Orthogonal Vector
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Orthogonal Set

• A set of vectors is called an orthogonal set if every pair of distinct vectors in the set is orthogonal.

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} \right\}$$

An orthogonal set?

By definition, a set with only one vector is an orthogonal set.

Is orthogonal set independent?

• Reference: Chapter 7.2
Independent?

- Any orthogonal set of nonzero vectors is linearly independent.

Let $S = \{v_1, v_2, \ldots, v_k\}$ be an orthogonal set $v_i \neq 0$ for $i = 1, 2, \ldots, k$.

Assume $c_1, c_2, \ldots, c_k$ make $c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$

$$(c_1v_1 + c_2v_2 + \cdots + c_iv_i + \cdots + c_kv_k) \cdot v_i = 0 \cdot v_i = 0$$

$= c_1v_1 \cdot v_i + c_2v_2 \cdot v_i + \cdots + c_iv_i \cdot v_i + \cdots + c_kv_k \cdot v_i$

$= c_i(v_i \cdot v_i) = c_i||v_i||^2$

$c_i = 0$

$c_1 = c_2 = \cdots = c_k = 0$
Orthonormal Set

- A set of vectors is called an orthonormal set if it is an orthogonal set, and the norm of all the vectors is 1

\[ S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} \right\} \]

A vector that has norm equal to 1 is called a unit vector.

Is orthonormal set independent?
Yes

\[ \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{42}} \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} \]
Orthogonal Basis

• A basis that is an orthogonal (orthonormal) set is called an orthogonal (orthonormal) basis

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Orthogonal basis of \( \mathbb{R}^3 \)

\[
\begin{bmatrix}
0 & 0 & 0 & 1
\end{bmatrix}
\]

Orthonormal basis of \( \mathbb{R}^3 \)
Orthogonal Projection

• Orthogonal projection of a vector onto a line

\[ z \cdot u = 0 \]

**v**: any vector  
**u**: any nonzero vector on \( \mathcal{L} \)  
**w**: orthogonal projection of \( v \) onto \( \mathcal{L} \), \( w = cu \)  
**z**: \( v - w \)

\[(v - w) \cdot u = (v - cu) \cdot u = v \cdot u - cu \cdot u = v \cdot u - c \|u\|^2 \]

\[ c = \frac{v \cdot u}{\|u\|^2} \quad w = cu = \frac{v \cdot u}{\|u\|^2} u \]

Distance from tip of \( v \) to \( \mathcal{L} \): \( \|z\| = \|v - w\| = \left\| v - \frac{v \cdot u}{\|u\|^2} u \right\| \)
Orthogonal Projection

• Example:

\[ c = \frac{v \cdot u}{\|u\|^2} \]
\[ w = cu = \frac{v \cdot u}{\|u\|^2} u \]

\[ \mathcal{L} \text{ is } y = (1/2)x \]
\[ v = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad u = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \]
Orthogonal Basis

Let $S = \{v_1, v_2, \cdots, v_k\}$ be an orthogonal basis for a subspace $V$, and let $u$ be a vector in $V$.

$$u = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

$u \cdot v_1$ $u \cdot v_2$ $u \cdot v_k$

$$\frac{u \cdot v_1}{\|v_1\|^2} \quad \frac{u \cdot v_2}{\|v_2\|^2} \quad \frac{u \cdot v_k}{\|v_k\|^2}$$

Proof

To find $c_i$

$$u \cdot v_i = (c_1 v_1 + c_2 v_2 + \cdots + c_i v_i + \cdots + c_k v_k) \cdot v_i$$

$$= c_1 v_1 \cdot v_i + c_2 v_2 \cdot v_i + \cdots + c_i v_i \cdot v_i + \cdots + c_k v_k \cdot v_i$$

$$= c_i (v_i \cdot v_i) = c_i \|v_i\|^2$$

$$c_i = \frac{u \cdot v_i}{\|v_i\|^2}$$
Example

• Example: $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for $\mathbb{R}^3$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$$

Let $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{u} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3$. 

\[
\begin{array}{ccc}
c_1 & c_2 & c_3 \\
\end{array}
\]
Orthogonal Basis

Let \(\{u_1, u_2, \cdots, u_k\}\) be a basis of a subspace \(V\). How to transform \(\{u_1, u_2, \cdots, u_k\}\) into an orthogonal basis \(\{v_1, v_2, \cdots, v_k\}\)?

Let \(\{u_1, u_2, \cdots, u_k\}\) be a basis for a subspace \(W\) of \(\mathbb{R}^n\). Define

\[
\begin{align*}
v_1 &= u_1, \\
v_2 &= u_2 - \frac{u_2 \cdot v_1}{\|v_1\|^2} v_1, \\
v_3 &= u_3 - \frac{u_3 \cdot v_1}{\|v_1\|^2} v_1 - \frac{u_3 \cdot v_2}{\|v_2\|^2} v_2, \\
&\vdots \\
v_k &= u_k - \frac{u_k \cdot v_1}{\|v_1\|^2} v_1 - \frac{u_k \cdot v_2}{\|v_2\|^2} v_2 - \cdots - \frac{u_k \cdot v_{k-1}}{\|v_{k-1}\|^2} v_{k-1}.
\end{align*}
\]

Then \(\{v_1, v_2, \cdots, v_i\}\) is an orthogonal set of nonzero vectors such that

\[
\text{Span} \ \{v_1, v_2, \cdots, v_i\} = \text{Span} \ \{u_1, u_2, \cdots, u_i\}
\]

for each \(i\). So \(\{v_1, v_2, \cdots, v_k\}\) is an orthogonal basis for \(W\).