How many solutions?

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Given a system of linear equations with \( m \) equations and \( n \) variables

\[
Ax = b \quad A: m \times n \quad x \in \mathbb{R}^n \quad b \in \mathbb{R}^m
\]

Is \( b \) a linear combination of columns of \( A \)?

Is \( b \) in the span of the columns of \( A \)?

\[\begin{align*}
\text{NO} & \quad \text{YES} \\
\text{No solution} & \quad \text{Have solution}
\end{align*}\]

We don’t know how many solutions
Today

Given a system of linear equations with $m$ equations and $n$ variables

$$Ax = b \quad A: m \times n \quad x \in R^n \quad b \in R^m$$

Is $b$ a linear combination of columns of $A$?

Is $b$ in the span of the columns of $A$?

---

**NO**

No solution

The columns of $A$ are **independent**.

- Rank $A = n$
- Nullity $A = 0$

**YES**

Unique solution

The columns of $A$ are **dependent**.

- Rank $A < n$
- Nullity $A > 0$

Other cases?

Infinite solution
How many solutions?

Dependent and Independent
Dependent and Independent

Linear Dependent

Given a vector set, \( \{ \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \} \), if there exists any \( \mathbf{a}_i \) that is a linear combination of other vectors

\[
\begin{align*}
\left\{ \begin{bmatrix} -4 \\ 12 \\ 6 \end{bmatrix}, \begin{bmatrix} -10 \\ 30 \\ 15 \end{bmatrix} \right\} & \quad \text{Dependent or Independent?} \\
\left\{ \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 11 \end{bmatrix} \right\} & \quad \text{Dependent or Independent?}
\end{align*}
\]
Dependent and Independent

Given a vector set, \{a_1, a_2, \ldots, a_n\}, if there exists any \(a_i\) that is a linear combination of other vectors:

\[
\left\{ \begin{bmatrix} 3 \\ -1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} \right\}
\]

Zero vector is the linear combination of any other vectors.

Any set contains zero vector would be linear dependent.

Flaw of the definition: How about a set with only one vector?
How to check?

Given a vector set, \( \{a_1, a_2, \ldots, a_n\} \), if there exists any \( a_i \) that is a linear combination of other vectors

\[
\begin{align*}
2a_i + a_j + 3a_k &= 0 \\
2a_i + a_j &= -3a_k \\
\left(-\frac{2}{3}\right)a_i + \left(-\frac{1}{3}\right)a_j &= a_k
\end{align*}
\]

\[
\begin{align*}
a_i' &= 3a_j' + 4a_k' \\
a_i' - 3a_j' - 4a_k' &= 0
\end{align*}
\]

Given a vector set, \( \{a_1, a_2, \ldots, a_n\} \), there exists scalars \( x_1, x_2, \ldots, x_n \), that are not all zero, such that

\[
x_1a_1 + x_2a_2 + \cdots + x_na_n = 0.
\]
Another Definition

• A set of n vectors \( \{a_1, a_2, \ldots, a_n\} \) is linear dependent
  • If there exist scalars \( x_1, x_2, \ldots, x_n \), not all zero, such that
    \[
    x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = 0
    \]
• A set of n vectors \( \{a_1, a_2, \ldots, a_n\} \) is linear independent
  • Only scalars such that
    \[
    x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = 0
    \]
    Only if \( x_1 = x_2 = \cdots = x_k = 0 \)

How about the vector with only one element?
Intuition

- Intuitive link between dependence and the number of solutions

\[
\begin{pmatrix}
6 & 1 & 7 \\
3 & 8 & 11 \\
3 & 3 & 6
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= 
\begin{pmatrix}
14 \\
22 \\
12
\end{pmatrix}

1 \cdot \begin{pmatrix}
6 \\
3 \\
3
\end{pmatrix}
+ 1 \cdot \begin{pmatrix}
1 \\
8 \\
3
\end{pmatrix}
= 
\begin{pmatrix}
7 \\
11 \\
6
\end{pmatrix}

1 \cdot \begin{pmatrix}
6 \\
3 \\
3
\end{pmatrix}
+ 1 \cdot \begin{pmatrix}
1 \\
8 \\
3
\end{pmatrix}
+ 1 \cdot \begin{pmatrix}
7 \\
11 \\
6
\end{pmatrix}
= 
\begin{pmatrix}
14 \\
22 \\
12
\end{pmatrix}

\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}

2 \cdot \begin{pmatrix}
6 \\
3 \\
3
\end{pmatrix}
+ 2 \cdot \begin{pmatrix}
1 \\
8 \\
3
\end{pmatrix}
= 
\begin{pmatrix}
14 \\
22 \\
12
\end{pmatrix}

\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= 
\begin{pmatrix}
2 \\
2 \\
0
\end{pmatrix}

Dependent:
Once we have solution, we have infinite.

