What can we know from RREF?

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Reference

- Textbook: Chapter 1.6, 1.7

Outline

- RREF v.s. Linear Combination
- RREF v.s. Independent
- RREF v.s. Rank
- RREF v.s. Span

RREF v.s. Linear Combination

Column Correspondence Theorem

$$A = [\boldsymbol{a_1} \quad \cdots \quad \boldsymbol{a_n}] \xrightarrow{\text{RREF}} R = [\boldsymbol{r_1} \quad \cdots \quad \boldsymbol{r_n}]$$

If $\boldsymbol{a_j}$ is a linear combination of other columns of A $\Rightarrow$ $\boldsymbol{r_j}$ is a linear combination of the ***corresponding*** columns of R with ***the same coefficients***

$\mathbf{a}_5 = -\mathbf{a}_1+\mathbf{a}_4$ $\qquad$ $\mathbf{r}_5 = -\mathbf{r}_1+\mathbf{r}_4$

$\boldsymbol{a_j}$ is a linear combination of the ***corresponding*** columns of A with ***the same coefficients*** $\Leftarrow$ If $\boldsymbol{r_j}$ is a linear combination of other columns of R

$\mathbf{a}_3 = 3\mathbf{a}_1-2\mathbf{a}_2$ $\qquad$ $\mathbf{r}_3 = 3\mathbf{r}_1-2\mathbf{r}_2$

Column Correspondence Theorem - Example

$$A = \begin{array}{c} \begin{array}{cccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 \end{array} \\ \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \end{array} \qquad R = \begin{array}{c} \begin{array}{cccccc} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{r}_4 & \mathbf{r}_5 & \mathbf{r}_6 \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{a}_2 = 2\mathbf{a}_1 \quad \Longleftrightarrow \quad \mathbf{r}_2 = 2\mathbf{r}_1$$

$$\mathbf{a}_5 = -\mathbf{a}_1 + \mathbf{a}_4 \quad \Longleftrightarrow \quad \mathbf{r}_5 = -\mathbf{r}_1 + \mathbf{r}_4$$

Column Correspondence Theorem – Intuitive Idea

$$a_1 + a_2 = a_3$$

$$A = \begin{bmatrix} 6 & 9 & 15 \\ 8 & 0 & 8 \\ 9 & 2 & 11 \end{bmatrix}$$

$$b_1 + b_2 = b_3$$

$$B = \begin{bmatrix} 9 & 2 & 11 \\ 8 & 0 & 8 \\ 6 & 9 & 15 \end{bmatrix}$$

$$c_1 + c_2 = c_3$$

$$C = \begin{bmatrix} 12 & 18 & 30 \\ 8 & 0 & 8 \\ 9 & 2 & 11 \end{bmatrix}$$

$$d_1 + d_2 = d_3$$

$$D = \begin{bmatrix} 6 & 9 & 15 \\ 8 & 0 & 8 \\ 3 & -7 & -4 \end{bmatrix}$$

Column Correspondence Theorem (Column 間的承諾)：
就算 row elementary operation 讓 column 變的不同，
他們之間的關係永遠不變。

Column Correspondence Theorem – Reason

- Before we start:

Coefficient Matrix: $A \xrightarrow{\text{RREF}} R$

Augmented Matrix: $[A \quad b] \xrightarrow{\text{RREF}} [R \quad b']$

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{array}\right] \Rightarrow \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$[A \quad b]$ $\qquad$ $[R \quad b']$

Column Correspondence Theorem – Reason

- The RREF of matrix A is R

  $Ax = b$ and $Rx = b$ have the same solution set?

- The RREF of augmented matrix $[A \quad b]$ is $[R \quad b']$

  $Ax = b$ and $Rx = b'$ have the same solution set

- The RREF of matrix A is R

  $Ax = 0$ and $Rx = 0$ have the same solution set

If $b = 0$, then $b' = 0$.

Column Correspondence Theorem – Reason

- The RREF of matrix A is R, $Ax = 0$ and $Rx = 0$ have the same solution set

$$
\begin{array}{c}
\begin{array}{cccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 \end{array} \\
A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix}
\end{array}
\qquad
\begin{array}{c}
\begin{array}{cccccc} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{r}_4 & \mathbf{r}_5 & \mathbf{r}_6 \end{array} \\
R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}
$$

$\mathbf{a}_2 = 2\mathbf{a}_1$

$-2\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{0}$

$\boxed{Ax = 0}$

$$x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{Rx = 0}$

$$x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\mathbf{r}_2 = 2\mathbf{r}_1$

