Classification: Logistic Regression

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有關分組

- 作業以個人為單位繳交
- 期末專題才需要分組
- 找不到組員也沒有關係，期末專題公告後找不到組員的同學助教會幫忙湊對

Step 1: Function Set

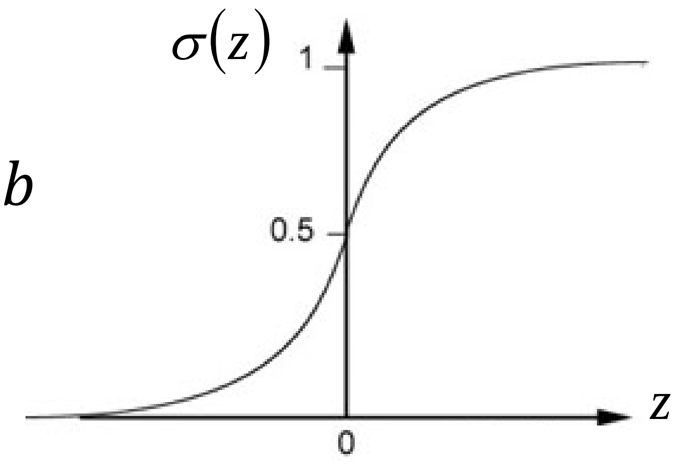
Function set: Including all different w and b

$$\begin{cases} z \geq 0 & \text{class 1} \\ z < 0 & \text{class 2} \end{cases}$$

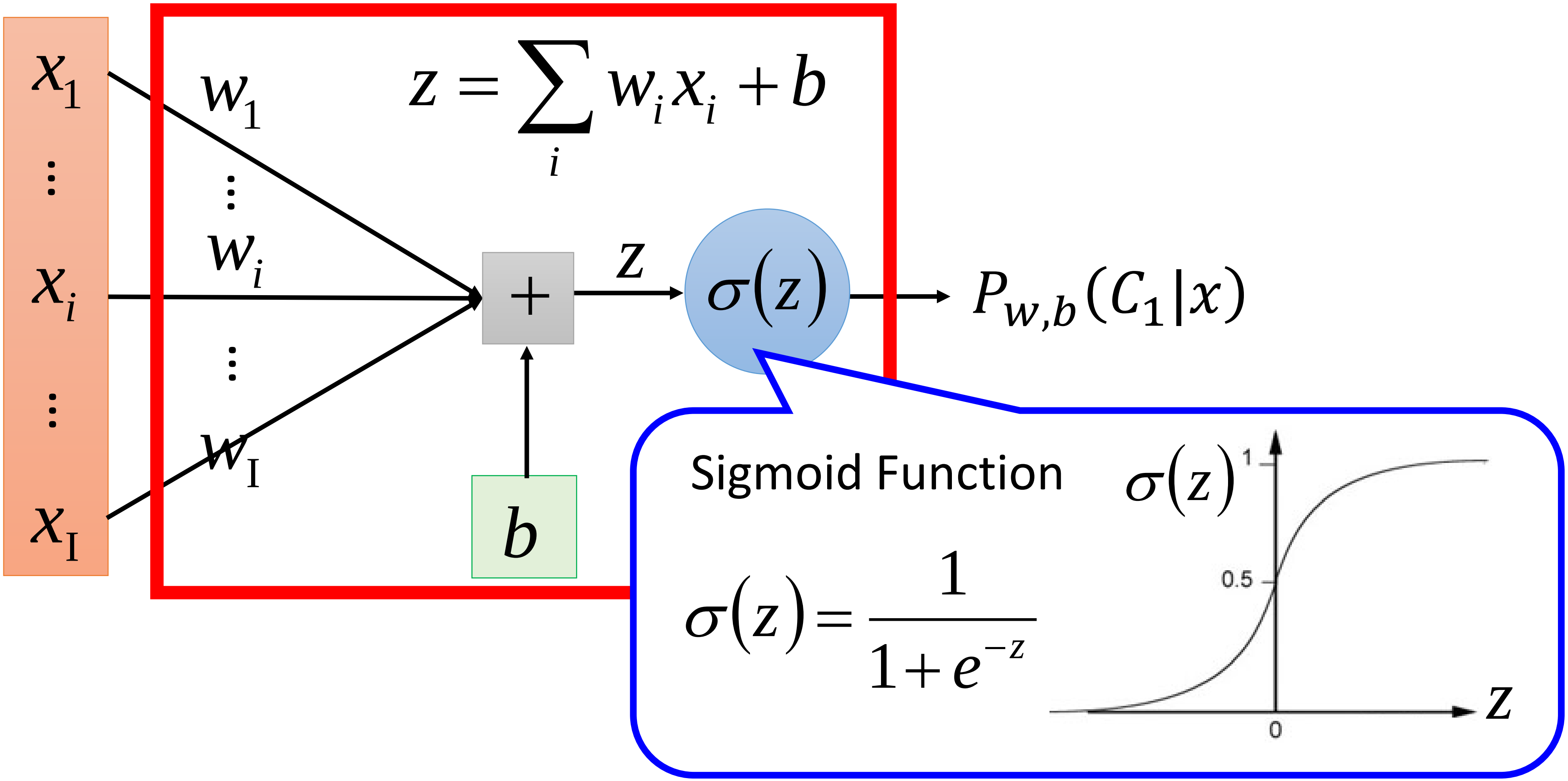
$$P_{w,b}(C_1|x) = \sigma(z)$$

$$z = w \cdot x + b = \sum_i w_i x_i + b$$

$$\sigma(z) = \frac{1}{1 + exp(-z)}$$



Step 1: Function Set



Step 2: Goodness of a Function

Training Data					
	x^1	x^2	x^3	……	x^N
	C_1	C_1	C_2		C_1

Assume the data is generated based on $f_{w,b}(x) = P_{w,b}(C_1|x)$

Given a set of w and b, what is its probability of generating the data?

$$L(w,b) = f_{w,b}(x^1) f_{w,b}(x^2) \left(1 - f_{w,b}(x^3)\right) \cdots f_{w,b}(x^N)$$

The most likely w^* and b^* is the one with the largest $L(w,b)$.

$$w^*, b^* = arg \max_{w,b} L(w,b)$$

x^1	x^2	x^3	……
C_1	C_1	C_2	

→

x^1	x^2	x^3	……
$\hat{y}^1 = 1$	$\hat{y}^2 = 1$	$\hat{y}^3 = 0$	

$\hat{y}^n$: 1 for class 1, 0 for class 2

$$L(w,b) = f_{w,b}(x^1) f_{w,b}(x^2)\left(1 - f_{w,b}(x^3)\right)\cdots$$

$$w^*, b^* = arg\max_{w,b} L(w,b) \quad = \quad w^*, b^* = arg\min_{w,b} -lnL(w,b)$$

$$-lnL(w,b)$$

$$= -lnf_{w,b}(x^1) \Rightarrow -[\,1\; lnf(x^1) + 0\; ln(1 - f(x^1))\,]$$

$$-lnf_{w,b}(x^2) \Rightarrow -[\,1\; lnf(x^2) + 0\; ln(1 - f(x^2))\,]$$

$$-ln\left(1 - f_{w,b}(x^3)\right) \Rightarrow -[\,0\; lnf(x^3) + 1\; ln(1 - f(x^3))\,]$$

⋮

Step 2: Goodness of a Function

$$L(w,b) = f_{w,b}(x^1) f_{w,b}(x^2) \left(1 - f_{w,b}(x^3)\right) \cdots f_{w,b}(x^N)$$

$$-lnL(w,b) = lnf_{w,b}(x^1) + lnf_{w,b}(x^2) + ln\left(1 - f_{w,b}(x^3)\right) \cdots$$

$\hat{y}^n$: 1 for class 1, 0 for class 2

$$= \sum_n -\left[\hat{y}^n lnf_{w,b}(x^n) + (1 - \hat{y}^n) ln\left(1 - f_{w,b}(x^n)\right)\right]$$

Cross entropy between two Bernoulli distribution

Distribution p:

$p(x = 1) = \hat{y}^n$

$p(x = 0) = 1 - \hat{y}^n$

cross entropy

Distribution q:

$q(x = 1) = f(x^n)$

$q(x = 0) = 1 - f(x^n)$

$$H(p,q) = -\sum_x p(x) ln\left(q(x)\right)$$

Step 2: Goodness of a Function

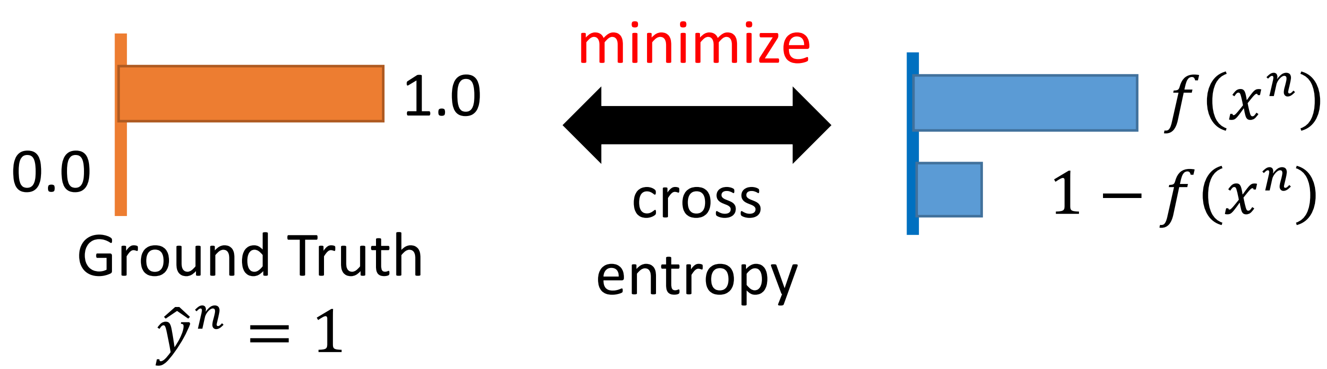
$$L(w,b) = f_{w,b}(x^1) f_{w,b}(x^2) \left(1 - f_{w,b}(x^3)\right) \cdots f_{w,b}(x^N)$$

$$-\ln L(w,b) = \ln f_{w,b}(x^1) + \ln f_{w,b}(x^2) + \ln\left(1 - f_{w,b}(x^3)\right) \cdots$$

