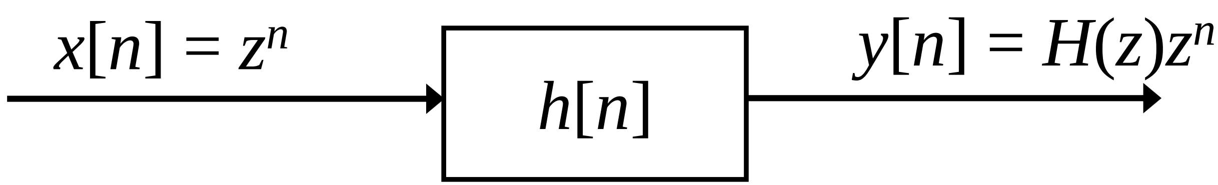
10.0 Z-Transform

10.1 General Principles of Z-Transform

Z-Transform

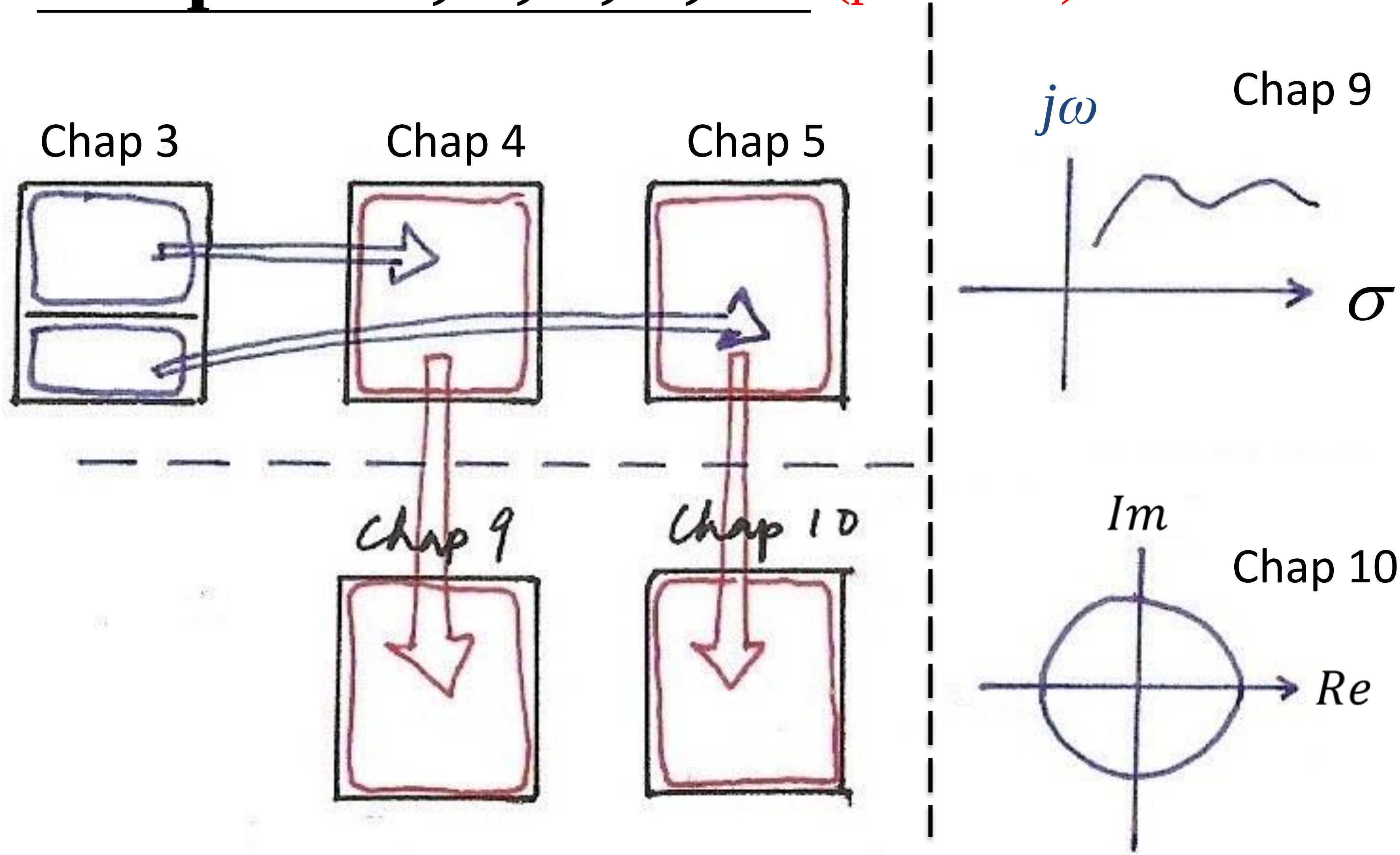
- Eigenfunction Property



linear, time-invariant

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

Chapters 3, 4, 5, 9, 10 (p.2 of 9.0)



Z-Transform

- Eigenfunction Property
  - applies for all complex variables z

$$z = e^{j\omega} \qquad z^n = e^{j\omega n}$$

$$H\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

Fourier Transform

$$z = re^{j\omega}$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

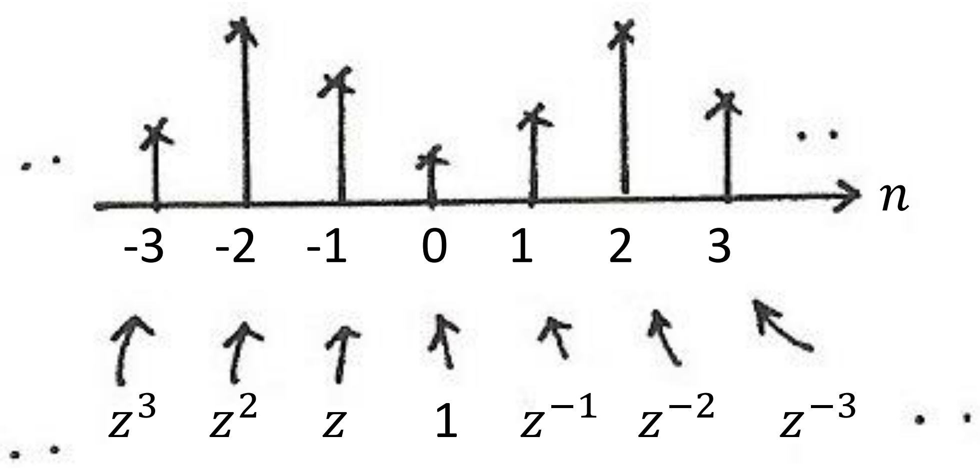
Z-Transform

- Z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$x[n] \overset{Z}{\longleftrightarrow} X(z)$$

Z-Transform



$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$x[n] \overset{Z}{\longleftrightarrow} X(z)$$

Laplace Transform (p.4 of 9.0)

- Laplace Transform

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt, \; s = \sigma + j\omega$$

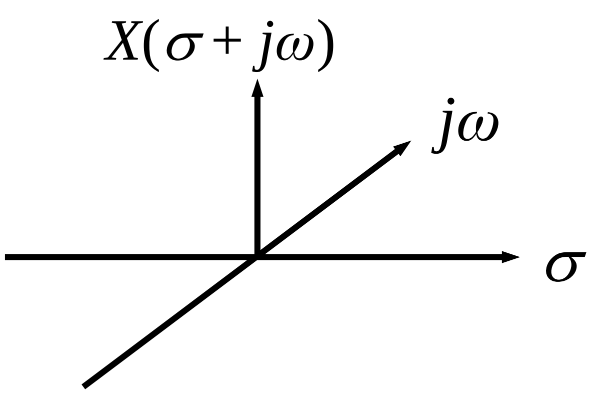
$$x(t) \overset{L}{\longleftrightarrow} X(s)$$

- A Generalization of Fourier Transform

from $s = j\omega$ to $s = \sigma + j\omega$

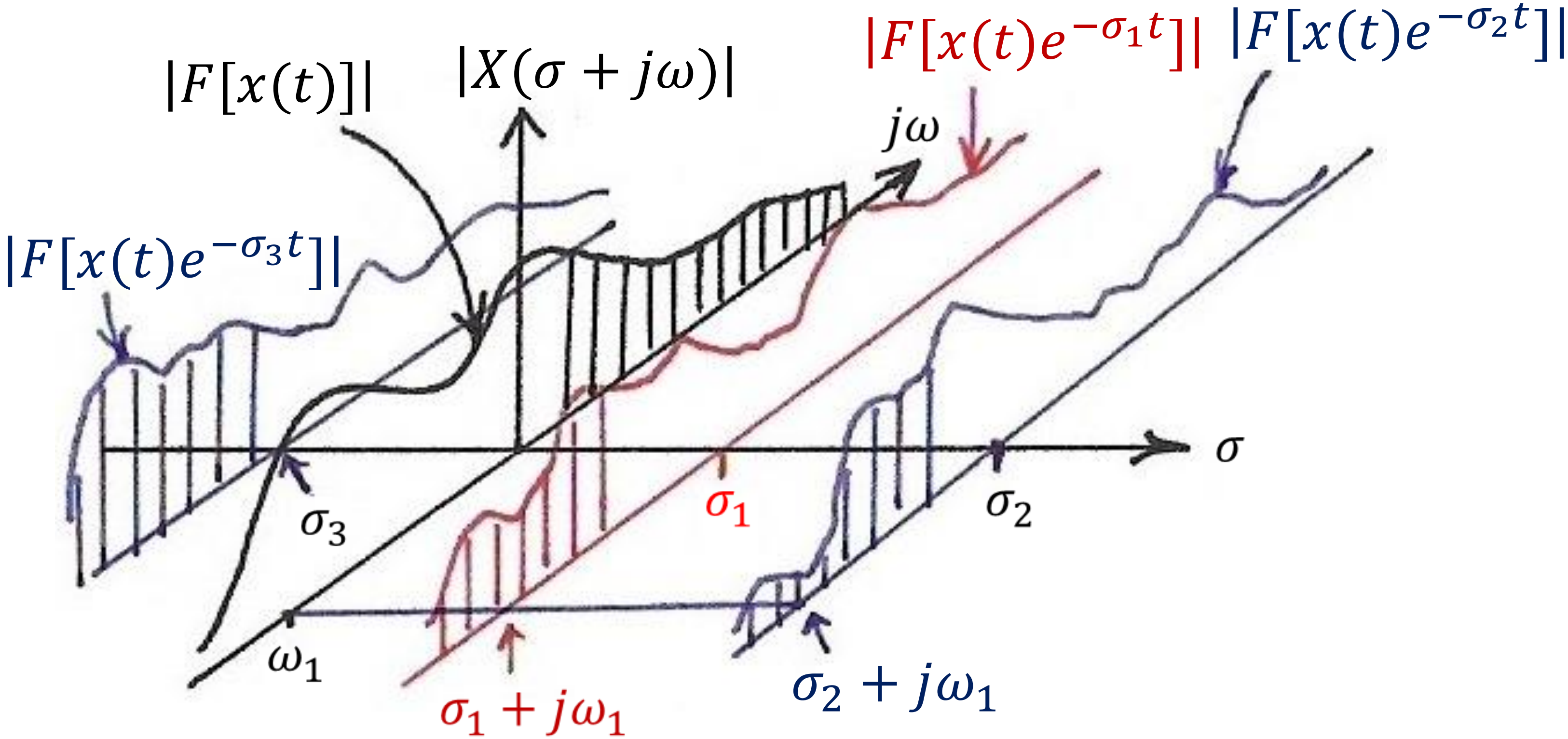
$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt$$

$$= \int_{-\infty}^{\infty} \left[x(t) \mathrm{e}^{-\sigma t} \right] e^{-j\omega t} dt$$



