Characteristic Polynomial
Hung-yi Lee
Outline

• Last lecture:
  • Given eigenvalues, we know how to find eigenvectors or eigenspaces
  • Check eigenvalues

• This lecture: How to find eigenvalues?

• Reference: Textbook 5.2
Looking for Eigenvalues

A scalar \( t \) is an eigenvalue of \( A \)

- Existing \( v \neq 0 \) such that \( Av = tv \)
- Existing \( v \neq 0 \) such that \( Av - tv = 0 \)
- Existing \( v \neq 0 \) such that \( (A - tI_n)v = 0 \)
- \( (A - tI_n)v = 0 \) has multiple solutions
- The columns of \( (A - tI_n) \) are Dependent
- \( \text{Rank } (A - tI_n) < n \) \( \iff \) \( (A - tI_n) \) is not invertible
- \( \text{det}(A - tI_n) = 0 \)
Characteristic Polynomial

A scalar $t$ is an eigenvalue of $A$ if
\[ \det(A - tI_n) = 0 \]

$A$ is the standard matrix of linear operator $T$

$\det(A - tI_n)$: Characteristic polynomial of $A$

linear operator $T$

$\det(A - tI_n) = 0$: Characteristic equation of $A$

linear operator $T$

Eigenvalues are the roots of characteristic polynomial or solutions of characteristic equation.
Looking for Eigenvalues

• Example 1: Find the eigenvalues of $A = \begin{bmatrix} -4 & -3 \\ 3 & 6 \end{bmatrix}$

A scalar $t$ is an eigenvalue of $A$ $\iff$ $\det(A - tI_n) = 0$

$A - tI_2 = \begin{bmatrix} -4 - t & -3 \\ 3 & 6 - t \end{bmatrix}$

$\det(A - tI_2) = 0$

$t = -3$ or $5$

The eigenvalues of $A$ are -3 or 5.
Looking for Eigenvalues

• Example 1: Find the eigenvalues of $A = \begin{bmatrix} -4 & -3 \\ 3 & 6 \end{bmatrix}$

The eigenvalues of $A$ are -3 or 5.

**Eigenspace of -3**

$Ax = -3x \quad \Rightarrow \quad (A + 3I)x = 0$  
find the solution

**Eigenspace of 5**

$Ax = 5x \quad \Rightarrow \quad (A - 5I)x = 0$  
find the solution
Looking for Eigenvalues

• Example 2: find the eigenvalues of linear operator

\[
T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} -x_1 \\ 2x_1 - x_2 - x_3 \\ -x_3 \end{bmatrix}
\]

standard matrix

A scalar \( t \) is an eigenvalue of \( A \) \( \iff \) \( \det(A - tI_n) = 0 \)

\[
A - tI_n = \begin{bmatrix} -1 - t & 0 & 0 \\ 2 & -1 - t & -1 \\ 0 & 0 & -1 - t \end{bmatrix}
\]

\[ \det(A - tI_n) = (-1 - t)^3 \]
Looking for Eigenvalues

- Example 3: linear operator on $\mathbb{R}^2$ that rotates a vector by 90°

A scalar $t$ is an eigenvalue of $A$ $\iff \det(A - tI_n) = 0$

Standard matrix of the 90°-rotation:

$$
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
$$

$$
\det \left( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - tI_2 \right)
$$

No eigenvalues, no eigenvectors
Characteristic Polynomial

• In general, a matrix $A$ and RREF of $A$ have different characteristic polynomials. Different Eigenvalues
• Similar matrices have the same characteristic polynomials The same Eigenvalues

$$det(B - tl) = det(P^{-1}AP - P^{-1}(tl)P)$$

$$= det(P^{-1}(A - tl)AP)$$

$$= det(P^{-1})det(A - tl)det(P)$$

$$= \left( \frac{1}{det(P)} \right) det(A - tl)det(P) = det(A - tl)$$
Characteristic Polynomial