$-2\mathbf{r}_1 + \mathbf{r}_2 = \mathbf{0}$

Column Correspondence Theorem – Reason

- The RREF of matrix A is R, $Ax = 0$ and $Rx = 0$ have the same solution set

$$A = \begin{array}{c} \begin{array}{cccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 \end{array} \\ \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \end{array}$$

$$R = \begin{array}{c} \begin{array}{cccccc} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{r}_4 & \mathbf{r}_5 & \mathbf{r}_6 \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\mathbf{a}_5 = -\mathbf{a}_1 + \mathbf{a}_4$

$\mathbf{a}_1 - \mathbf{a}_4 + \mathbf{a}_5 = 0$

$\boxed{Ax = 0}$

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$\boxed{Rx = 0}$

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$\mathbf{r}_5 = -\mathbf{r}_1 + \mathbf{r}_4$

$\mathbf{r}_1 - \mathbf{r}_4 + \mathbf{r}_5 = 0$

How about Rows?

- Are there row correspondence theorem? NO

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} — \boldsymbol{a}_1^T — \\ — \boldsymbol{a}_2^T — \\ — \boldsymbol{a}_3^T — \\ — \boldsymbol{a}_4^T — \end{bmatrix} \qquad R = \begin{bmatrix} — \boldsymbol{r}_1^T — \\ — \boldsymbol{r}_2^T — \\ — \boldsymbol{r}_3^T — \\ — \boldsymbol{r}_4^T — \end{bmatrix}$$

$$Span\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\} = Span\{\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \boldsymbol{r}_4\}$$

Are they the same?

Span of Columns

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = [\boldsymbol{a_1} \quad \cdots \quad \boldsymbol{a_6}] \qquad R = [\boldsymbol{r_1} \quad \cdots \quad \boldsymbol{r_6}]$$

$$Span\{\boldsymbol{a_1}, \quad \cdots \quad ,\boldsymbol{a_6}\} \qquad Span\{\boldsymbol{r_1}, \quad \cdots \quad ,\boldsymbol{r_6}\}$$

Are they the same?

The elementary row operations change the span of columns.

NOTE

- Original Matrix v.s. RREF
  - Columns:
    - The relations between the columns are the same.
    - The span of the columns are different.
  - Rows:
    - The relations between the rows are changed.
    - The span of the rows are the same.

RREF v.s. Independent

Column Correspondence Theorem

pivot columns

$$A = \begin{bmatrix} \boxed{1} & 2 & \boxed{-1} & \boxed{2} & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix}$$

linear independent

Leading entries

$$R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

linear independent

The pivot columns are linear independent.

Column Correspondence Theorem

pivot columns

$$A = \begin{bmatrix} \boxed{\begin{matrix}1\\-1\\2\\-3\end{matrix}} & \begin{matrix}2\\-2\\4\\-6\end{matrix} & \boxed{\begin{matrix}-1\\1\\-3\\2\end{matrix}} & \boxed{\begin{matrix}2\\2\\2\\0\end{matrix}} & \begin{matrix}1\\3\\0\\3\end{matrix} & \begin{matrix}2\\6\\3\\9\end{matrix} \end{bmatrix}$$

$\mathbf{a}_2 = 2\mathbf{a}_1$

$\mathbf{a}_5 = -\mathbf{a}_1 + \mathbf{a}_4$

$\mathbf{a}_6 = -5\mathbf{a}_1 - 3\mathbf{a}_3 + 2\mathbf{a}_4$

Leading entries

$$R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathbf{r}_2 = 2\mathbf{r}_1$

$\mathbf{r}_5 = -\mathbf{r}_1 + \mathbf{r}_4$

$\mathbf{r}_6 = -5\mathbf{r}_1 - 3\mathbf{r}_3 + 2\mathbf{r}_4$

The non-pivot columns are the linear combination of the previous pivot columns.

Independent

All columns are independent

↓

Every column is a pivot column

↓

Every column in RREF(A) is standard vector.

3X3

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

Columns are linear independent

RREF ↓

Identity matrix

Independent

All columns are independent

↓

Every column is a pivot column

↓

Every column in RREF(A) is standard vector.

4X3

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

Columns are linear independent

RREF ↓

$$\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} \qquad \begin{bmatrix} I \\ O \end{bmatrix}$$

Independent

All columns are independent

Every column is a pivot column

Every column in RREF(A) is standard vector.