Fourier transform of $x(t) e^{-\sigma t}$

Laplace Transform (p.5 of 9.0)



$(e^{j\omega_1 t}) \perp (e^{j\omega_2 t})$ orthogonal

$\left(e^{(\sigma_1 + j\omega_1)t}\right) \not\perp \left(e^{(\sigma_2 + j\omega_1)t}\right)$ Not orthogonal

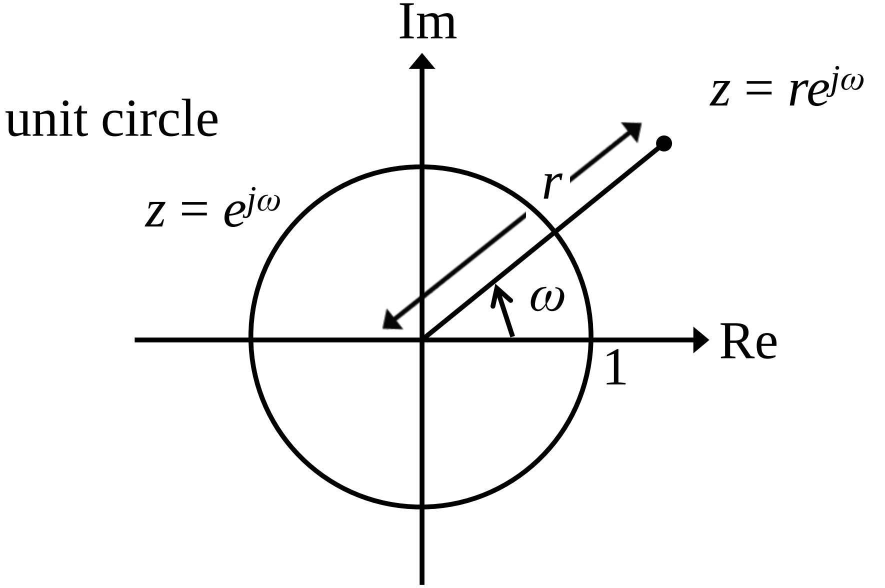
Z-Transform

- A Generalization of Fourier Transform

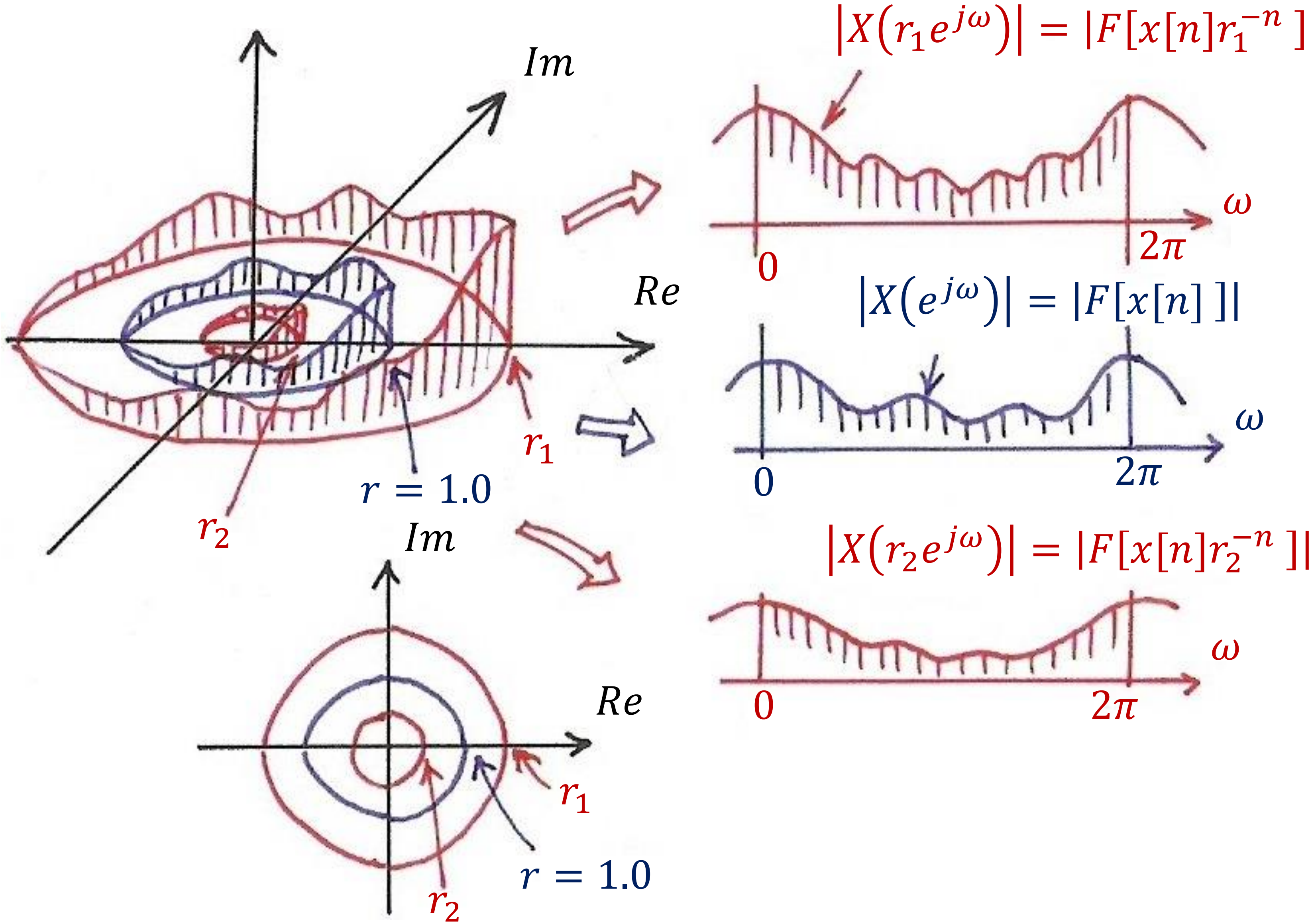
from $z = e^{j\omega}$ to $z = re^{j\omega}$

$$X\left(re^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\left(re^{j\omega}\right)^{-n} = \sum_{n=-\infty}^{\infty} \left\{ x[n] r^{-n} \right\} e^{-j\omega n}$$

Fourier Transform of $x[n] r^{-n}$

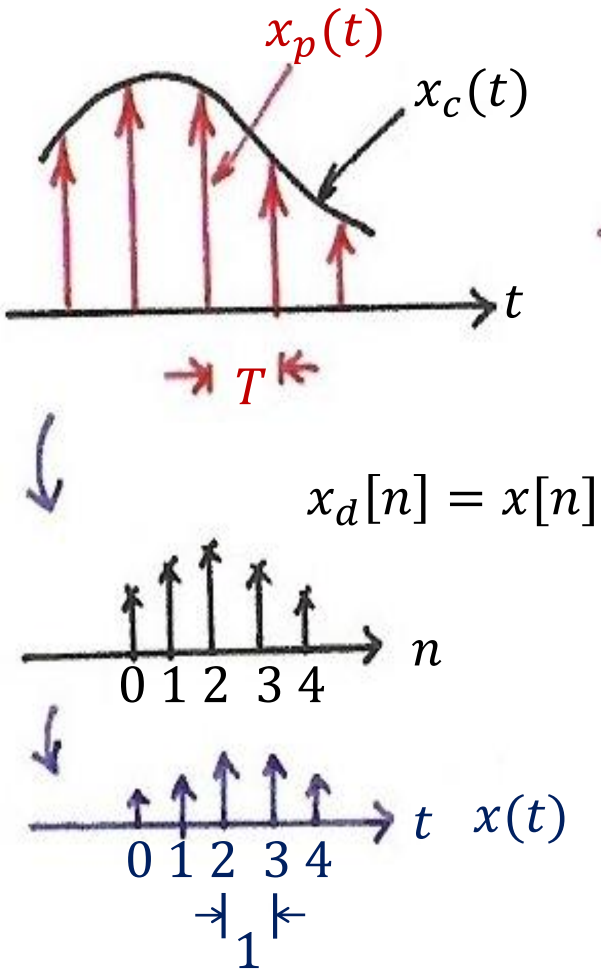


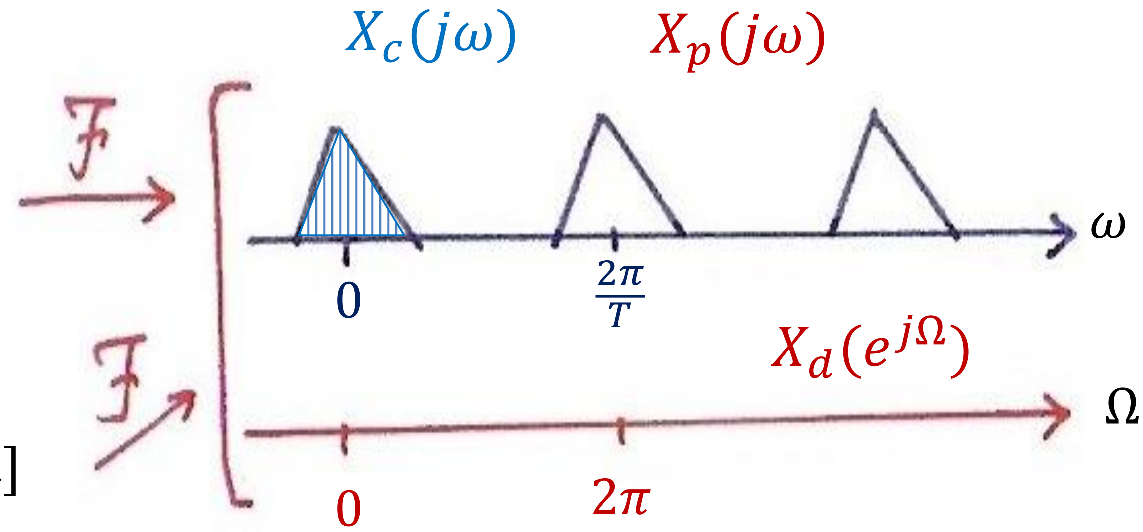
Z-Transform



Im
Re
r₁
r₂
r = 1.0
Im
Re
r₁
r₂
r = 1.0
|X(r₁e^{jω})| = |F[x[n]r₁^{−n}]|
ω
0
2π
|X(e^{jω})| = |F[x[n]]|
ω
0
2π
|X(r₂e^{jω})| = |F[x[n]r₂^{−n}]|
ω
0
2π