$\hat{y}^n$: 1 for class 1, 0 for class 2

$$= \sum_n -\left[\hat{y}^n \ln f_{w,b}(x^n) + (1-\hat{y}^n) \ln\left(1 - f_{w,b}(x^n)\right)\right]$$

Cross entropy between two Bernoulli distribution



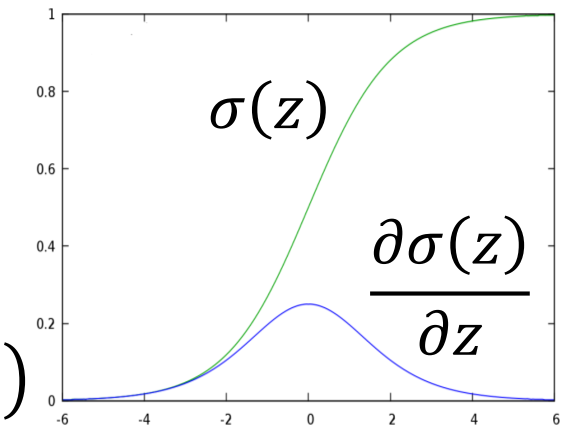
Step 3: Find the best function

$$\frac{-\ln L(w,b)}{\partial w_i} = \sum_n -\left[\hat{y}^n \frac{\ln f_{w,b}(x^n)}{\partial w_i} + (1-\hat{y}^n)\frac{\ln\left(1-f_{w,b}(x^n)\right)}{\partial w_i}\right]$$

$$\left(1-f_{w,b}(x^n)\right)x_i^n$$

$$\frac{\partial \ln f_{w,b}(x)}{\partial w_i} = \frac{\partial \ln f_{w,b}(x)}{\partial z}\frac{\partial z}{\partial w_i} \qquad \frac{\partial z}{\partial w_i} = x_i$$

$$\frac{\partial \ln \sigma(z)}{\partial z} = \frac{1}{\sigma(z)}\frac{\partial \sigma(z)}{\partial z} = \frac{1}{\sigma(z)}\sigma(z)\left(1-\sigma(z)\right)$$



$$f_{w,b}(x) = \sigma(z)$$
$$= 1/1+exp(-z)$$

$$z = w \cdot x + b = \sum_i w_i x_i + b$$

Step 3: Find the best function

$$\frac{-lnL(w,b)}{\partial w_i} = \sum_n -\left[\hat{y}^n \frac{ln f_{w,b}(x^n)}{\partial w_i} + (1-\hat{y}^n)\frac{ln\left(1-f_{w,b}(x^n)\right)}{\partial w_i}\right]$$

$$\left(1-f_{w,b}(x^n)\right)x_i^n \qquad -f_{w,b}(x^n)x_i^n$$

$$\frac{\partial ln\left(1-f_{w,b}(x)\right)}{\partial w_i} = \frac{\partial ln\left(1-f_{w,b}(x)\right)}{\partial z}\frac{\partial z}{\partial w_i} \qquad \frac{\partial z}{\partial w_i} = x_i$$

$$\frac{\partial ln\left(1-\sigma(z)\right)}{\partial z} = -\frac{1}{1-\sigma(z)}\frac{\partial \sigma(z)}{\partial z} = -\frac{1}{1-\sigma(z)}\sigma(z)\left(1-\sigma(z)\right)$$

$$f_{w,b}(x) = \sigma(z)$$
$$= 1/1+exp(-z)$$

$$z = w \cdot x + b = \sum_i w_i x_i + b$$

Step 3: Find the best function

$$\left(1 - f_{w,b}(x^n)\right) x_i^n \qquad\qquad -f_{w,b}(x^n) x_i^n$$

$$\frac{-\ln L(w,b)}{\partial w_i} = \sum_n -\left[\hat{y}^n \frac{\ln f_{w,b}(x^n)}{\partial w_i} + (1-\hat{y}^n)\frac{\ln\left(1 - f_{w,b}(x^n)\right)}{\partial w_i}\right]$$

$$= \sum_n -\left[\hat{y}^n \left(1 - f_{w,b}(x^n)\right) x_i^n - (1-\hat{y}^n) f_{w,b}(x^n) x_i^n\right]$$

$$= \sum_n -\left[\hat{y}^n - \hat{y}^n f_{w,b}(x^n) - f_{w,b}(x^n) + \hat{y}^n f_{w,b}(x^n)\right] x_i^n$$

$$= \sum_n -\left(\hat{y}^n - f_{w,b}(x^n)\right) x_i^n$$

Larger difference, larger update

$$w_i \leftarrow w_i - \eta \sum_n -\left(\hat{y}^n - f_{w,b}(x^n)\right) x_i^n$$

Logistic Regression + Square Error

Step 1: $f_{w,b}(x) = \sigma\left(\sum_i w_i x_i + b\right)$

Step 2: Training data: $(x^n, \hat{y}^n)$, $\hat{y}^n$: 1 for class 1, 0 for class 2

$$L(f) = \frac{1}{2}\sum_n \left(f_{w,b}(x^n) - \hat{y}^n\right)^2$$

Step 3:

$$\frac{\partial\,(f_{w,b}(x) - \hat{y})^2}{\partial w_i} = 2\left(f_{w,b}(x) - \hat{y}\right) \frac{\partial f_{w,b}(x)}{\partial z} \frac{\partial z}{\partial w_i}$$

$$= 2\left(f_{w,b}(x) - \hat{y}\right) f_{w,b}(x)\left(1 - f_{w,b}(x)\right) x_i$$

$\hat{y}^n = 1$

If $f_{w,b}(x^n) = 1$ (close to target) → $\partial L / \partial w_i = 0$

If $f_{w,b}(x^n) = 0$ (far from target) → $\partial L / \partial w_i = 0$

Logistic Regression + Square Error

Step 1: $$f_{w,b}(x) = \sigma\left(\sum_i w_i x_i + b\right)$$

Step 2: Training data: $(x^n, \hat{y}^n)$, $\hat{y}^n$: 1 for class 1, 0 for class 2

$$L(f) = \frac{1}{2}\sum_n \left(f_{w,b}(x^n) - \hat{y}^n\right)^2$$

Step 3:

$$\frac{\partial\,(f_{w,b}(x) - \hat{y})^2}{\partial w_i} = 2\left(f_{w,b}(x) - \hat{y}\right)\frac{\partial f_{w,b}(x)}{\partial z}\frac{\partial z}{\partial w_i}$$