3X4

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

Columns are ~~linea~~
~~independent~~

RREF

Cannot be a pivot column

$$\begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \end{bmatrix}$$

Independent

$$\begin{bmatrix}1 & 2\\ 1 & 2\\ 1 & 2\end{bmatrix}$$

因為太胖了，自己走不動

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

(矮胖型)

The columns are dependent

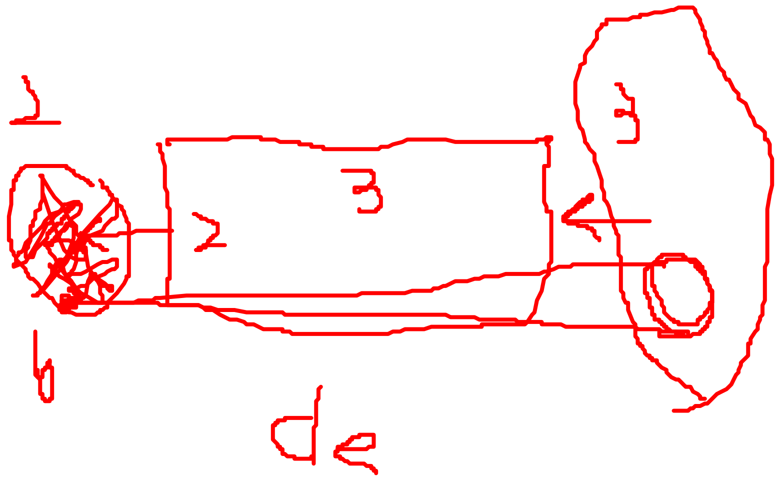
$$\mathcal{S} = \left\{ \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\4\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix} \right\}$$

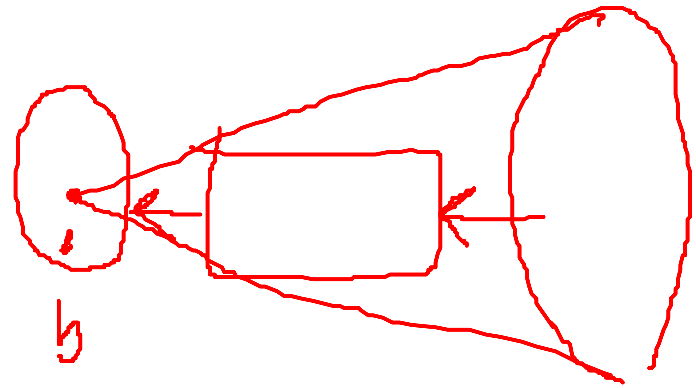
Dependent or Independent?

More than 3 vectors in R^3 must be dependent.

More than m vectors in R^m must be dependent.

Independent – Intuition





RREF v.s. Rank

Rank

Maximum number of Independent Columns

||

Number of Pivot Column

||

Number of Non-zero rows

$$R(A) = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix}$$

Rank = ? **3**

||

$$\mathrm{Rank}(R) = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = ? **3**

Properties of Rank from RREF

Maximum number of Independent Columns

||

Number of Pivot Column

||

Number of Non-zero rows

Rank A ≤ Number of columns

Rank A ≤ Min(Number of columns, Number of rows)

Rank A ≤ Number of rows

Properties of Rank from RREF

- Given a mxn matrix A:
  - Rank A $\leq$ min(m, n)
  - Because "the columns of A are independent" is equivalent to "rank A = n"
    - If m < n, the columns of A is dependent.

Matrix A is ***full rank*** if Rank A = min(m,n)

m   n = 4

3

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

3 X 4

Rank A $\leq$ 3

$$\left\{ \begin{bmatrix} * \\ * \\ * \end{bmatrix}, \begin{bmatrix} * \\ * \\ * \end{bmatrix}, \begin{bmatrix} * \\ * \\ * \end{bmatrix}, \begin{bmatrix} * \\ * \\ * \end{bmatrix} \right\}$$

rank A ≤ 3

A matrix set has 4 vectors belonging to R^3 is dependent

- In R^m, you cannot find more than m vectors that are independent.

Basic, Free Variables v.s. Rank

3 useful equations

$$Ax = b$$

$$\begin{aligned} x_1 + 2x_2 - x_3 + 2x_4 + 5x_5 &= 2 \\ -x_1 - 2x_2 + x_3 + 2x_4 + 3x_5 &= 6 \\ 2x_1 + 4x_2 - 3x_3 + 2x_4 &= 3 \\ -3x_1 - 6x_2 + 2x_3 + 3x_5 &= 9 \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 - x_5 &= -5 \\ x_3 &= -3 \\ x_4 + x_5 &= 2 \\ \cancel{0} &= \cancel{0} \end{aligned}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 2 & 5 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{array}\right] \qquad \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