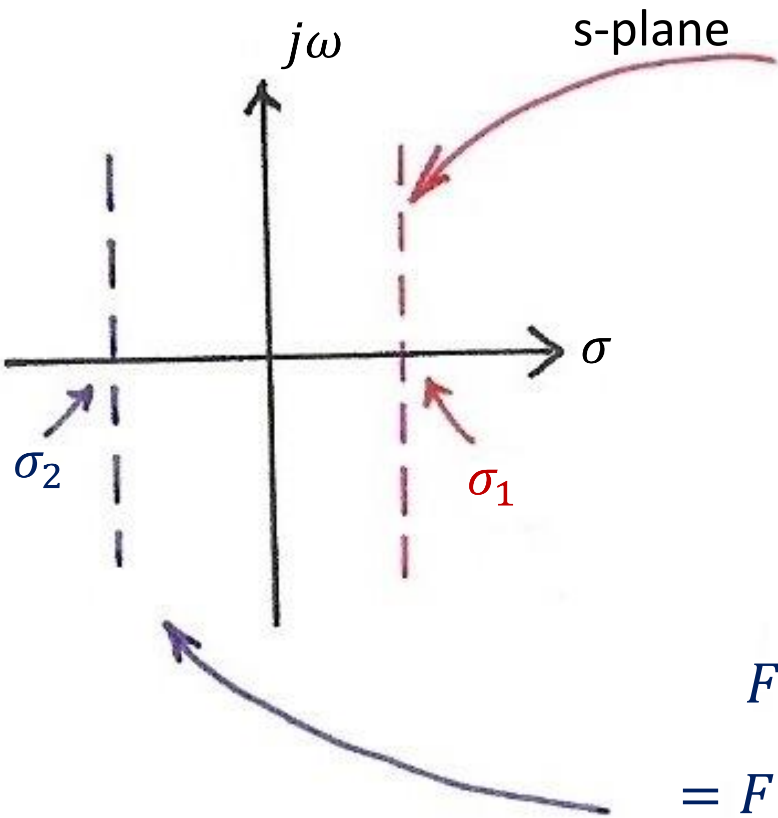
Z-Transform





$$x[n] \rightarrow x(t) = \sum_{n=-\infty}^{\infty} x[n]\ \delta(t-n)$$

Z-Transform



$$L\{x(t)\}\Big|_{\sigma=\sigma_1}$$

$$= F\{x(t)e^{-\sigma_1 t}\}$$

$$= F\{(\sum_n x[n]\,\delta(t-n))\cdot e^{-\sigma_1 t}\}$$

$$= F\{\sum_n x[n]\,\boxed{e^{-\sigma_1 n}}\,\delta(t-n)\}$$

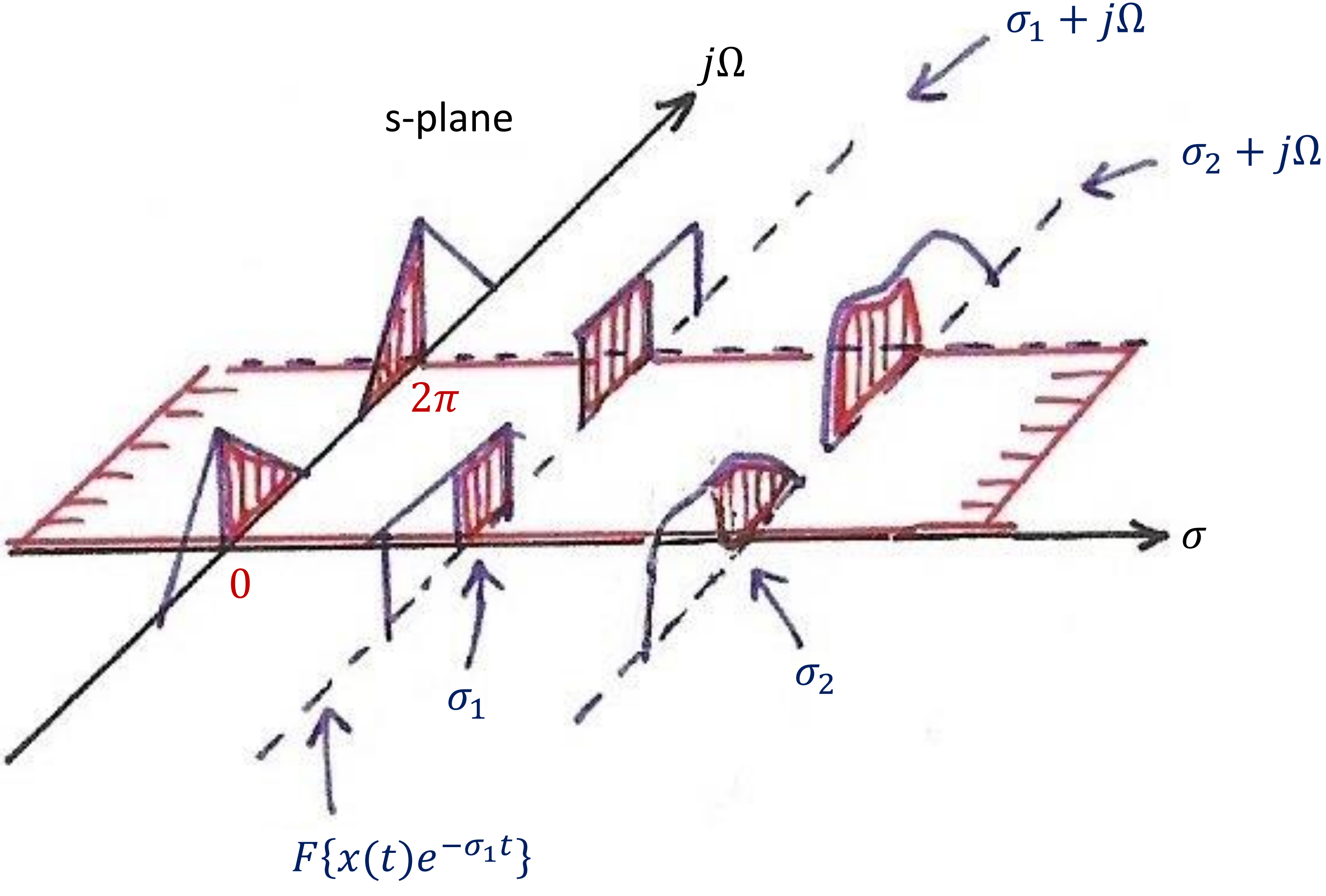
$$= r_1^{-n},\ r_1 = e^{\sigma_1}$$

$$F\{x(t)e^{-\sigma_2 t}\}$$

$$= F\{\sum_n x[n]\,\boxed{e^{-\sigma_2 n}}\,\delta(t-n)\}$$

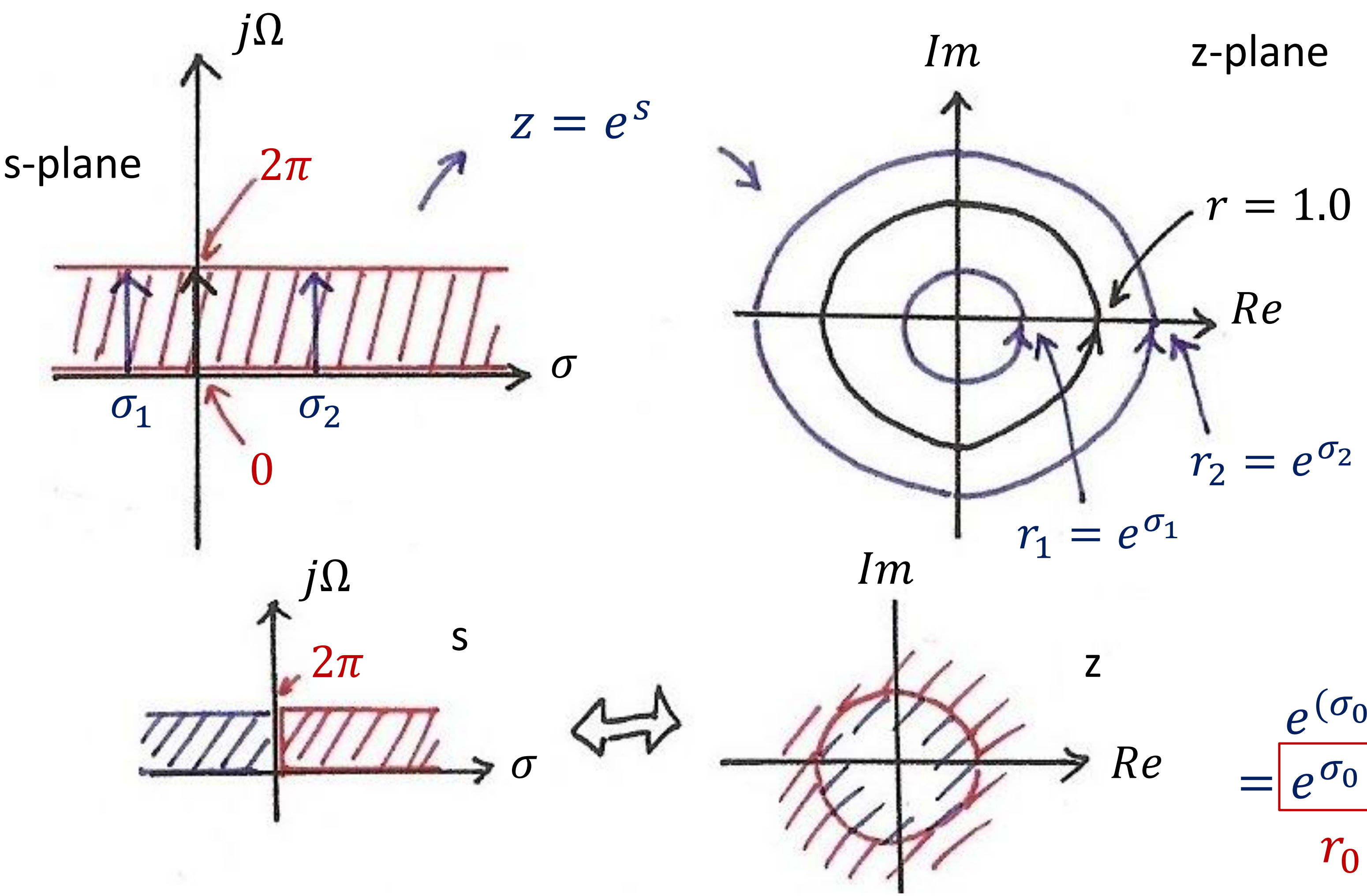
$$= r_2^{-n},\ r_2 = e^{\sigma_2}$$

Z-Transform



$\sigma_1 + j\Omega$
$j\Omega$
s-plane
$\sigma_2 + j\Omega$
2π
σ
0
σ_1
σ_2
$F\{x(t)e^{-\sigma_1 t}\}$