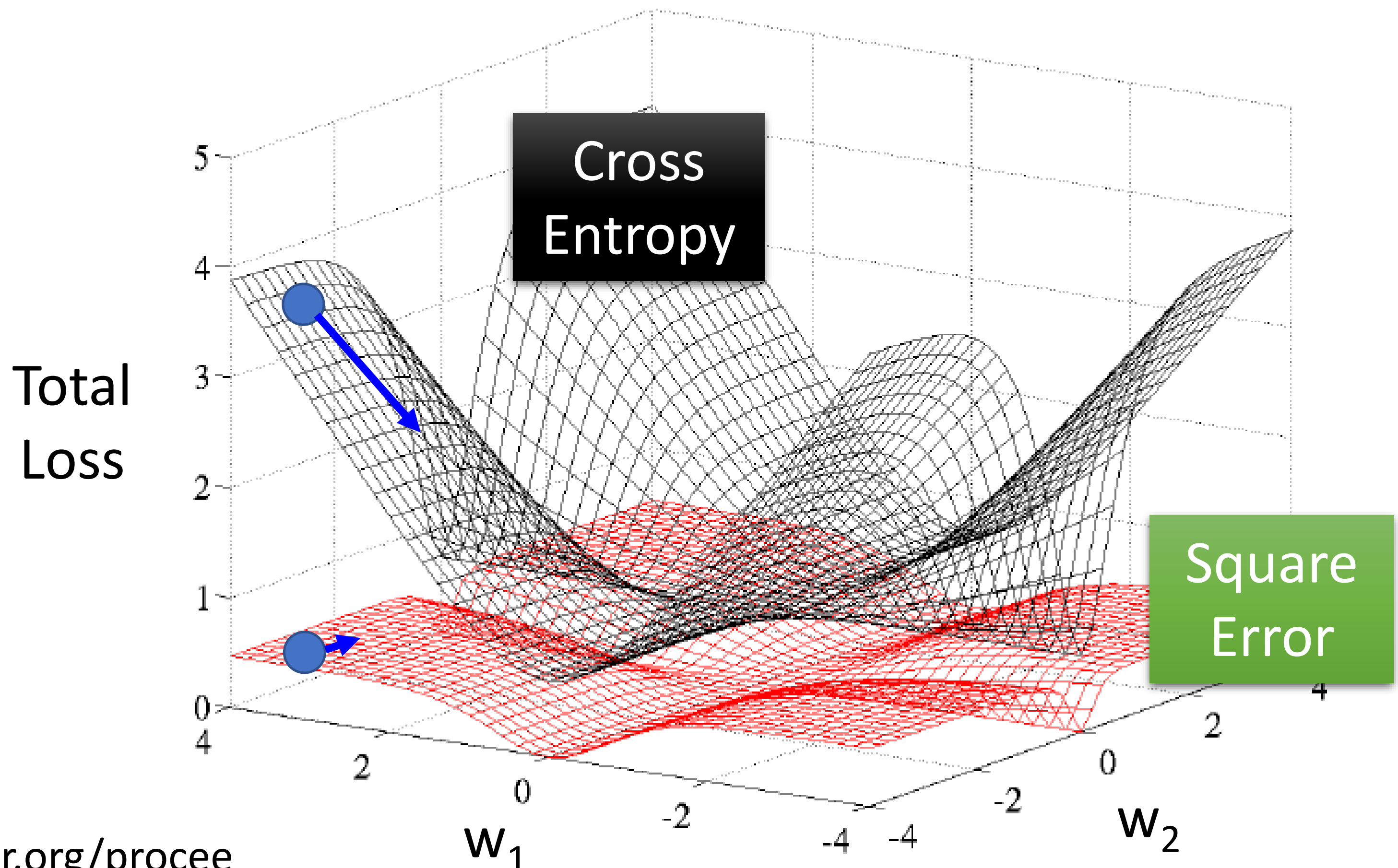
$$= 2\left(f_{w,b}(x) - \hat{y}\right) f_{w,b}(x)\left(1 - f_{w,b}(x)\right) x_i$$

$\hat{y}^n = 0$

If $f_{w,b}(x^n) = 1$ (far from target) ➡ $\partial L / \partial w_i = 0$

If $f_{w,b}(x^n) = 0$ (close to target) ➡ $\partial L / \partial w_i = 0$

Cross Entropy v.s. Square Error



http://jmlr.org/proceedings/papers/v9/glorot10a/glorot10a.pdf

	Logistic Regression	***Linear Regression***
Step 1:	$f_{w,b}(x) = \sigma\left(\sum_i w_i x_i + b\right)$ Output: between 0 and 1	$f_{w,b}(x) = \sum_i w_i x_i + b$ Output: any value
Step 2:		
Step 3:		

	Logistic Regression	***Linear Regression***
Step 1:	$f_{w,b}(x) = \sigma\left(\sum_i w_i x_i + b\right)$ Output: between 0 and 1	$f_{w,b}(x) = \sum_i w_i x_i + b$ Output: any value
Step 2:	Training data: $(x^n, \hat{y}^n)$ $\hat{y}^n$: 1 for class 1, 0 for class 2 $L(f) = \sum_n l(f(x^n), \hat{y}^n)$	Training data: $(x^n, \hat{y}^n)$ $\hat{y}^n$: a real number $L(f) = \frac{1}{2}\sum_n (f(x^n) - \hat{y}^n)^2$

Cross entropy:

$$l(f(x^n), \hat{y}^n) = -\left[\hat{y}^n lnf(x^n) + (1 - \hat{y}^n) ln\left(1 - f(x^n)\right)\right]$$

	Logistic Regression	***Linear Regression***
Step 1:	$f_{w,b}(x) = \sigma\left(\sum_i w_i x_i + b\right)$ Output: between 0 and 1	$f_{w,b}(x) = \sum_i w_i x_i + b$ Output: any value
Step 2:	Training data: $(x^n, \hat{y}^n)$ $\hat{y}^n$: 1 for class 1, 0 for class 2 $L(f) = \sum_n l(f(x^n), \hat{y}^n)$	Training data: $(x^n, \hat{y}^n)$ $\hat{y}^n$: a real number $L(f) = \frac{1}{2}\sum_n (f(x^n) - \hat{y}^n)^2$

Step 3:

Logistic regression: $w_i \leftarrow w_i - \eta \sum_n -\left(\underline{\hat{y}^n - f_{w,b}(x^n)}\right) x_i^n$

Linear regression: $w_i \leftarrow w_i - \eta \sum_n -\left(\underline{\hat{y}^n - f_{w,b}(x^n)}\right) x_i^n$

Discriminative v.s. Generative

$$P(C_1|x) = \sigma(w \cdot x + b)$$

directly find ***w*** and b

Find $\mu^1, \mu^2, \Sigma^{-1}$

$$w^T = (\mu^1 - \mu^2)^T \Sigma^{-1}$$

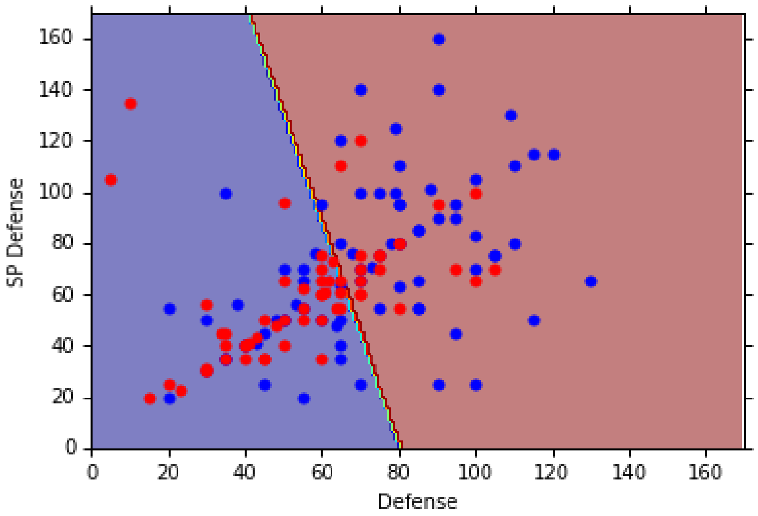
$$b = -\frac{1}{2}(\mu^1)^T(\Sigma^1)^{-1}\mu^1 + \frac{1}{2}(\mu^2)^T(\Sigma^2)^{-1}\mu^2 + ln\frac{N_1}{N_2}$$

Will we obtain the same set of w and b?

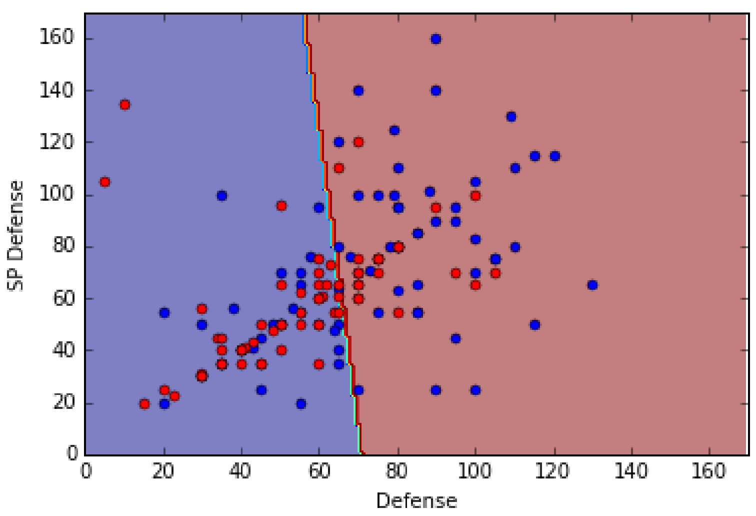
The same model (function set), but different function may be selected by the same training data.

Generative v.s. Discriminative

Generative



Discriminative



All: hp, att, sp att, de, sp de, speed

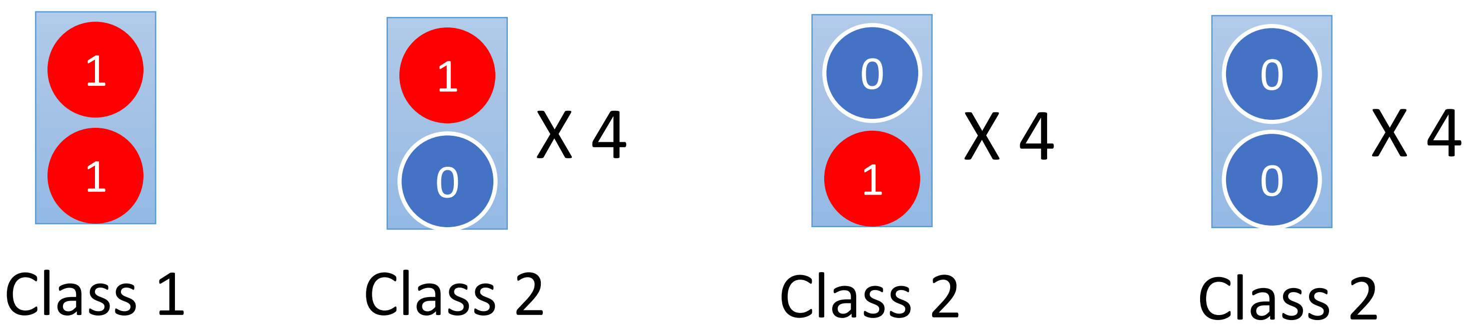
73% accuracy

79% accuracy

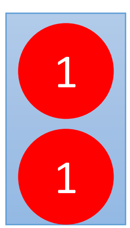
Generative v.s. Discriminative

- Example

Training Data



Testing Data



Class 1?