Infinite Solution
Homogeneous Equations

\[ x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = 0 \quad \iff \quad Ax = 0 \]

\[ A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \]

\[ x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \]

**Homogeneous linear equations**

Always having 0 as solution

A set of n vectors \( \{a_1, a_2, \cdots, a_n\} \) is linear dependent

Ax = 0 have non-zero solution

A set of n vectors \( \{a_1, a_2, \cdots, a_n\} \) is linear independent

Ax = 0 only have zero solution

infinite
Homogeneous Equations

- Columns of $A$ are \textbf{dependent} $\rightarrow$ If $Ax=b$ have solution, it will have \textbf{Infinite Solutions}
- We can find non-zero solution $u$ such that $Au = 0$
- There exists $v$ such that $Av = b$

\begin{align*}
A(u + v) &= b \\
&u + v \text{ is another solution different to } v
\end{align*}

- If $Ax=b$ have \textbf{Infinite solutions} $\rightarrow$ Columns of $A$ are dependent
\begin{align*}
Au &= b \\
Av &= b
\end{align*}
\begin{align*}
A(u - v) &= 0 \\
&\text{Non-zero}
\end{align*}
How many solutions?

Rank and Nullity
Intuitive Definition

• The **rank** of a matrix is defined as the maximum number of *linearly independent columns* in the matrix.

• **Nullity** = Number of columns - **rank**

\[
\begin{bmatrix}
-3 & 2 & -1 \\
7 & 9 & 0 \\
0 & 0 & 2
\end{bmatrix}
\quad
\begin{bmatrix}
1 & 3 & 10 \\
2 & 6 & 20 \\
3 & 9 & 30
\end{bmatrix}
\quad
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
Intuitive Definition

• The **rank** of a matrix is defined as the maximum number of *linearly independent columns* in the matrix.

• **Nullity** = Number of columns - **rank**

\[
\begin{bmatrix}
1 & 3 & 4 \\
2 & 6 & 8
\end{bmatrix}
\quad
\begin{bmatrix}
0 & 3 \\
0 & 5
\end{bmatrix}
\quad
\begin{bmatrix}
5 \\
2
\end{bmatrix}
\quad
[6]
\]
Intuitive Definition

• The **rank** of a matrix is defined as the maximum number of *linearly independent columns* in the matrix.

• **Nullity** = Number of columns - **rank**

If A is a mxn matrix:

- Rank A = n
- Nullity A = 0
- Columns of A are independent
How many solutions?

Concluding Remarks
Conclusion

Is $b$ a linear combination of columns of $A$?

No solution

NO

Is $b$ in the span of the columns of $A$?

The columns of $A$ are **independent**.

- Rank $A = n$
- Nullity $A = 0$

Unique solution

YES

The columns of $A$ are **dependent**.

- Rank $A < n$
- Nullity $A > 0$

Infinite solution

$Ax = b$

$A: m \times n \quad x \in R^n \quad b \in R^m$
Conclusion

The columns of $A$ are *independent*.

- Rank $A = n$
- Nullity $A = 0$

Is $b$ a linear combination of columns of $A$?

- NO

Is $b$ in the span of the columns of $A$?

- NO

*No solution*

- YES

*Infinite solution*

$A: m \times n$

- $x \in R^n$
- $b \in R^m$

Is $b$ a linear combination of columns of $A$?

- NO

Is $b$ in the span of the columns of $A$?

- YES

*Unique solution*
Question

• True or False

• If the columns of \( A \) are linear independent, then \( Ax=b \) has unique solution.

• If the columns of \( A \) are linear independent, then \( Ax=b \) has at most one solution.

• If the columns of \( A \) are linear dependent, then \( Ax=b \) has infinite solution.

• If the columns of \( A \) are linear independent, then \( Ax=0 \) (homogeneous equation) has unique solution.

• If the columns of \( A \) are linear dependent, then \( Ax=0 \) (homogeneous equation) has infinite solution.
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