Z-Transform



$$e^{(\sigma_0 + j\omega)} = e^{\sigma_0} \cdot e^{j\omega}$$

r_0 Ω

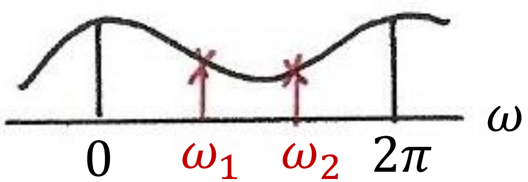
Z-Transform

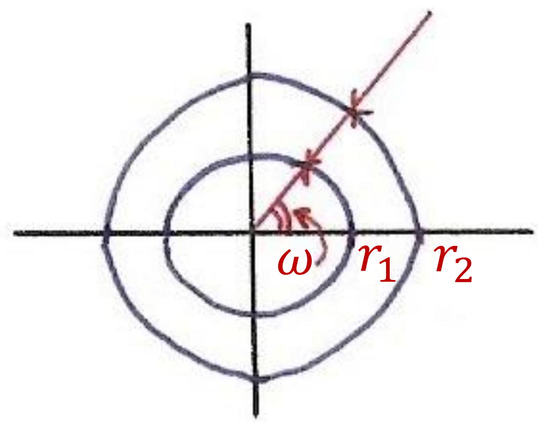
$$L[x(t)]$$

$$= \int_{-\infty}^{\infty} [\sum_n x[n]\,\delta(t-n)]\, e^{-st} dt$$

$$= \sum_n x[n] \boxed{e^{-s}}^{n} \quad = \sum_n x[n] z^{-n} \quad = Z[x[n]]$$

$= z^{-1}$





$$\left(e^{j\omega_1 n}\right) \perp \left(e^{j\omega_2 n}\right)$$

orthogonal

$$(r_1 e^{j\omega})^n \not\perp (r_2 e^{j\omega})^n$$

Not orthogonal

Z-Transform

- A Generalization of Fourier Transform

  $$X(z)\Big|_{z=e^{j\omega}} = X\left(e^{j\omega}\right)$$ reduces to Fourier Transform

  – $X(z)$ may not be well defined (or converged) for all z

  – $X(z)$ may converge at some region of z-plane, while $x[n]$ doesn't have Fourier Transform

  – covering broader class of signals, performing more analysis for signals/systems

Z-Transform

- Rational Expressions and Poles/Zeros

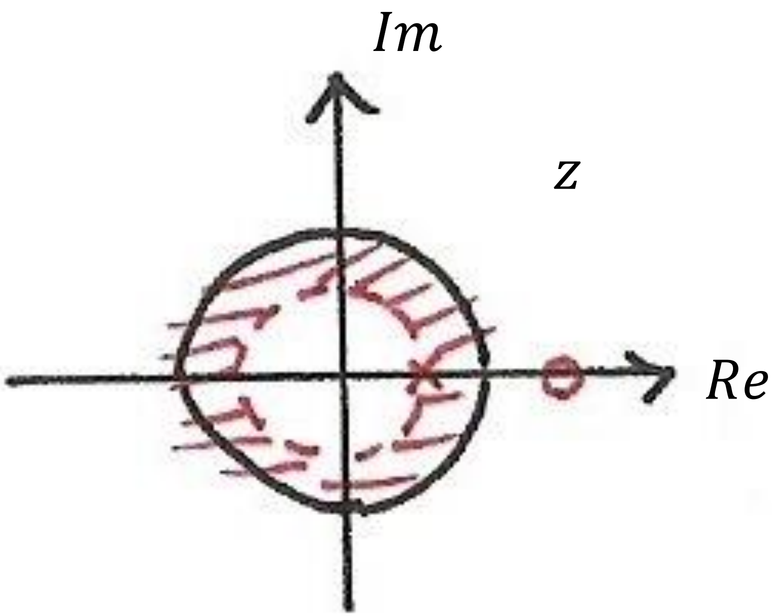
$$X(z) = \frac{N(z)}{D(z)} \begin{array}{l} \longrightarrow \text{roots} \longrightarrow \text{zeros} \\ \longrightarrow \text{roots} \longrightarrow \text{poles} \end{array}$$

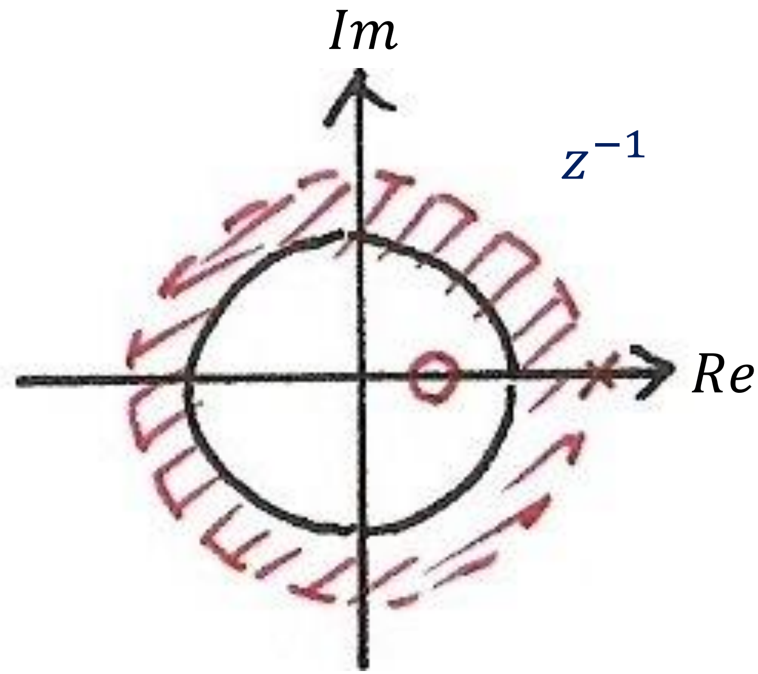
In terms of z, not z^{-1}

– Pole-Zero Plots

specifying $X(z)$ except for a scale factor

Z-Transform





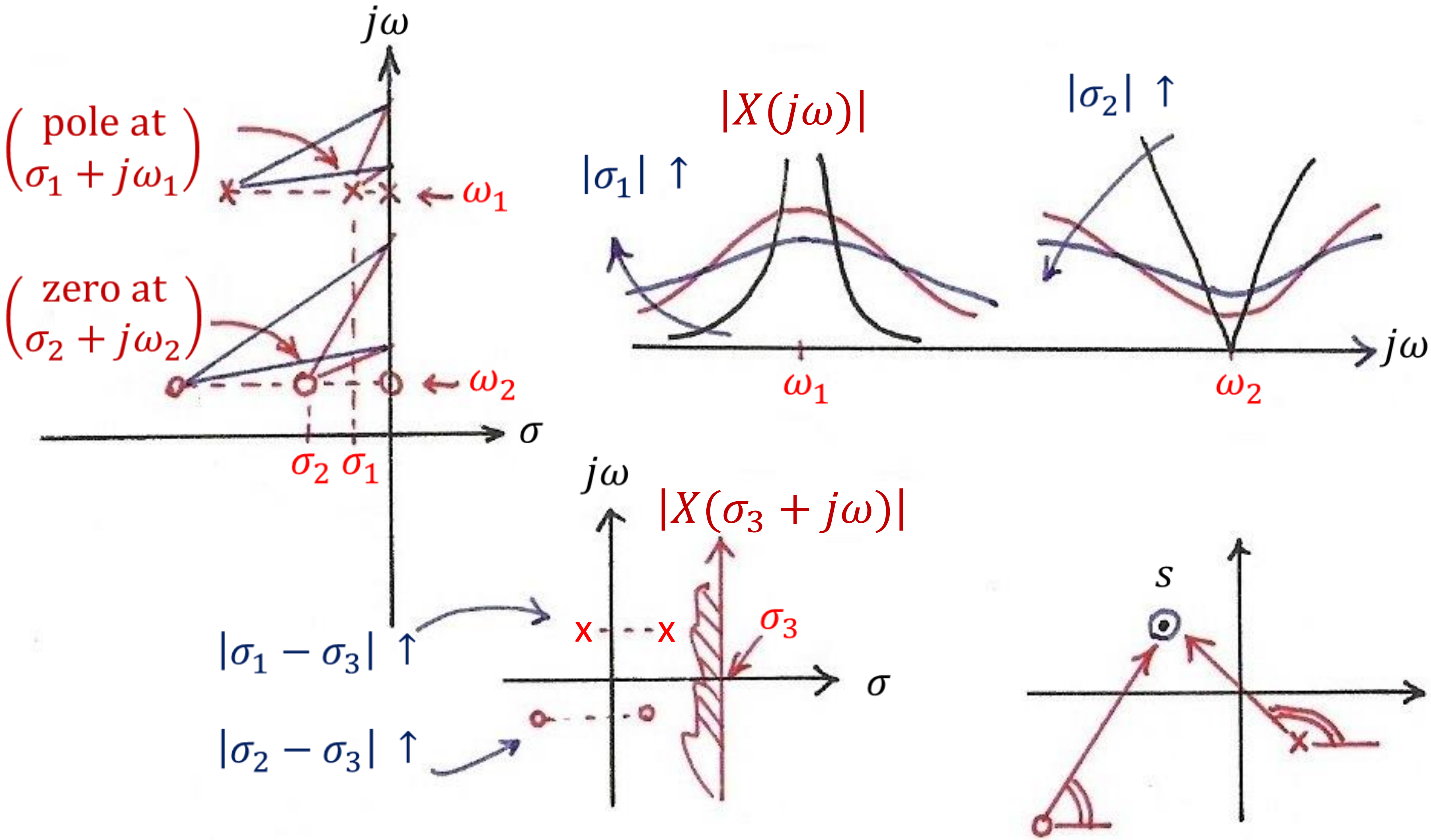
Z-Transform

- Rational Expressions and Poles/Zeros
  - Geometric evaluation of Fourier/Z-Transform from pole-zero plots