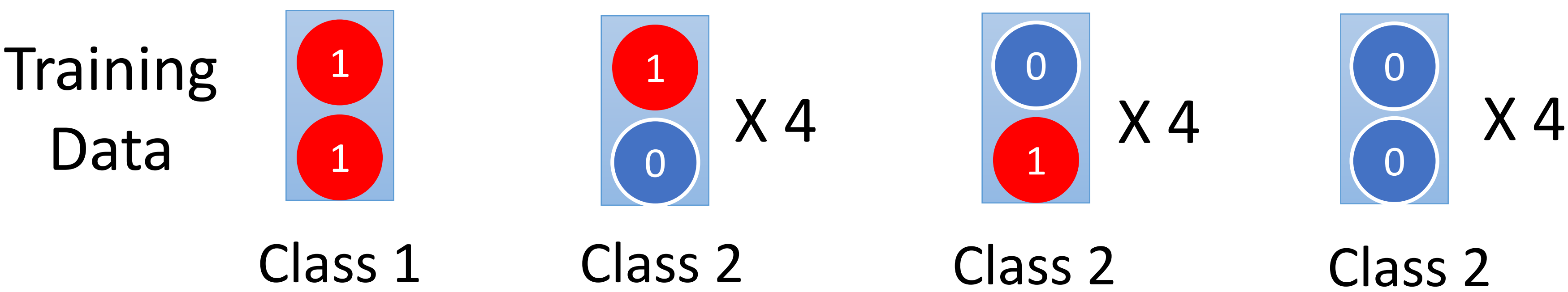
Class 2?

How about Naïve Bayes?

$$P(x|C_i) = P(x_1|C_i)P(x_2|C_i)$$

Generative v.s. Discriminative

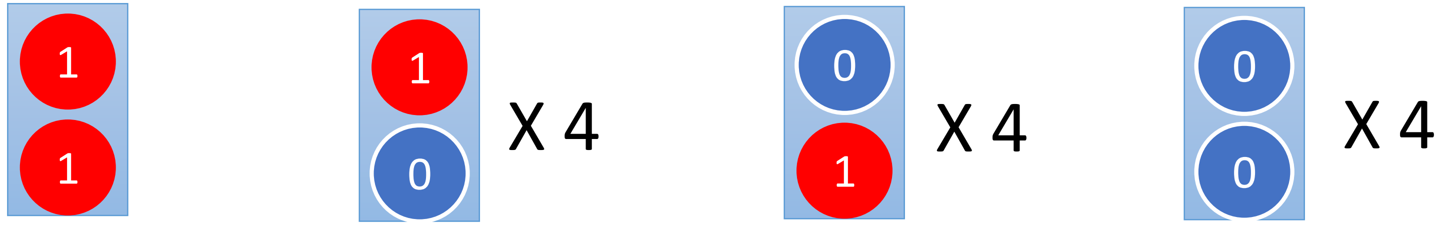
- Example



$$P(C_1) = \frac{1}{13} \qquad P(x_1 = 1|C_1) = 1 \qquad P(x_2 = 1|C_1) = 1$$

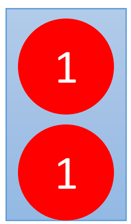
$$P(C_2) = \frac{12}{13} \qquad P(x_1 = 1|C_2) = \frac{1}{3} \qquad P(x_2 = 1|C_2) = \frac{1}{3}$$

Training Data



Class 1   Class 2   Class 2   Class 2

Testing Data



<0.5

$$P(C_1|x) = \frac{P(x|C_1)P(C_1)}{P(x|C_1)P(C_1) + P(x|C_2)P(C_2)}$$

Numerator: $P(x|C_1) \rightarrow 1 \times 1$, $P(C_1) \rightarrow \frac{1}{13}$

Denominator: $P(x|C_1) \rightarrow 1 \times 1$, $P(C_1) \rightarrow \frac{1}{13}$, $P(x|C_2) \rightarrow \frac{1}{3} \times \frac{1}{3}$, $P(C_2) \rightarrow \frac{12}{13}$

$$P(C_1) = \frac{1}{13} \qquad P(x_1 = 1|C_1) = 1 \qquad P(x_2 = 1|C_1) = 1$$

$$P(C_2) = \frac{12}{13} \qquad P(x_1 = 1|C_2) = \frac{1}{3} \qquad P(x_2 = 1|C_2) = \frac{1}{3}$$

Generative v.s. Discriminative

- Usually people believe discriminative model is better
- Benefit of generative model
  - With the assumption of probability distribution
    - less training data is needed
    - more robust to the noise
  - Priors and class-dependent probabilities can be estimated from different sources.

Multi-class Classification (3 classes as example)

C_1: w^1, b_1 $\quad z_1 = w^1 \cdot x + b_1$

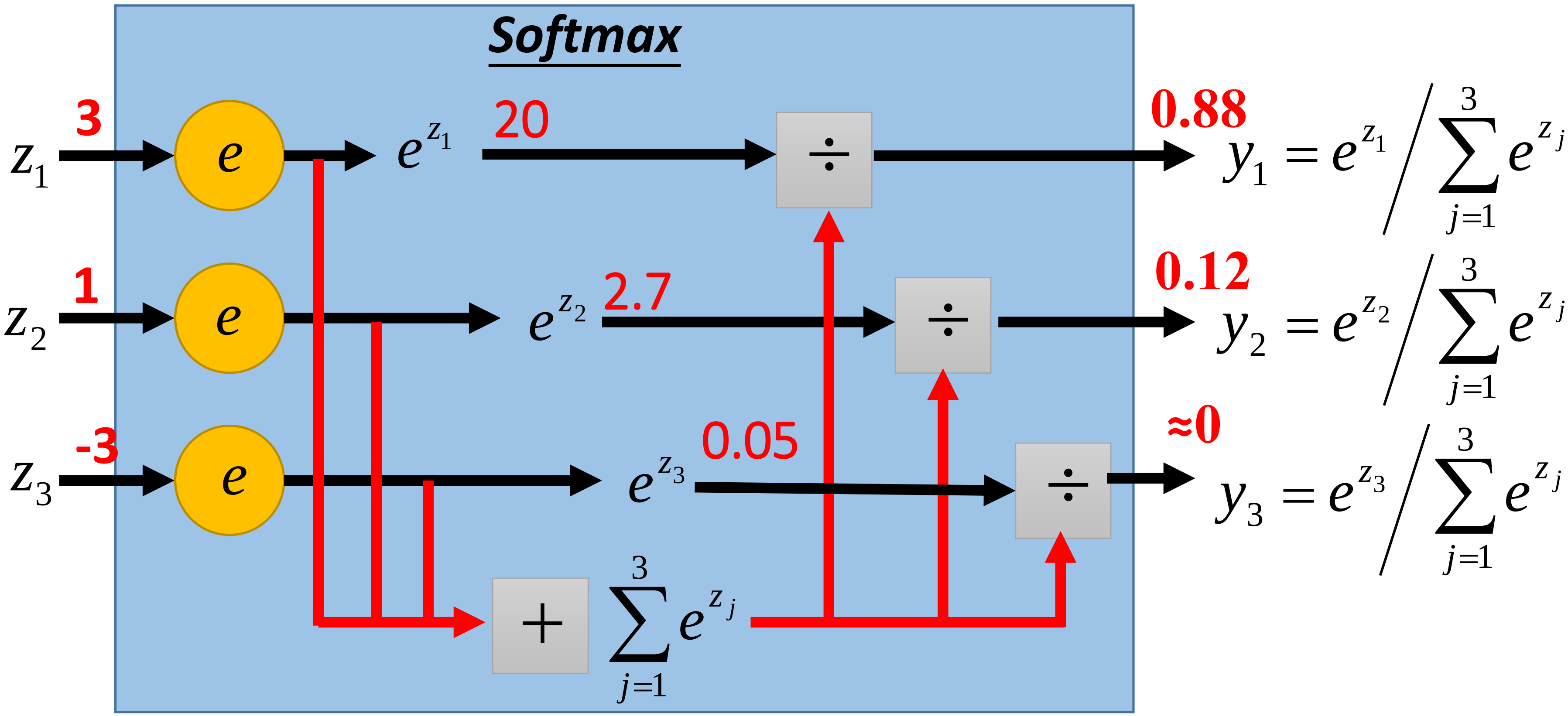
C_2: w^2, b_2 $\quad z_2 = w^2 \cdot x + b_2$

C_3: w^3, b_3 $\quad z_3 = w^3 \cdot x + b_3$

Probability:

