How many solutions?

Hung-yi Lee

Reference

- Textbook: Chapter 1.7

Review

Given a system of linear equations with m equations and n variables

$A\mathbf{x} = \mathbf{b}$ $\quad A: m \times n \quad x \in R^n \quad b \in R^m$

Is b a linear combination of columns of A?

Is b in the span of the columns of A?

NO

No solution

YES

Have solution

How many solutions?

Today

Given a system of linear equations with m equations and n variables

$\mathbf{A}\mathbf{x} = \mathbf{b}$ $A: m \times n$ $x \in R^n$ $b \in R^m$

Is b a linear combination of columns of A?

Is b in the span of the columns of A?

NO

No solution

Other cases?

YES

The columns of A are ***independent***.

Rank A = n

Nullity A = 0

Unique solution

The columns of A are ***dependent***.

Rank A < n

Nullity A > 0

Infinite solution

(依賴的、不獨立的)

Dependent and Independent

(獨立的、自主的)

Definition

- A set of n vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\}$ is linear dependent

Find one → Obtain many

  - If there exist scalars $x_1, x_2, \cdots, x_n$, **not all zero**, such that

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \boldsymbol{0}$$

- A set of n vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\}$ is linear independent

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \boldsymbol{0}$$

Only if $x_1 = x_2 = \cdots = x_n = 0$

unique

Dependent and Independent

Linear Dependent

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, if there exists any $\mathbf{a}_i$ that is a linear combination of other vectors

$$\left\{\begin{bmatrix}-4\\12\\6\end{bmatrix}, \begin{bmatrix}-10\\30\\15\end{bmatrix}\right\}$$

Dependent or Independent?

$$\left\{\begin{bmatrix}6\\3\\3\end{bmatrix}, \begin{bmatrix}1\\8\\3\end{bmatrix}, \begin{bmatrix}7\\11\\6\end{bmatrix}\right\}$$

Dependent or Independent?

Dependent and Independent

Linear Dependent

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, if there exists any $\mathbf{a}_i$ that is a linear combination of other vectors

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} \right\}$$

Dependent or Independent?

Zero vector is the linear combination of any other vectors

Any set contains zero vector would be linear dependent

How about a set with only one vector?

Dependent and Independent

Linear Dependent

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, if there exists any $\mathbf{a}_i$ that is a linear combination of other vectors

$$2\boldsymbol{a}_i + \boldsymbol{a}_j + 3\boldsymbol{a}_k = \boldsymbol{0}$$
$$2\boldsymbol{a}_i + \boldsymbol{a}_j = -3\boldsymbol{a}_k$$
$$\left(-\frac{2}{3}\right)\boldsymbol{a}_i + \left(-\frac{1}{3}\right)\boldsymbol{a}_j = \boldsymbol{a}_k$$

$$\Updownarrow$$

$$\boldsymbol{a}_{i'} = 3\boldsymbol{a}_{j'} + 4\boldsymbol{a}_{k'}$$
$$\boldsymbol{a}_{i'} - 3\boldsymbol{a}_{j'} - 4\boldsymbol{a}_{k'} = \boldsymbol{0}$$

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, there exists scalars $x_1, x_2, \ldots, x_n$, that are **not all zero**, such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$.

Intuition

Dependent:
Once we have solution, we have infinite.

- Intuitive link between dependence and the number of solutions

$$\begin{bmatrix} 6 & 1 & 7 \\ 3 & 8 & 11 \\ 3 & 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 22 \\ 12 \end{bmatrix} \qquad 1 \cdot \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix}$$

$$1 \cdot \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix} = \begin{bmatrix} 14 \\ 22 \\ 12 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$2 \cdot \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 22 \\ 12 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

Infinite Solution

Proof

- Columns of A are **dependent** → If Ax=b have solution, it will have Infinite Solutions

- If Ax=b have Infinite solutions → Columns of A are dependent

Proof

Homogeneous linear equations

$$Ax = \mathbf{0} \qquad A = [a_1 \quad a_2 \quad \cdots \quad a_n] \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

(always having $\mathbf{0}$ as solution)

Based on the definition

A set of n vectors $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots, \boldsymbol{a_n}\}$ is linear dependent $\Leftrightarrow$ $Ax = \mathbf{0}$ have non-zero solution

infinite

A set of n vectors $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots, \boldsymbol{a_n}\}$ is linear independent $\Leftrightarrow$ $Ax = \mathbf{0}$ only have zero solution

Proof

- Columns of A are **dependent** → If Ax=b have solution, it will have Infinite solutions

We can find non-zero solution u such that $Au = \mathbf{0}$

There exists v such that $Av = \mathrm{b}$

$$A(u+v) = \mathrm{b}$$

$u+v$ is another solution different to v

- If Ax=b have Infinite solutions → Columns of A are dependent

$u \neq v$

$$Au = \mathrm{b}$$

$$Av = \mathrm{b}$$

$$A\underbrace{(u-v)}_{\text{Non-zero}} = \mathbf{0}$$

Rank and Nullity

Rank and Nullity

- The **rank** of a matrix is defined as the maximum number of *linearly independent columns* in the matrix.
- **Nullity** = Number of columns - **rank**

$$\begin{bmatrix} -3 & 2 & -1 \\ 7 & 9 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank and Nullity

- The **rank** of a matrix is defined as the maximum number of *linearly independent columns* in the matrix.
- **Nullity** = Number of columns - **rank**

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix} \qquad \begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix} \qquad \begin{bmatrix} 5 \\ 2 \end{bmatrix} \qquad [6]$$

Rank and Nullity

- The **rank** of a matrix is defined as the maximum number of *linearly independent columns* in the matrix.
- **Nullity** = Number of columns - **rank**

If A is a mxn matrix:

Rank A = n

Nullity A = 0

Columns of A are independent

Conclusion

$$A\mathbf{x} = \mathbf{b}$$

A: $m \times n$ $x \in R^n$ $b \in R^m$

Is b a linear combination of columns of A?

Is b in the span of the columns of A?

NO

No solution

YES

The columns of A are ***independent***.

Rank A = n

Nullity A = 0

Unique solution

The columns of A are ***dependent***.

Rank A < n

Nullity A > 0

Infinite solution

Conclusion

The columns of A are ***independent***.

Rank A = n

Nullity A = 0

$A: m \times n$

$x \in R^n \quad b \in R^m$

NO

Is b a linear combination of columns of A?

Is b in the span of the columns of A?

NO

No solution

YES

Infinite solution

YES

Is b a linear combination of columns of A?

Is b in the span of the columns of A?

NO

No solution

YES

Unique solution

Question

- True or False
  - If the columns of A are linear independent, then Ax=b has unique solution.
  - If the columns of A are linear independent, then Ax=b has at most one solution.
  - If the columns of A are linear dependent, then Ax=b has infinite solution.
  - If the columns of A are linear independent, then Ax=0 (homogeneous equation) has unique solution.
  - If the columns of A are linear dependent, then Ax=0 (homogeneous equation) has infinite solution.