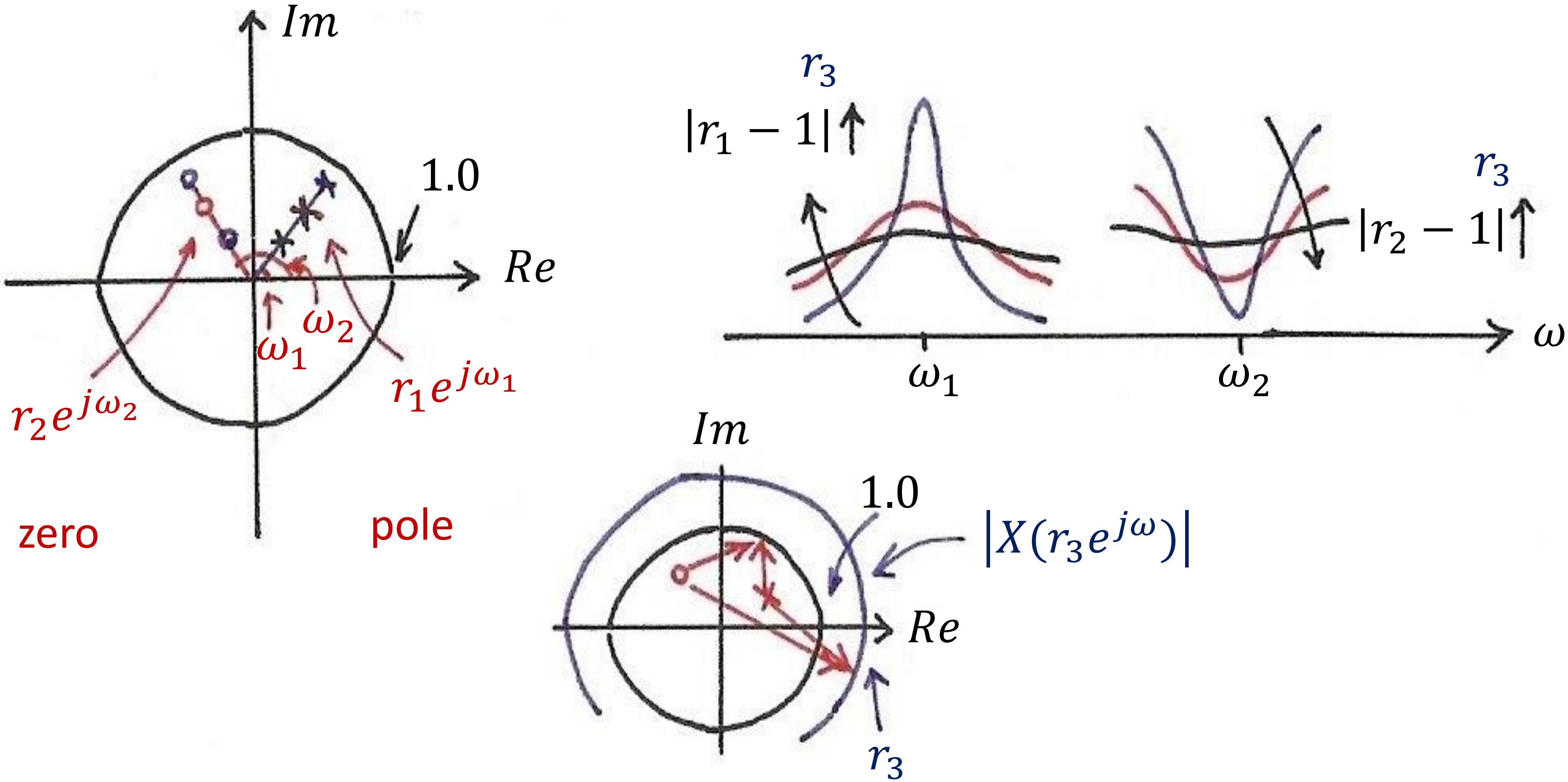
$$X(z) = M \frac{\Pi_i (z - \beta_i)}{\Pi_j (z - \alpha_j)}$$

each term (z-β_i) or (z-α_j) represented by a vector with magnitude/phase

Poles & Zeros (p.9 of 9.0)

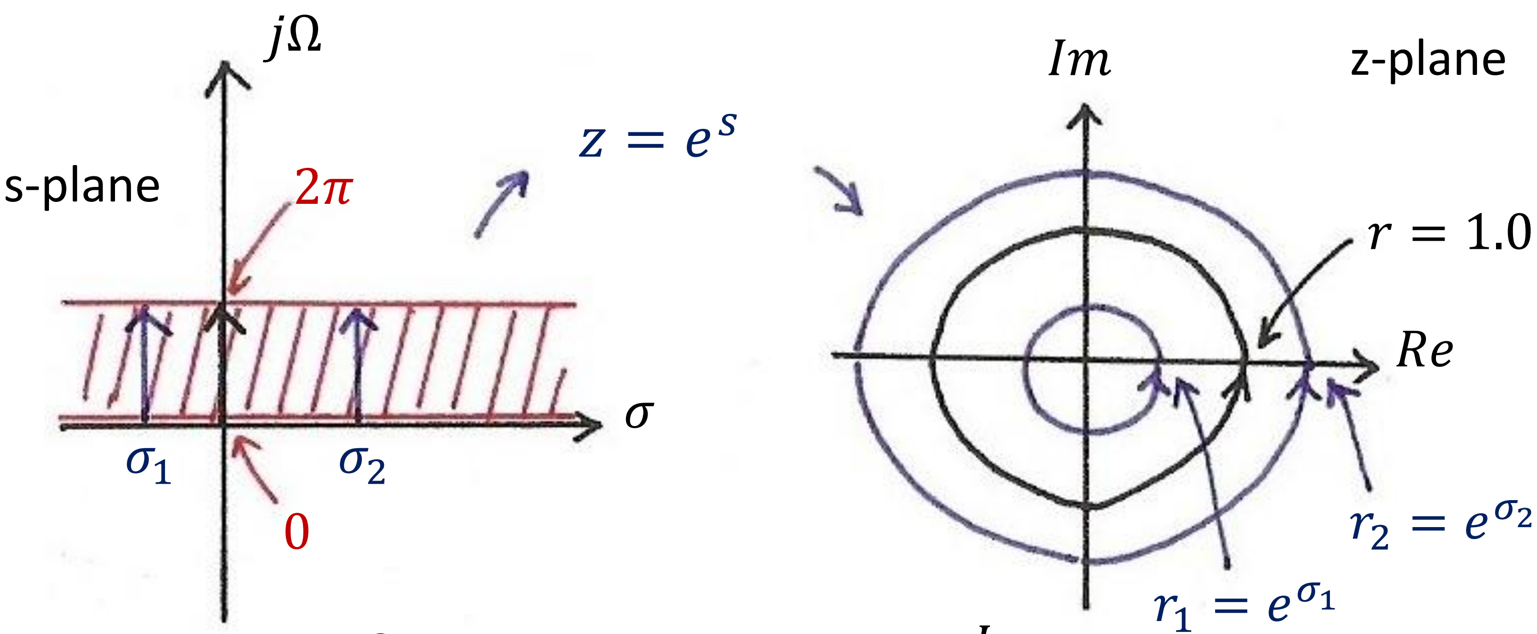


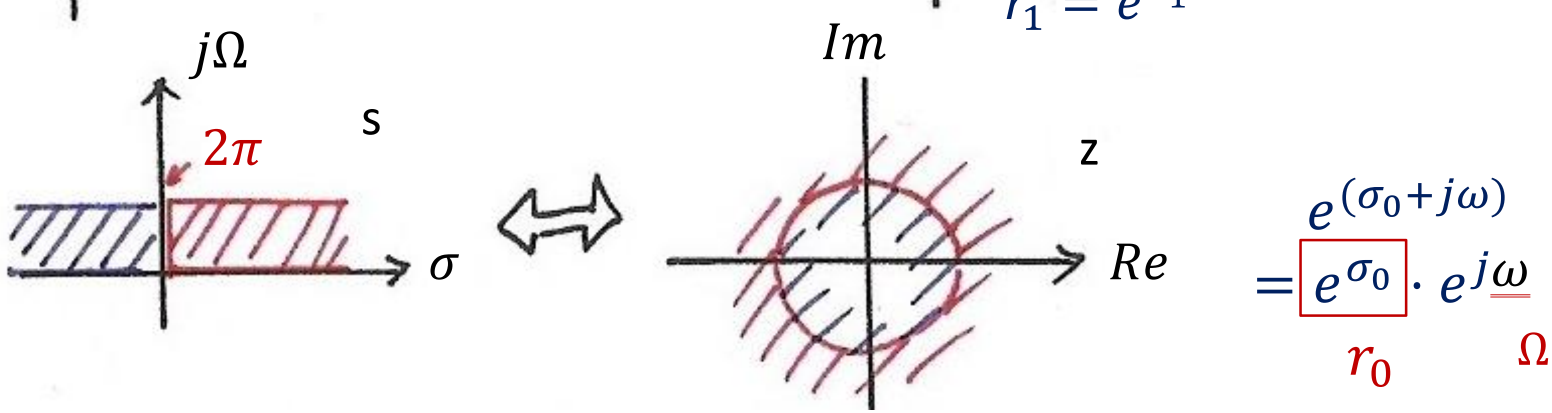
Poles & Zeros



Im
Re
1.0
r₁e^{jω₁}
r₂e^{jω₂}
ω₁
ω₂
zero
pole
r₃
|r₁ − 1|↑
|r₂ − 1|↑
r₃
ω
ω₁
ω₂
Im
Re
1.0
|X(r₃e^{jω})|
r₃

Z-Transform (p.12 of 10.0)