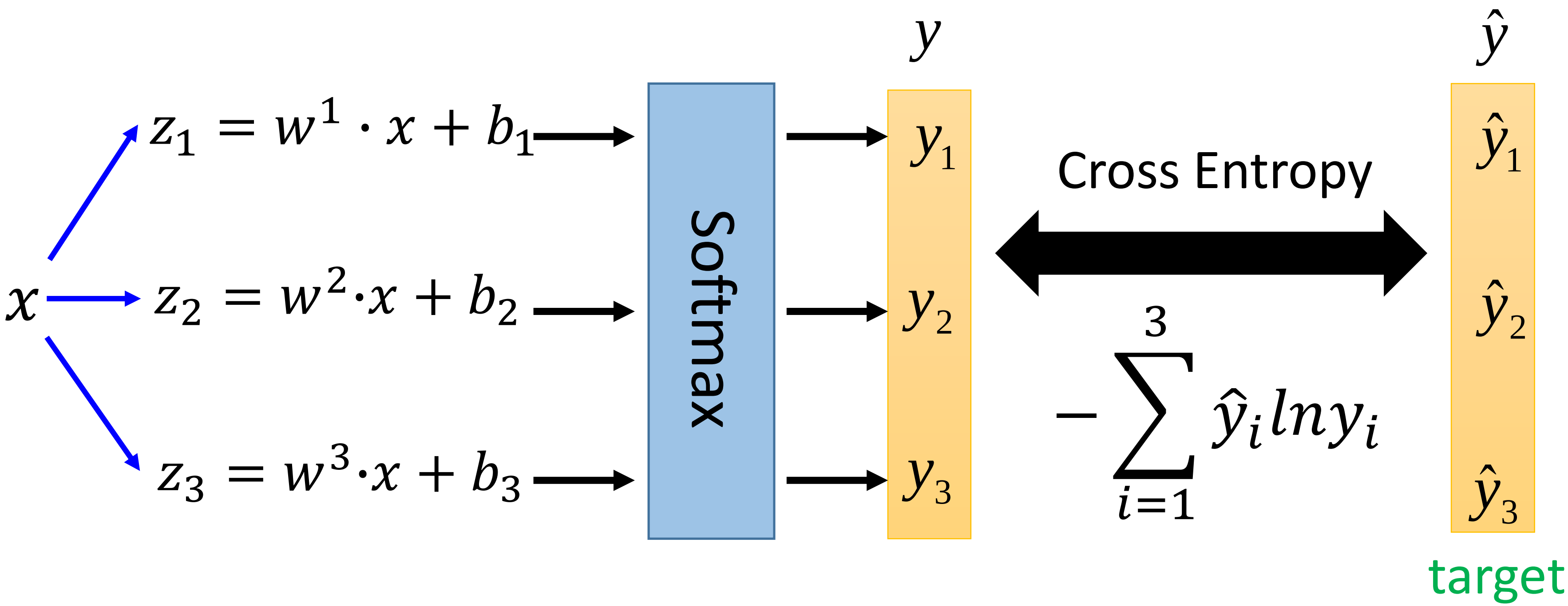
- $1 > y_i > 0$
- $\sum_i y_i = 1$

$$y_i = P(C_i \mid x)$$



Multi-class Classification (3 classes as example)

[Bishop, P209-210]



If x ∈ class 1

$$\hat{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$-ln y_1$$

If x ∈ class 2

$$\hat{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$-ln y_2$$

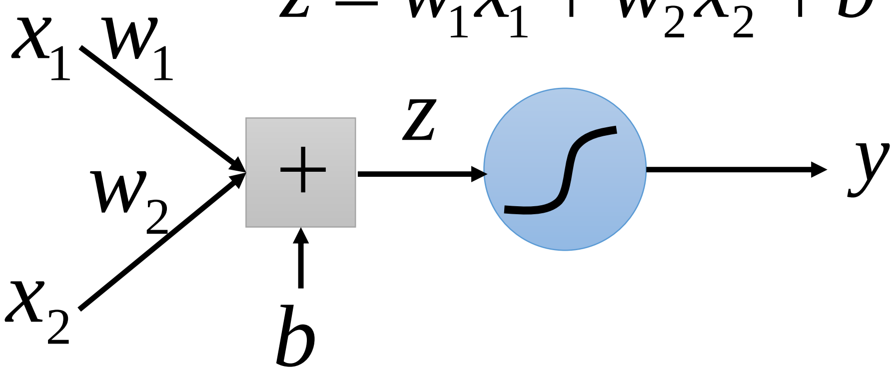
If x ∈ class 3

$$\hat{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$-ln y_3$$

Limitation of Logistic Regression

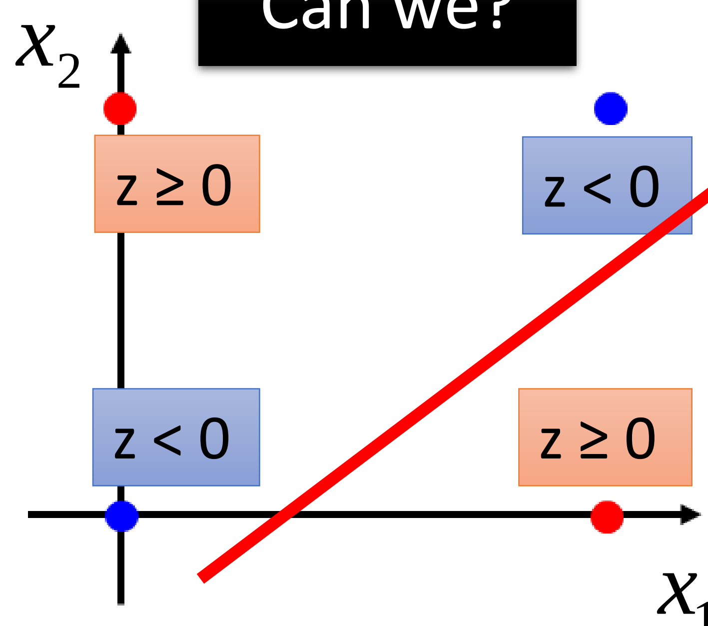
$$z = w_1 x_1 + w_2 x_2 + b$$



$$\begin{cases} Class1 & y \geq 0.5 \quad (z \geq 0) \\ Class2 & y < 0.5 \quad (z < 0) \end{cases}$$

Input Feature		Label
x_1	x_2	
0	0	Class 2
0	1	Class 1
1	0	Class 1
1	1	Class 2

Can we?



Limitation of Logistic Regression

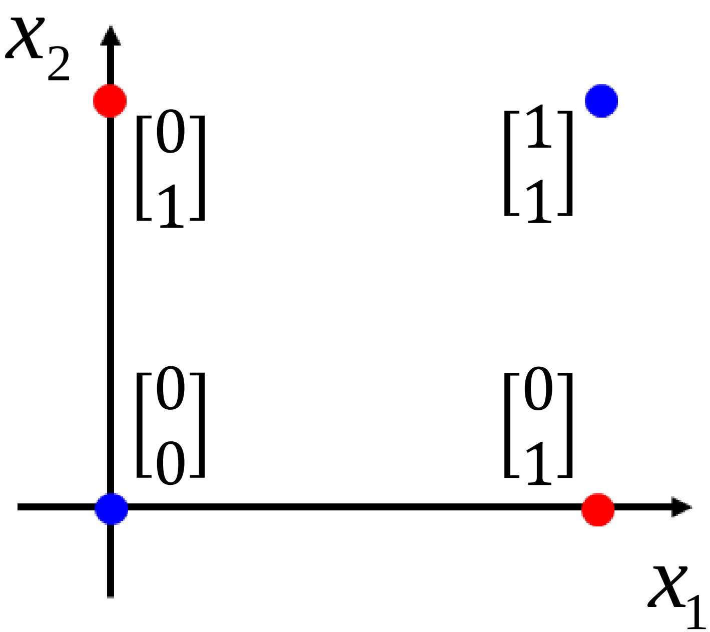
- ***Feature transformation***

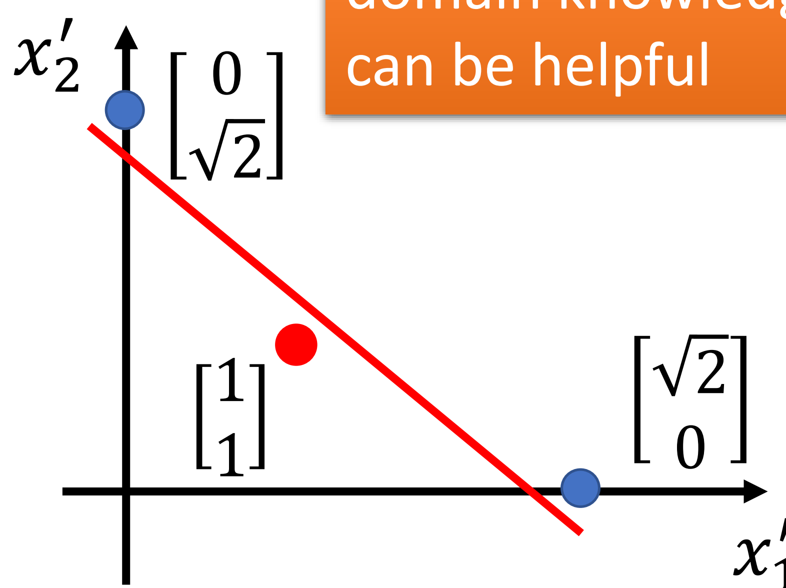
x_1': distance to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