• Question: What is the order of the characteristic polynomial of an $n \times n$ matrix $A$?
  • The characteristic polynomial of an $n \times n$ matrix is indeed a polynomial with degree $n$
  • Consider $\text{det}(A - tl_n)$

• Question: What is the number of eigenvalues of an $n \times n$ matrix $A$?
  • Fact: An $n \times n$ matrix $A$ have less than or equal to $n$ eigenvalues
  • Consider complex roots and multiple roots
Characteristic Polynomial

• If nxn matrix A has n eigenvalues (including multiple roots)

\[ A = \begin{bmatrix} -4 & -3 \\ 3 & 6 \end{bmatrix} \]

Eigenvalues: -3, 5
Characteristic Polynomial

• The eigenvalues of an upper triangular matrix are its diagonal entries.

Characteristic Polynomial:

\[
\begin{bmatrix}
a & * & * \\
0 & b & * \\
0 & 0 & c
\end{bmatrix}
\]

\[
\det
\begin{bmatrix}
a - t & * & * \\
0 & b - t & * \\
0 & 0 & c - t
\end{bmatrix}
\]

\[
= (a - t)(b - t)(c - t)
\]

The determinant of an upper triangular matrix is the product of its diagonal entries.
Characteristic Polynomial v.s. Eigenspace

- Characteristic polynomial of $A$ is

$$det(A - tI_n) = (t - \lambda_1)^{m_1}(t - \lambda_2)^{m_2} \ldots (t - \lambda_k)^{m_k}$$

Eigenvalue: $\lambda_1, \lambda_2, \ldots, \lambda_k$
Eigenspace: $d_1, d_2, \ldots, d_k$
(dimension) $\leq m_1, \leq m_2, \ldots, \leq m_k$
Characteristic Polynomial v.s. Eigenspace

• Example 1:

\[
A = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 2 \\
0 & 2 & 1 \\
\end{bmatrix}
\]

characteristic polynomials:

\[-(t + 1)^2(t – 3)\]

Eigenvalue -1

Multiplierity of “-1” is 2

Dim of eigenspace is 1 or 2

\text{Dim} = 2

Eigenvalue 3

Multiplierity of “3” is 1

Dim of eigenspace must be 1
Characteristic Polynomial v.s. Eigenspace

• Example 2:

\[ B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \]

characteristic polynomials:

\[-(t + 1)(t - 3)^2\]

Eigenvalue -1

Multiplicity of “-1” is 1  
Dim of eigenspace must be 1

Eigenvalue 3

Multiplicity of “3” is 2  
Dim of eigenspace is 1 or 2

\[ \text{Dim} = 2 \]
Characteristic Polynomial v.s. Eigenspace

• Example 3:

\[ C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \]

characteristic polynomials:

\[-(t + 1)(t - 3)^2\]

Eigenvalue -1

Multiplicity of “-1” is 1

Dim of eigenspace must be 1

Eigenvalue 3

Multiplicity of “3” is 2

Dim of eigenspace is 1 or 2

Dim = 1
<table>
<thead>
<tr>
<th></th>
<th>Characteristic polynomial</th>
<th>Eigenvalues</th>
<th>Eigenspaces</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A</strong></td>
<td>$-(t + 1)^2(t - 3)$</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>$-(t + 1)(t - 3)^2$</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td><strong>C</strong></td>
<td>$-(t + 1)(t - 3)^2$</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>
Summary

• Characteristic polynomial of $A$ is

$$\text{det}(A - tI_n) = (t - \lambda_1)^{m_1}(t - \lambda_2)^{m_2} \ldots (t - \lambda_k)^{m_k}$$

**Factorization**  \hspace{1cm} **multiplicity**

- **Eigenvalue:** $\lambda_1, \lambda_2, \ldots, \lambda_k$
- **Eigenspace:** $d_1, d_2, \ldots, d_k$
  - (dimension) $\leq m_1, \leq m_2, \ldots, \leq m_k$
Homework