Z-Transform

- Rational Expressions and Poles/Zeros
  - Geometric evaluation of Fourier/Z-transform from pole-zero plots
    - Example : 1st-order

$$h[n] = a^n u[n]$$

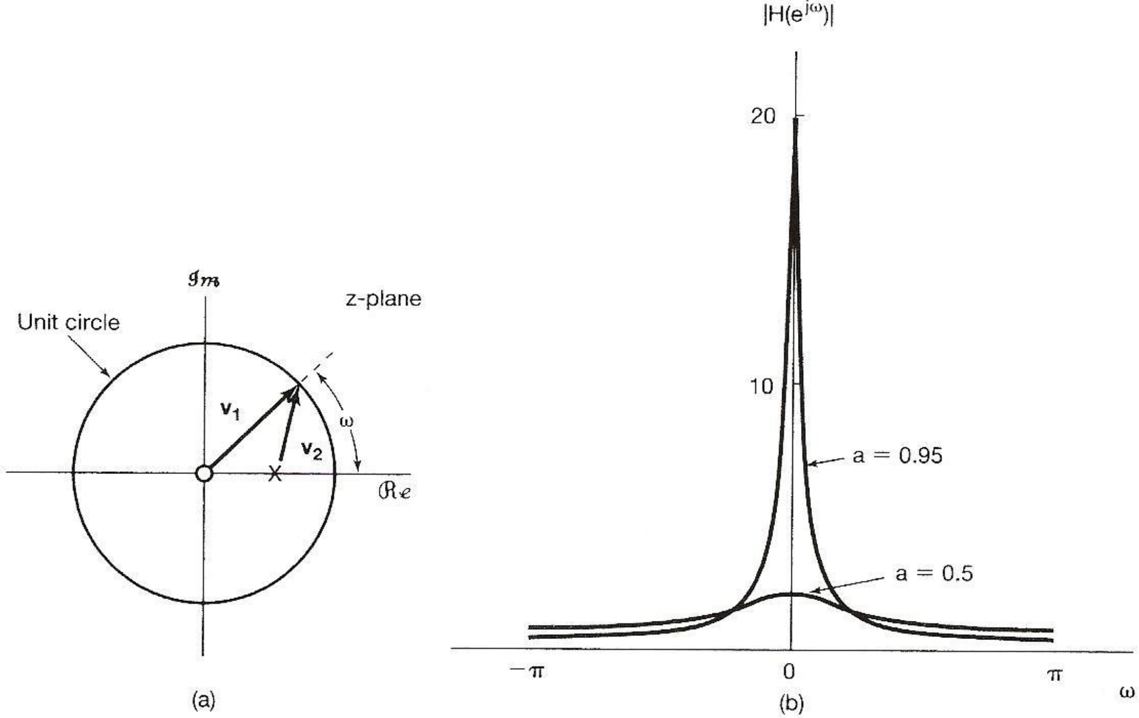
$$H[z] = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad |z| > |a|$$

$$\text{pole} : z = a, \ \text{zero} : z = 0$$

See Example 10.1, p.743~744 of text

$$H\left(e^{j\omega}\right) = \frac{1}{1 - ae^{-j\omega}}$$

See Fig. 10.13, p.764 of text



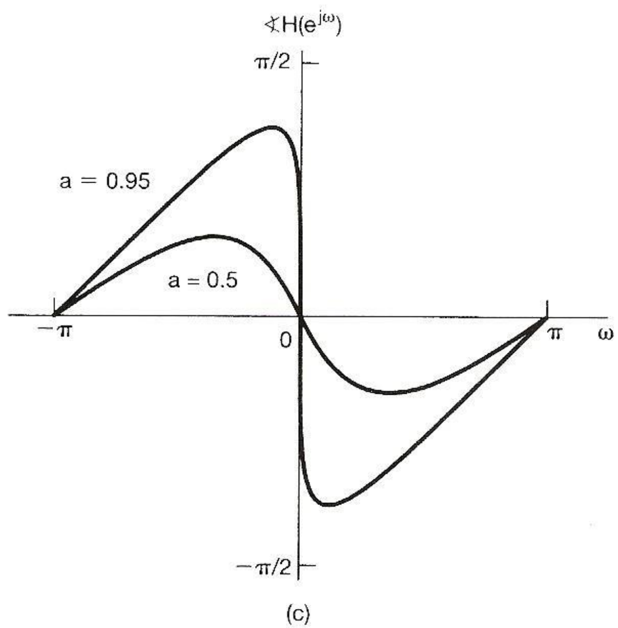


Figure 10.13 (a) Pole and zero vectors for the geometric determination of the frequency response for a first-order system for a value of a between 0 and 1; (b) magnitude of the frequency response for $a = 0.95$ and $a = 0.5$; (c) phase of the frequency response for $a = 0.95$ and $a = 0.5$.

Z-Transform

- Rational Expressions and Poles/Zeros
  - Geometric evaluation of Fourier/Z-transform from pole-zero plots
    - Example : 2nd-order

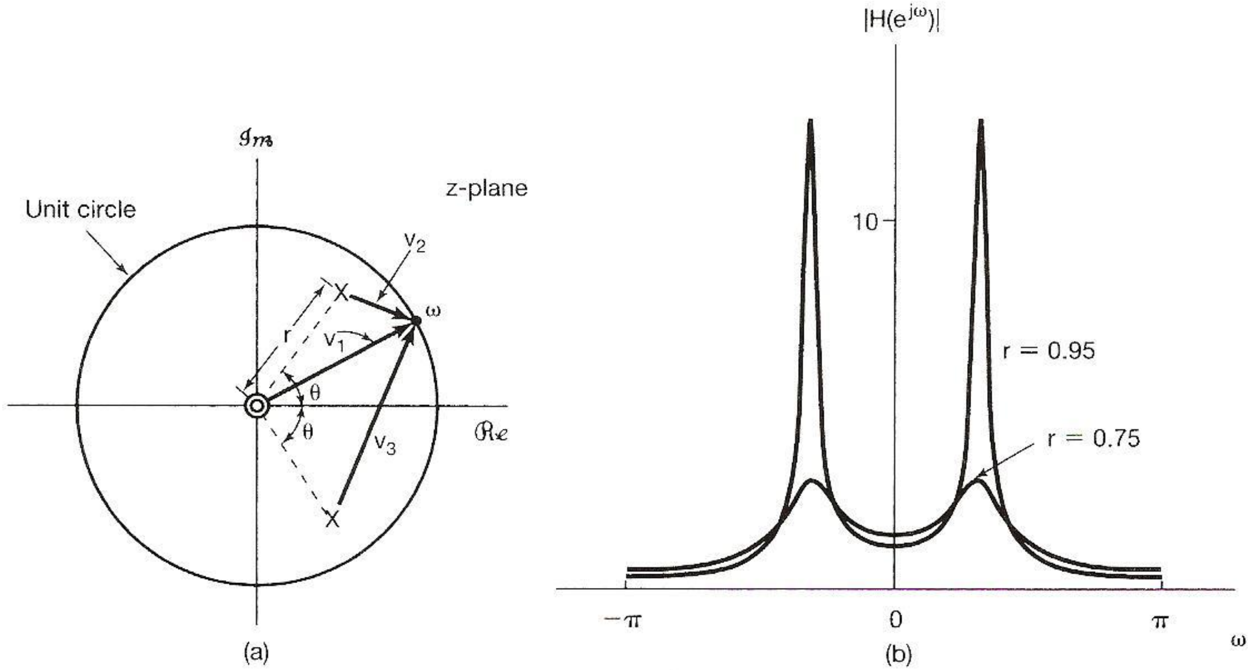
$$h[n] = r^n \frac{\sin(n+1)\theta}{\sin\theta} u[n]$$

$$H(z) = \frac{1}{1-(2r\cos\theta)z^{-1} + r^2 z^{-2}}$$

$$\text{pole}: z_1 = re^{j\theta},\ \ z_2 = re^{-j\theta}$$

$$\text{double zero}: z = 0$$

See Fig. 10.14, p.766 of text



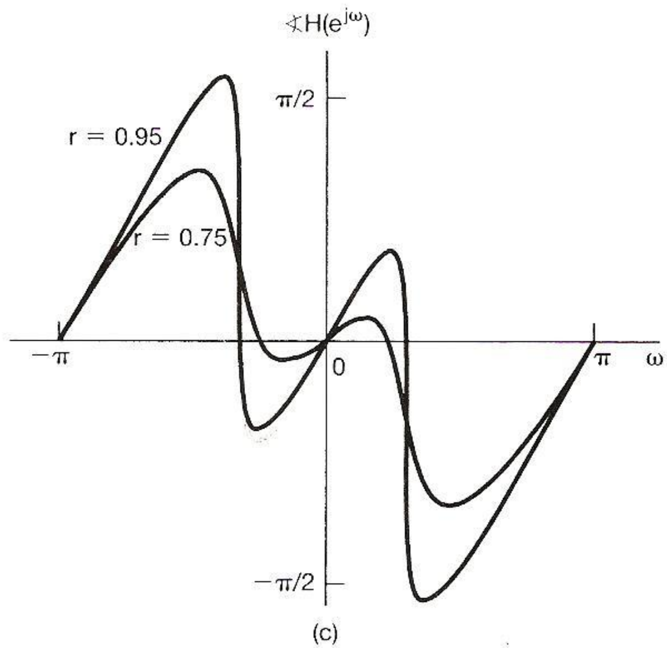


Figure 10.14 (a) Zero vector $\mathbf{v}_1$ and pole vectors $\mathbf{v}_2$ and $\mathbf{v}_3$ used in the geometric calculation of the frequency responses for a second-order system; (b) magnitude of the frequency response corresponding to the reciprocal of the product of the lengths of the pole vectors for $r = 0.95$ and $r = 0.75$; (c) phase of the frequency response for $r = 0.95$ and $r = 0.75$.