x_2': distance to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

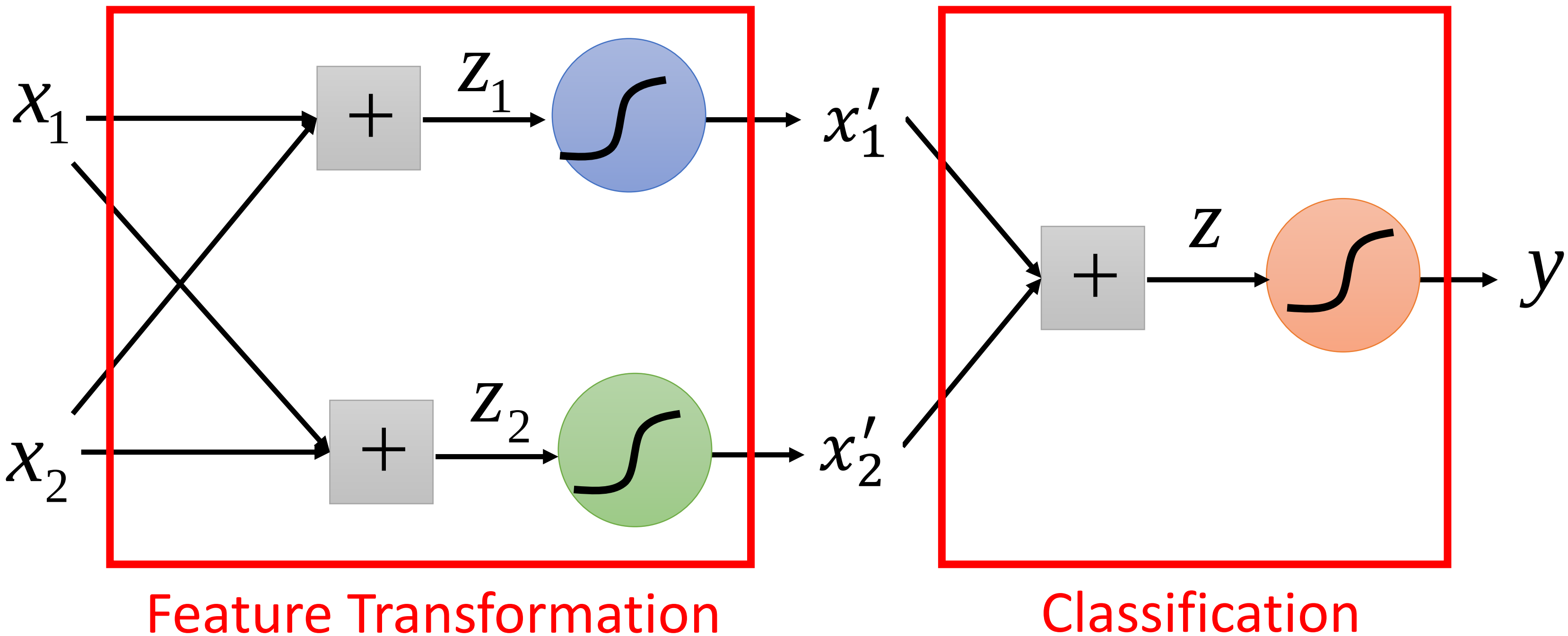
Not always easy ….. domain knowledge can be helpful