A $\quad b$ $\qquad$ RREF(A) $\quad b'$

rank = non-zero row = 3 basic variables

nullity = No. column – non-zero row = 2 free variables

$$x_1 = -5 - 2x_2 + x_5$$

$$x_3 = -3$$

$$x_4 = 2 - x_5$$

Rank

Rank

Maximum number of Independent Columns

Number of Pivot Column

Number of Non-zero rows of RREF

Number of Basic Variables

+

Nullity = no. column - rank

~~Number of zero rows of RREF~~

Number of Free Equations

=

C

RREF v.s. Span

Consistent or not

- Given Ax=b, if the reduced row echelon form of [*A* b] is

$$\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Consistent

b is in the span of the columns of A

- Given Ax=b, if the reduced row echelon form of [*A* b] is

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$$

inconsistent

b is NOT in the span of the columns of A

Consistent or not

Ax =b is inconsistent (no solution)

The RREF of [A b] is

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad d \neq 0$$

Only the last column is non-zero

Rank A $\neq$ rank [A b]

Need to know b

Consistent or not

Ax =b is consistent for ***every*** b

$\Updownarrow$

RREF of [A b] cannot have a row whose only non-zero entry is at the last column

$\Updownarrow$

RREF of A cannot have zero row

$\Updownarrow$

Rank A = no. of rows

Consistent or not

e.g.

$$A = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

3 independent columns

$m \times n$ $\quad n \quad m \quad m=3,\ n=4$

Ax =b is consistent for ***every*** b ⟺ Rank A = no. of rows $= m$

$m \times n$

Every b is in the span of the columns of

A= $[a_1 \quad \cdots \quad a_n]$

Every b belongs to $Span\{a_1, \quad \cdots \quad, a_n\}$

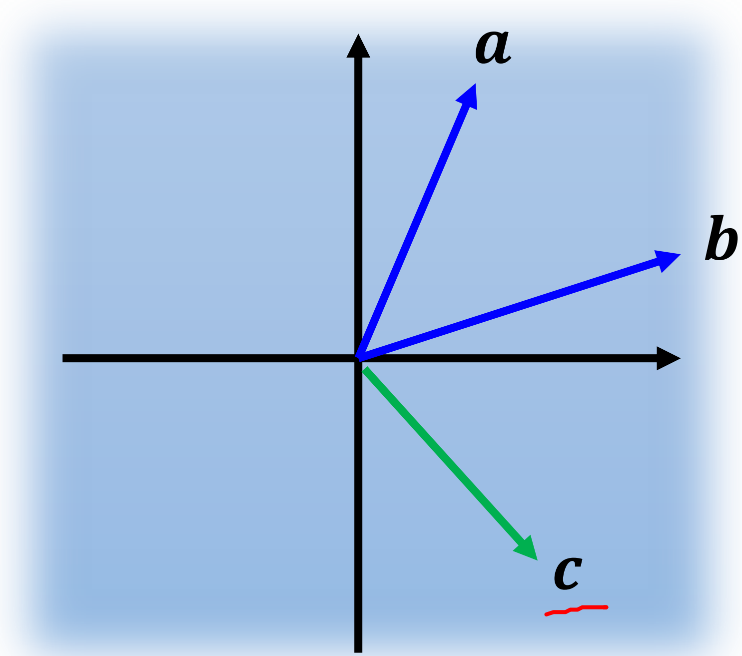
$Span\{a_1, \quad \cdots \quad, a_n\} = R^m$ $\quad \in R^m$

m independent vectors can span R^m

➡ More than m vectors in R^m must be dependent.

m independent vectors can span R^m → More than m vectors in R^m must be dependent.

- Consider R^2



$$\text{Does } \mathcal{S} = \left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}} \right\} \text{ generate } \mathcal{R}^3?$$

yes

independent

Full Rank: Rank = n & Rank = m

- The size of A is mxn n

  Rank A = n m

  A is square or 高瘦

$$\underbrace{\begin{bmatrix} \boxed{1} & \boxed{0} & * & * & * & * \\ \boxed{0} & \boxed{1} & * & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \boxed{0} & \boxed{0} & * & * & * & * \\ \boxed{0} & \boxed{0} & * & * & * & * \end{bmatrix}}$$

*A***x** = **b** has at most one solution

The columns of *A* are linearly independent.

All columns are pivot columns.

RREF of A: $\boxed{\begin{bmatrix} I \\ O \end{bmatrix}}$

Full Rank: Rank = n & Rank = m

- The size of A is mxn

$n \geq m$

Rank A = m

A is square or 矮胖

$$\begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 1 & * & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Every row of R contains a pivot position (leading entry).

$A\mathbf{x} = \mathbf{b}$ always have solution (at least one solution) for every $\mathbf{b}$ in $\mathcal{R}^m$.

The columns of A generate $\mathcal{R}^m$.