Z-Transform

- Rational Expressions and Poles/Zeros
  - Specification of Z-Transform includes the region of convergence (ROC)
    - Example :

$$x_1[n] = a^n u[n]$$

$$X_1(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \ |z| > |a|$$

$$x_2[n] = -a^n u[-n-1]$$

$$X_2(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \ |z| < |a|$$

pole : $z = a$, zero : $z = 0$ in both cases

See Example 10.1, 10.2, p.743~745 of text

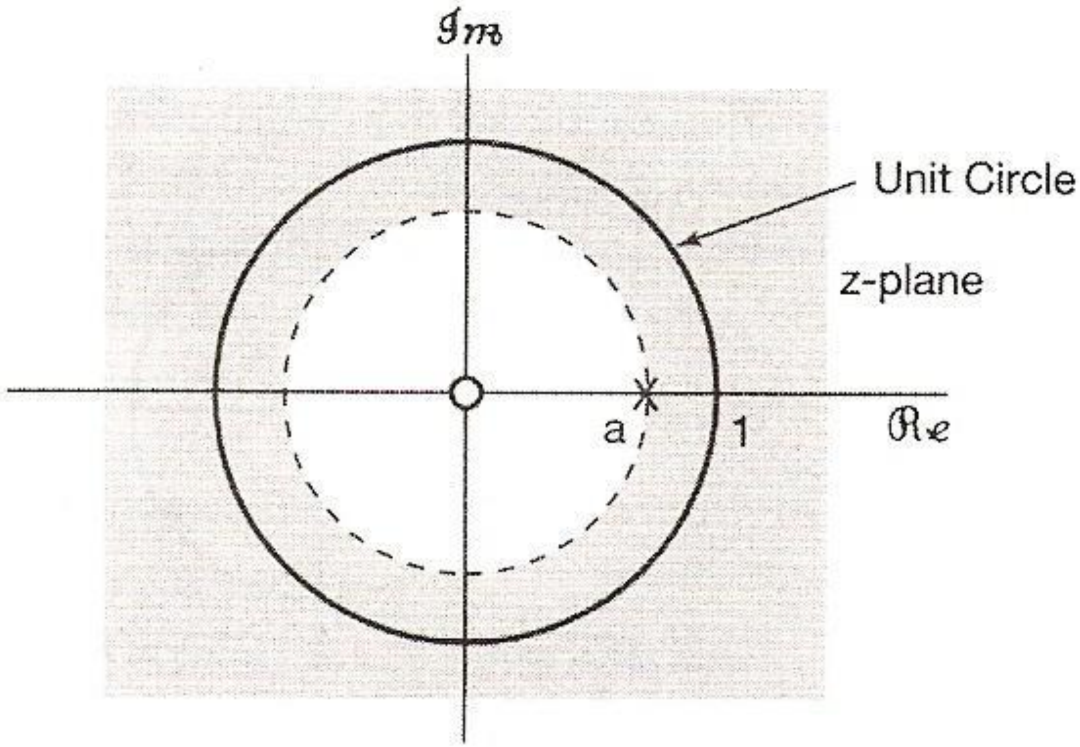


Figure 10.2 Pole-zero plot and region of convergence for Example 10.1 for $0 < a < 1$.

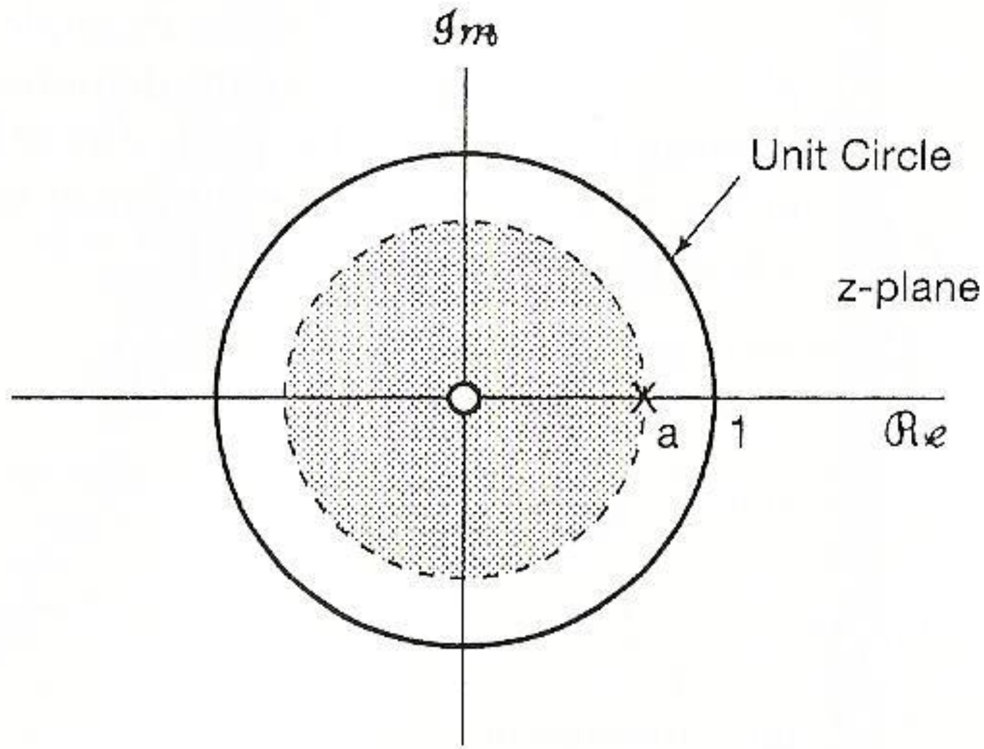


Figure 10.3 Pole-zero plot and region of convergence for Example 10.2 for $0 < a < 1$.

Z-Transform

- Rational Expressions and Poles/Zeros
  - $x[n] = 0, \; n < 0$

    $X(z)$ involes only negative powers of z initially

    $x[n] = 0, \; n > 0$

    $X(z)$ involes only positive powers of z initially

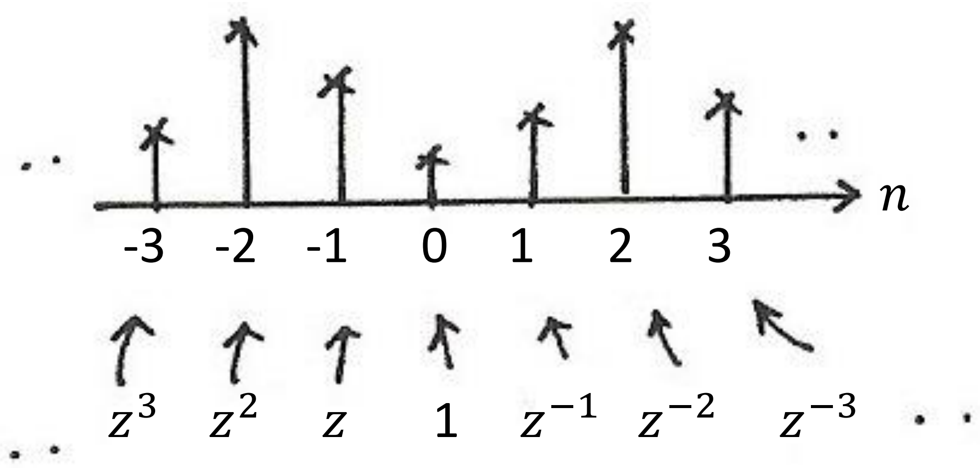
  - poles at infinity if

    degree of $N(z)$ > degree of $D(z)$

    zeros at infinity if

    degree of $D(z)$ > degree of $N(z)$

Z-Transform (p.4 of 10.0)



$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$x[n] \overset{Z}{\longleftrightarrow} X(z)$$

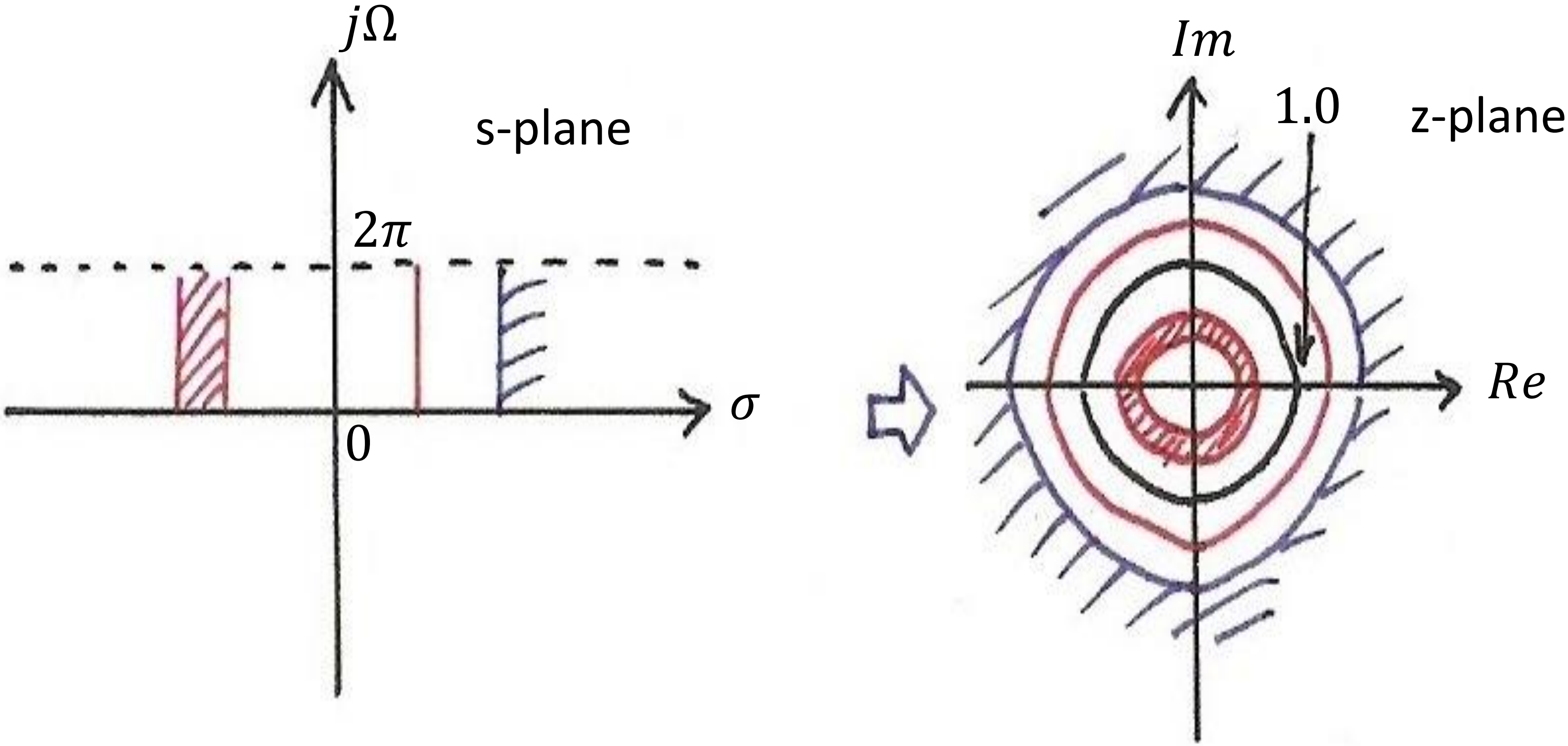
Region of Convergence (ROC)

- Property 1 : The ROC of $X(z)$ consists of a ring in the z-plane centered at the origin
  - for the Fourier Transform of $x[n]r^{-n}$ to converge

    $$\sum_{n=-\infty}^{\infty} |x[n]| r^{-n} < \infty,$$ depending on r only, not on ω
  - the inner boundary may extend to include the origin, and the outer boundary may extend to infinity