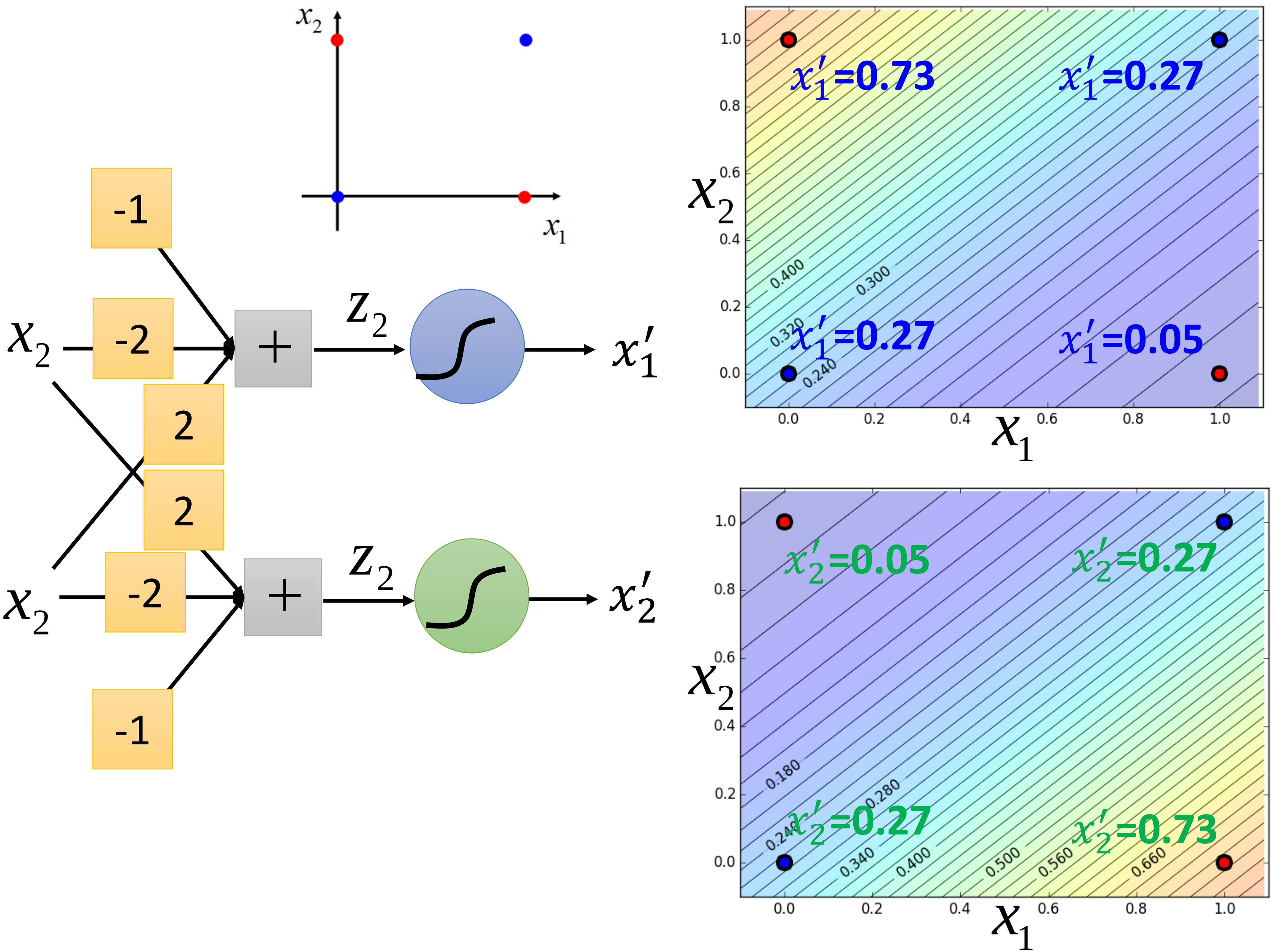


Limitation of Logistic Regression

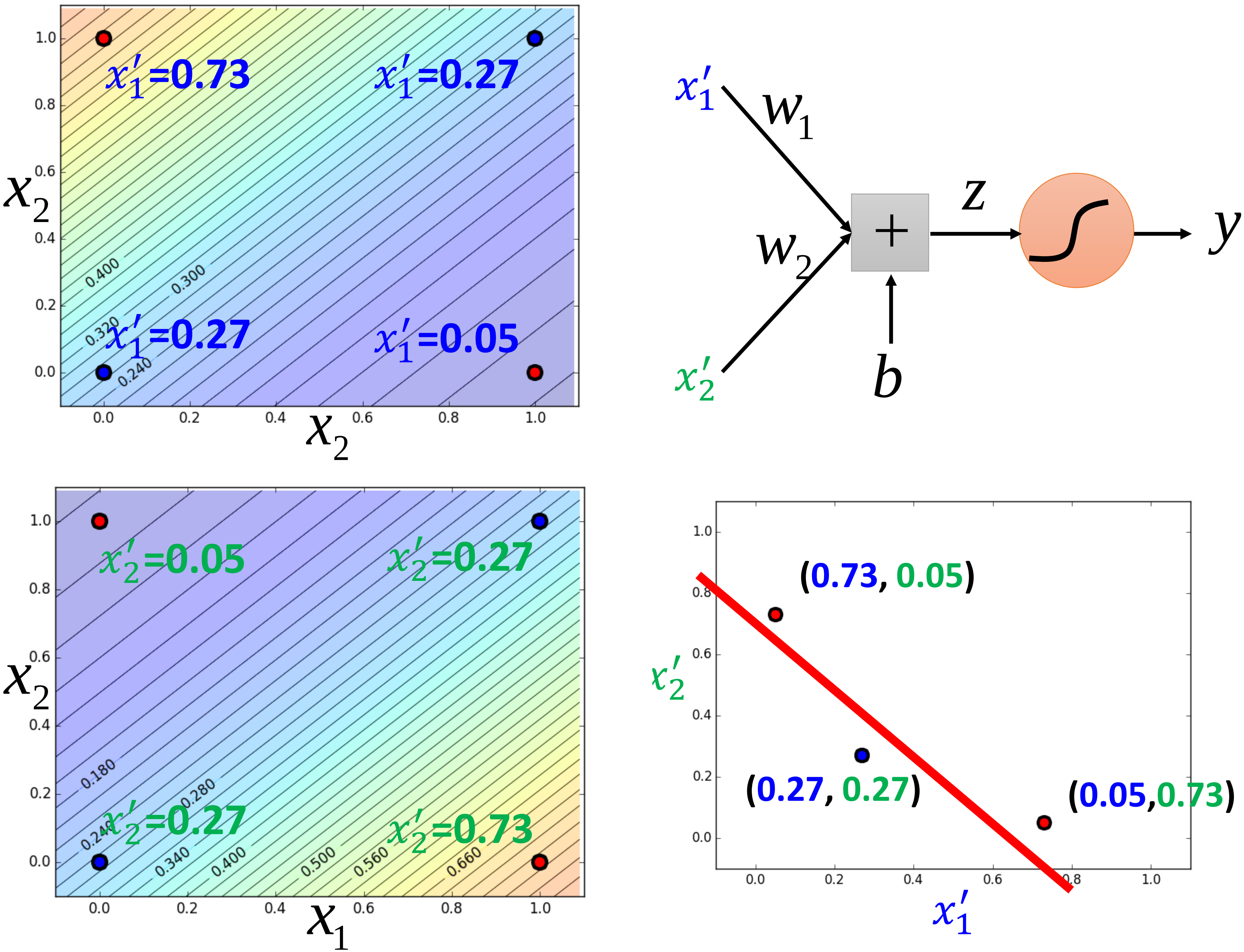
- Cascading logistic regression models



(ignore bias in this figure)



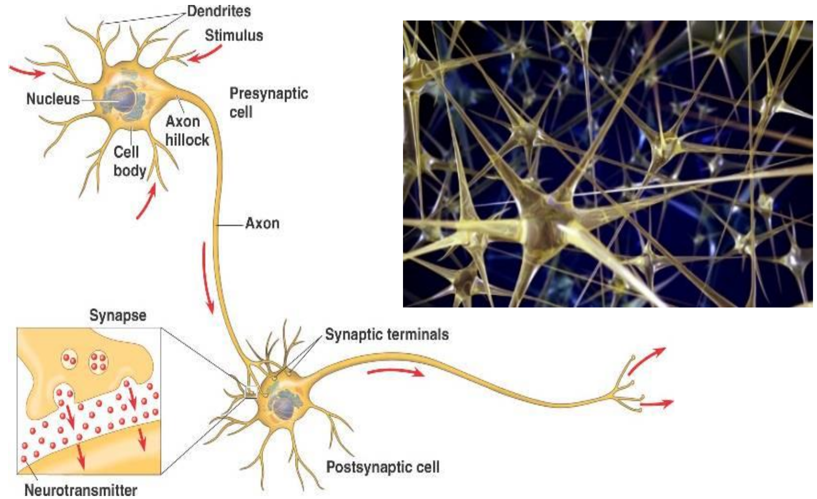
x2
x1
-1
-2
2
2
-2
-1
x2
x2
z2
z2
+
+
x'1
x'2
x'1=0.73
x'1=0.27
x'1=0.27
x'1=0.05
0.400
0.300
0.320
0.240
x2
x1
x'2=0.05
x'2=0.27
x'2=0.27
x'2=0.73
0.180
0.280
0.340
0.400
0.500
0.560
0.660
x2
x1

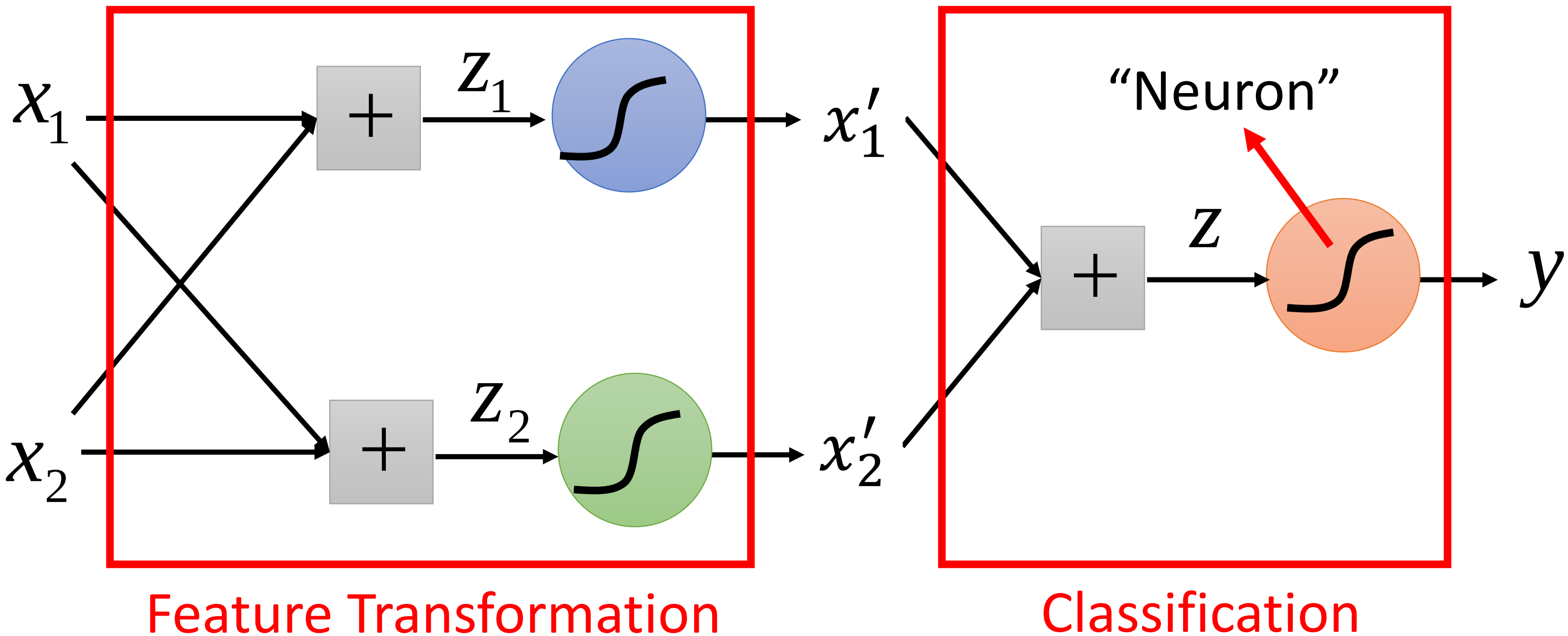


x'1=0.73
x'1=0.27
x'1=0.27
x'1=0.05
0.400
0.300
0.320
0.240
x2
x2
w1
w2
x'1
x'2
z
y
b
+
x'2=0.05
x'2=0.27
x'2=0.27
x'2=0.73
0.180
0.280
0.240
0.340
0.400
0.500
0.560
0.660
x2
x1
(0.73, 0.05)
(0.27, 0.27)
(0.05,0.73)
x'1
x'2

Deep Learning!

All the parameters of the logistic regressions are jointly learned.





Neural Network

Reference

- Bishop: Chapter 4.3

Acknowledgement

- 感謝 林恩妤 發現投影片 上的錯誤

Appendix

Three Steps

x^1	x^2	x^3	……	x^n
$\hat{y}^1$	$\hat{y}^2$	$\hat{y}^3$		$\hat{y}^n$

$$\hat{y}^n = class\ 1, class\ 2$$

- Step 1. Function Set (Model)

x feature → If $P(C_1|x) > 0.5$, output: y = class 1
Otherwise, output: y = class 2 → y class

$$P(C_1|x) = \sigma(w \cdot x + b)$$

w and b are related to $N_1, N_2, \mu^1, \mu^2, \Sigma$

- Step 2. Goodness of a function

$$L(f) = \sum_n \delta(f(x^n) \neq \hat{y}^n) \Rightarrow L(f) = \sum_n l(f(x^n) \neq \hat{y}^n)$$

- Step 3. Find the best function: gradient descent

Step 2: Loss function

$$f_{w,b}(x) = \begin{cases} +1 & z \geq 0 \\ -1 & z < 0 \end{cases}$$

Ideal loss:

