ det(A) ≠ 0

A → R

Elementary row operation

$det(A)$

$$det(R) = \pm k_1 k_2 \cdots det(A)$$

Exchange: Change sign

Scaling: Multiply k

Add row: nothing

If A is invertible, R is identity

$det(R) = 1$ ⇒ $det(A) \neq 0$

If A is not invertible, R has zero row

$det(R) = 0$ ⇒ $det(A) = 0$

Invertible

We collect one more properties for invertible!

- Let A be an n x n matrix. A is invertible if and only if

onto

- The columns of A span R^n
- For every b in R^n, the system Ax=b is consistent
- The rank of A is n

One-on-one

- The columns of A are linear independent
- The only solution to Ax=0 is the zero vector
- The nullity of A is zero
- The reduced row echelon form of A is I_n
- A is a product of elementary matrices
- There exists an n x n matrix B such that BA = I_n
- There exists an n x n matrix C such that AC = I_n
- det(A) $\neq$ 0

Example

A is invertible ⟷ $\det(A) \neq 0$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & c \\ 2 & 1 & 7 \end{bmatrix}$$

For what scalar c is the matrix not invertible?

$\det(A) = 0$

$$detA = \quad 1 \cdot 0 \cdot 7 \quad +(-1) \cdot c \cdot 2 \quad +2 \cdot (-1) \cdot 1$$

$$-2 \cdot 0 \cdot 2 \quad -(-1) \cdot (-1) \cdot 7 \quad -1 \cdot c \cdot 1$$

$$= 0 - 2c - 2 - 7 - c \;= -3c - 9$$

not invertible ⇒ $-3c - 9 = 0$ ⇒ $c = -3$

More Properties of Determinants

$det(A+B) \neq det(A)+det(B)$

- $det(AB) = det(A)det(B)$

  Q: find $det(A^{-1})$

  $\because A^{-1}A = I \quad \therefore det(A^{-1})det(A) = det(I) = 1$

  $\therefore\ det(A^{-1}) = 1/det(A)$

  Q: find $det(A^2)$

  $$det(A^2) = det(A)det(A) = det(A)^2$$

- $det(A^T) = det(A)$
  - Zero row → zero column
  - Same row → same column ……

P212 - 215

Cramer’s Rule

Formula for A^{-1}

- $A^{-1} = \frac{1}{det(A)} C^T$
  - $det(A)$: scalar
  - C: cofactors of A (C has the same size as A, so does C^T)
  - C^T is adjugate of A (adj A, 伴隨矩陣)

$$C = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$det(A) = ad - bc$$

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$C^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Formula for A^{-1}

$$A^{-1} = \frac{1}{det(A)} C^T$$

- $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, A^{-1} =?$

$$det(A) = aei + bfg + cdh - ceg - bdi - afh$$

$$C = \begin{bmatrix} +\begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & +\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & +\begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ +\begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & +\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

Formula for A^{-1}

$$A^{-1} = \frac{1}{det(A)} C^T$$

- Proof: $AC^T = det(A) I_n$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} c_{11} & \cdots & c_{n1} \\ \vdots & \ddots & \vdots \\ c_{1n} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} det(A) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & det(A) \end{bmatrix}$$

transpose

Diagonal: By definition of determinants

Not Diagonal:

(Exercise 82, P221)

Cramer's Rule

$$A^{-1} = \frac{1}{det(A)} C^T$$

$$Ax = b$$

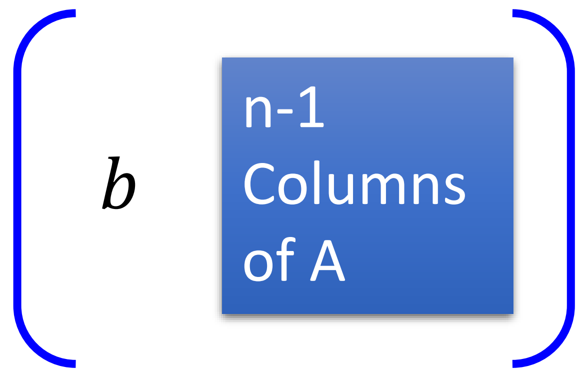
$$x = A^{-1}b$$

$$= \frac{1}{det(A)} C^T b$$

$$x_1 = \frac{det(B_1)}{det(A)}$$

B_1 =with column 1 replaced by b

$$x_2 = \frac{det(B_2)}{det(A)}$$



⋮

$$x_j = \frac{det(B_j)}{det(A)}$$

B_j =with column j replaced by b

Appendix

Formula from Three Properties

1 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$ **2** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} + \det \begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$$ **3-b**

$$= \det \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} + \det \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} + \det \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} + \det \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}$$ **3-b**

3-a $= 0$ **3-a** $= ad$ **3-a** $= -\mathrm{bc}$ **3-a** $= 0$

$$= ad - bc$$

Finally, we get 3 x 3 x 3 matrices

Most of them have zero determinants

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \det\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \det\begin{bmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \det\begin{bmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= det\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + det\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + det\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= det\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{bmatrix} + det\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{bmatrix} + det\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

3! matrices have non-zero rows

$$= \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$a_{11}a_{22}a_{33} \qquad -a_{11}a_{23}a_{32} \qquad -a_{12}a_{21}a_{33}$$

$$+ \begin{bmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{bmatrix}$$

$$a_{12}a_{23}a_{31} \qquad a_{13}a_{21}a_{32} \qquad -a_{13}a_{22}a_{31}$$

Pick an element at each row,
but they can not be in the same column.

Formula from Three Properties

- Given an n x n matrix A

$$det(A) = \sum n!\, terms$$

Format of each term: $a_{\underline{1}\underline{\alpha}} a_{\underline{2}\underline{\beta}} a_{\underline{3}\underline{\gamma}} \cdots a_{\underline{n}\underline{\omega}}$

Find an element in each row

permutation of 1,2, …, n

Example

$$det\left(\begin{bmatrix}0 & 0 & 1 & 1\\0 & 1 & 1 & 0\\1 & 1 & 0 & 0\\1 & 0 & 0 & 1\end{bmatrix}\right)$$

$$= det\left(\begin{bmatrix}0 & 0 & 1 & 0\\0 & 1 & 0 & 0\\1 & 0 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix}\right) + det\left(\begin{bmatrix}0 & 0 & 0 & 1\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\\1 & 0 & 0 & 0\end{bmatrix}\right)$$

-1 +1

Formulas for Determinants

$$detA = \sum n!\, terms$$

Format of each term: $a_{1\alpha}a_{2\beta}a_{3\gamma}\cdots a_{n\omega}$

$$detA = a_{11}c_{11} + a_{12}c_{12} + \cdots + a_{1n}c_{1n}$$

All terms including a_{11}

All terms including a_{12}

All terms including a_{1n}