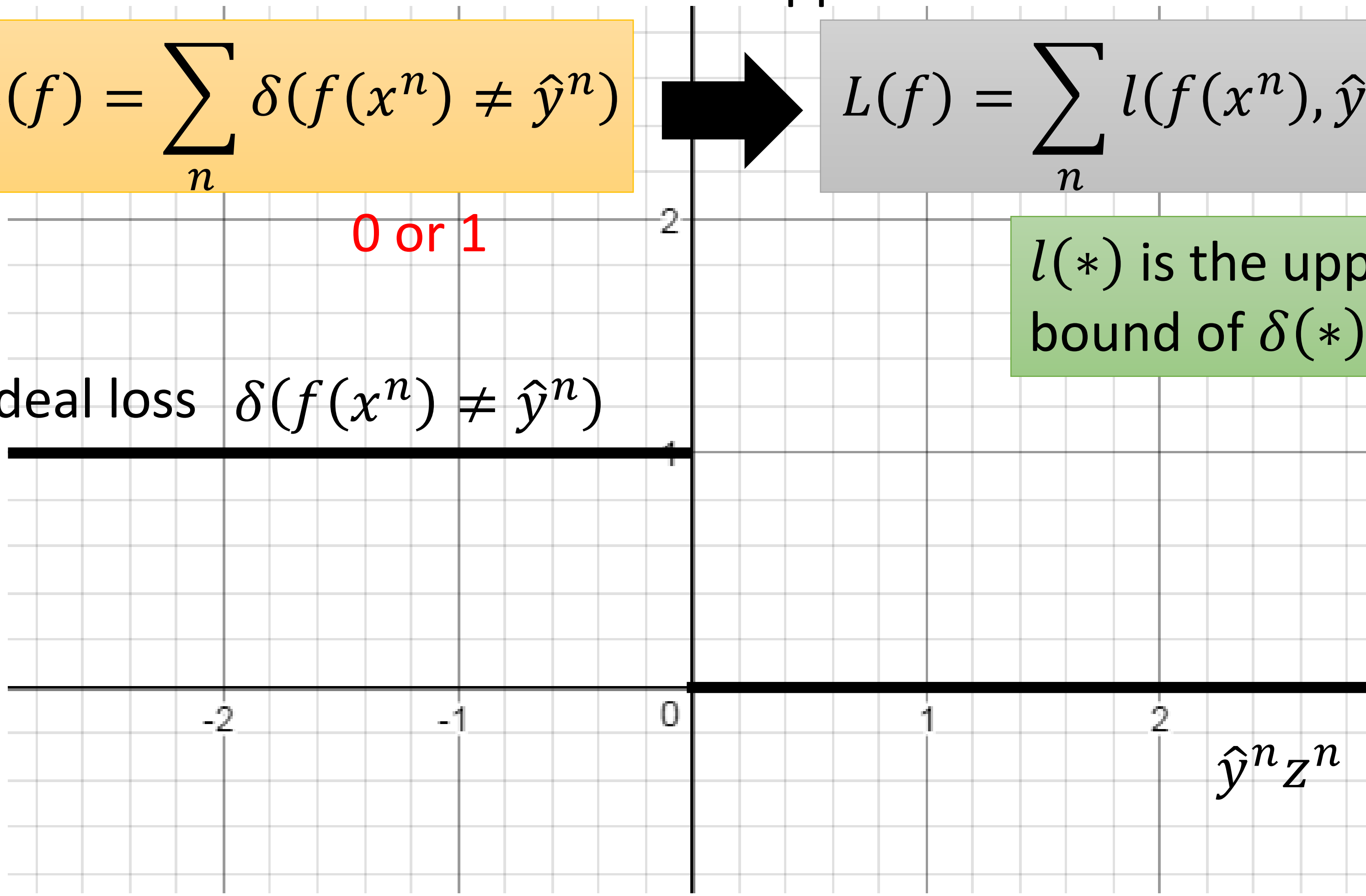
$$L(f) = \sum_n \delta(f(x^n) \neq \hat{y}^n)$$

0 or 1

Approximation:

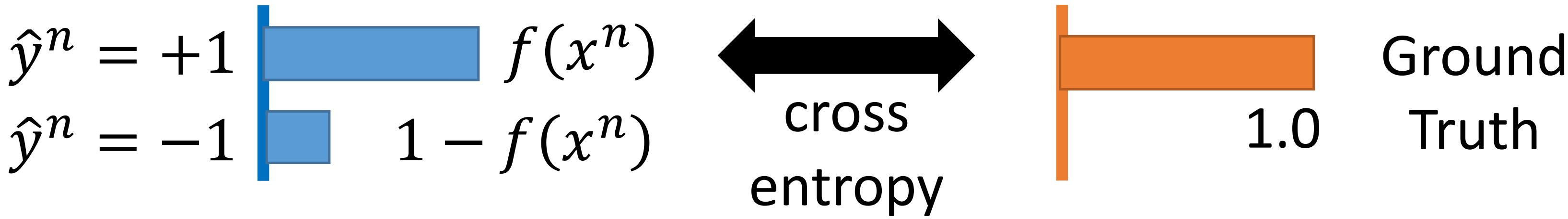
$$L(f) = \sum_n l(f(x^n), \hat{y}^n)$$

$l(*)$ is the upper bound of $\delta(*)$



Step 2: Loss function

$l(f(x^n),\hat{y}^n)$: cross entropy



If $\hat{y}^n = +1$:

$$l(f(x^n),\hat{y}^n) = -\ln f(x^n) = -\ln \sigma(z^n) = -\ln \frac{1}{1+exp(-z^n)}$$

$$= \ln\left(1+exp(-z^n)\right) = \underline{\ln\left(1+exp(-\hat{y}^n z^n)\right)}$$

If $\hat{y}^n = -1$:

$$l(f(x^n),\hat{y}^n) = -\ln\left(1-f(x^n)\right)$$

$$= -\ln\left(1-\sigma(x^n)\right) = -\ln \frac{exp(-z^n)}{1+exp(-z^n)} = -\ln \frac{1}{1+exp(z^n)}$$

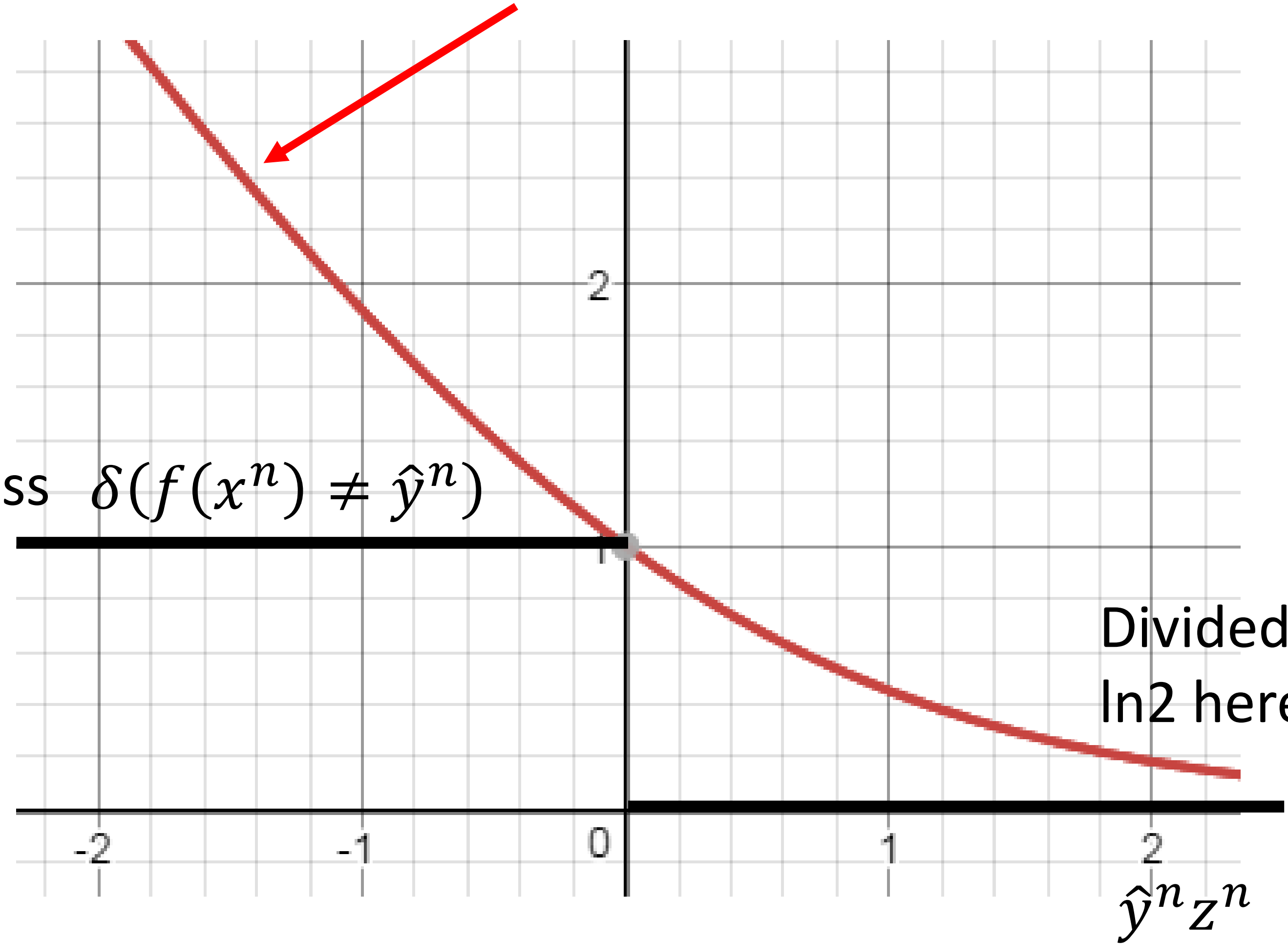
$$= \ln\left(1+exp(z^n)\right) = \underline{\ln\left(1+exp(-\hat{y}^n z^n)\right)}$$

Step 2: Loss function

$l(f(x^n), \hat{y}^n)$: cross entropy

$$l(f(x^n), \hat{y}^n) = \ln\left(1 + exp(-\hat{y}^n z^n)\right)$$

Ideal loss $\delta(f(x^n) \neq \hat{y}^n)$



Divided by ln2 here