- Property 2 : The ROC of $X(z)$ doesn't include any poles

Property 1



Region of Convergence (ROC)

- Property 3 : If $x[n]$ is of finite duration, the ROC is the entire z-plane, except possibly for $z = 0$ and/or $z = \infty$

$$x[n] = 0,\ n < N_1,\ n > N_2$$

$$X(z) = \sum_{n=N_1}^{N_2} x[n] z^{-n}$$

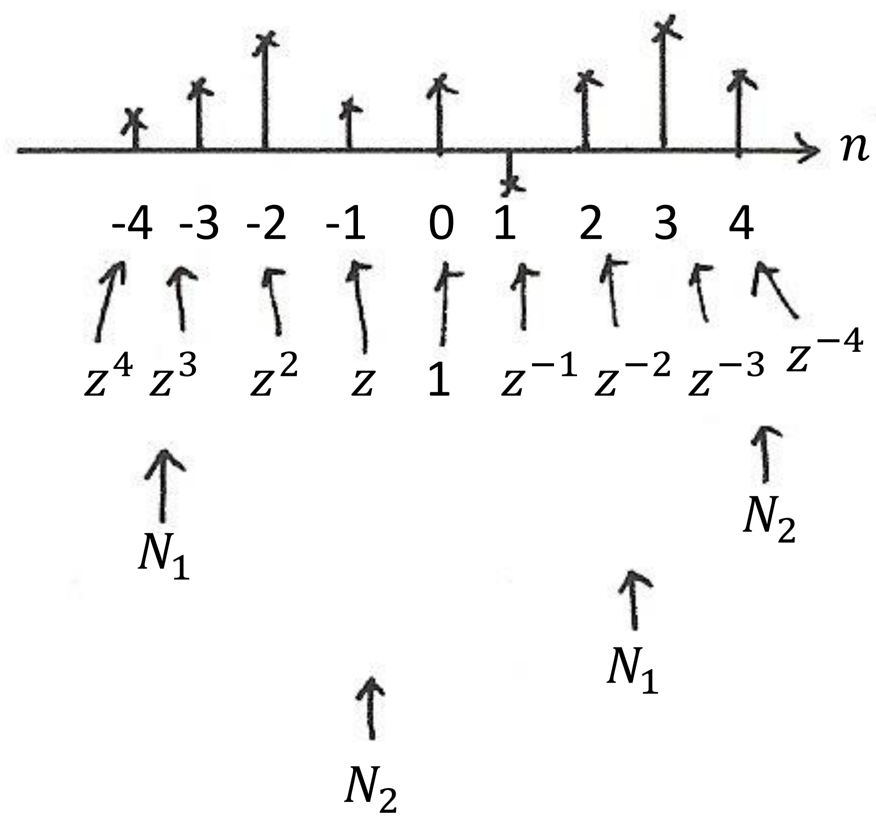
– If $N_1 < 0,\ N_2 > 0$

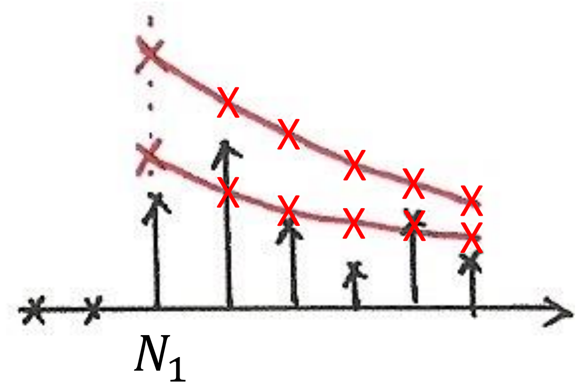
$$z = 0 \notin \text{ROC},\ z = \infty \notin \text{ROC}$$

If $N_1 \geq 0,\ z = \infty \in \text{ROC}$

If $N_2 \leq 0,\ z = 0 \in \text{ROC}$

Property 3, 4, 5





$$\sum_n |x[n]|\, r_1^{-n} \qquad r_1 > r_0$$

$$= \left(\frac{r_1}{r_0}\right)^{-n} \left\{ \sum_n |x[n]|\, r_0^{-n} \right\}$$

$$< \left(\frac{r_1}{r_0}\right)^{-N_1} \cdot \left\{ \sum_n |x[n]|\, r_0^{-n} \right\}$$

$\frac{r_1}{r_0} > 1$ $\sum_n |x[n]|\, r_0^{-n} < \infty$

Region of Convergence (ROC)

- Property 4 : If $x[n]$ is right-sided, and $\{z \mid |z| = r_0\} \subset$ROC, then $\{z \mid \infty > |z| > r_0\} \subset$ROC

$$\sum_{n=N_1}^{\infty} |x[n]|\, r_0^{-n} < \infty$$

$$\text{If } N_1 < 0,\ z = \infty \notin \text{ROC}$$

$$\text{If } N_1 \geq 0,\ z = \infty \in \text{ROC}$$

Region of Convergence (ROC)

- Property 5 : If $x[n]$ is left-sided and $\{z \mid |z| = r_0\} \subset \text{ROC}$, then $\{z \mid 0 < |z| < r_0\} \subset \text{ROC}$

$$\sum_{n=-\infty}^{N_2} x[n] r_0^{-n} < \infty$$

If $N_2 > 0, \; z = 0 \notin \text{ROC}$

If $N_2 \le 0, \; z = 0 \in \text{ROC}$

Region of Convergence (ROC)

- Property 6 : If $x[n]$ is two-sided, and $\{z \mid |z| = r_0\} \in$ ROC, then ROC consists of a ring that includes $\{z \mid |z| = r_0\}$

$$x[n] = x_R[n] + x_L[n]$$

  – a two-sided $x[n]$ may not have ROC

Region of Convergence (ROC)

- Property 7 : If $X(z)$ is rational, then ROC is bounded by poles or extends to zero or infinity
- Property 8 : If $X(z)$ is rational, and $x[n]$ is right-sided, then ROC is the region outside the outermost pole, possibly includes $z = \infty$. If in addition $x[n] = 0$, $n < 0$, ROC also includes $z = \infty$

Region of Convergence (ROC)

- Property 9 : If $X(z)$ is rational, and $x[n]$ is left-sided, the ROC is the region inside the innermost pole, possibly includes $z = 0$. If in addition $x[n] = 0$, $n > 0$, ROC also includes $z = 0$

See Example 10.8, p.756~757 of text

Fig. 10.12, p.757 of text

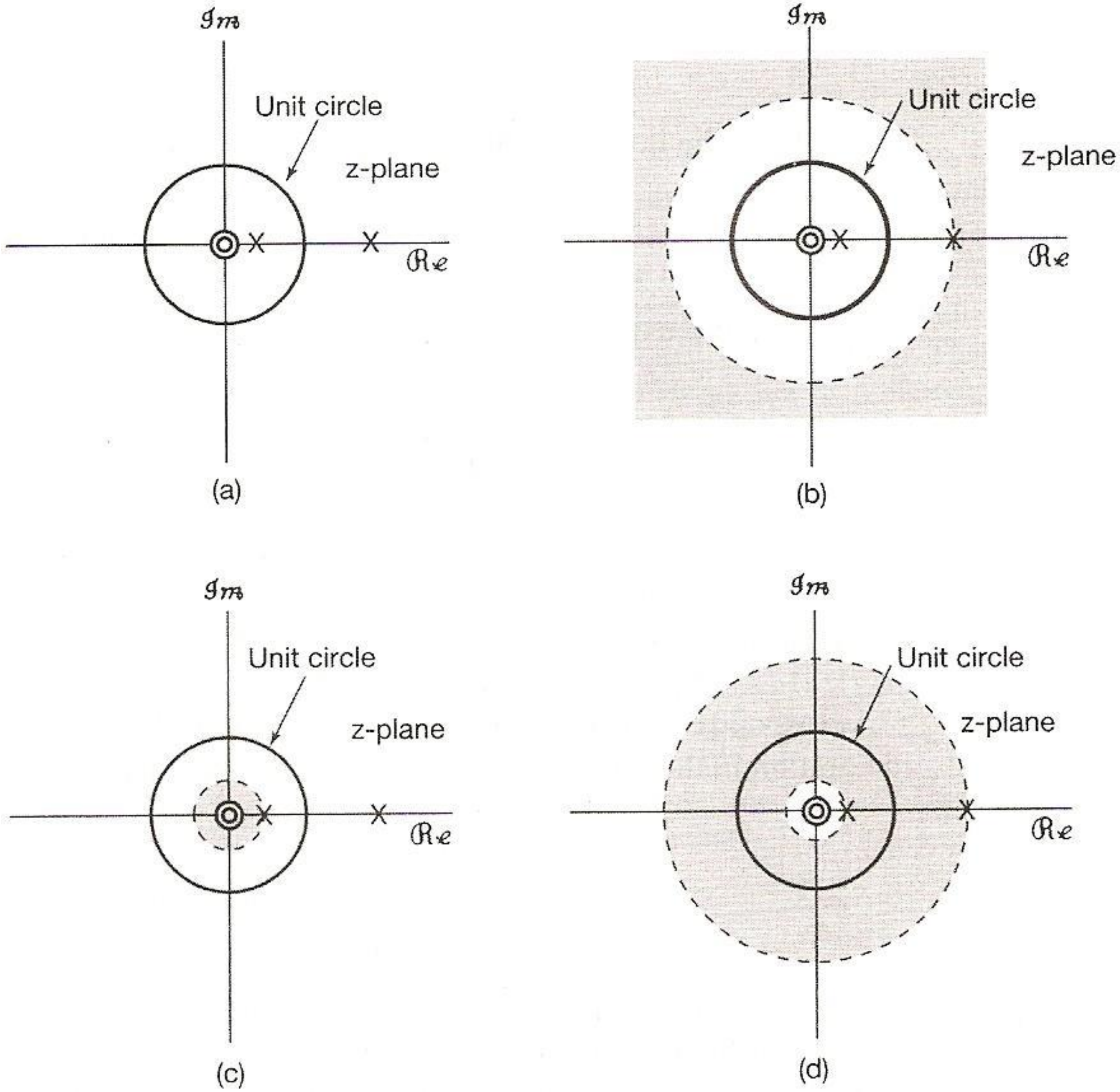


Figure 10.12 The three possible ROCs that can be connected with the expression for the z-transform in Example 10.8: (a) pole-zero pattern for $X(z)$; (b) pole-zero pattern and ROC if $x[n]$ is right sided; (c) pole-zero pattern and ROC if $x[n]$ is left sided; (d) pole-zero pattern and ROC if $x[n]$ is two sided. In each case, the zero at the origin is a second-order zero.

Inverse Z-Transform

$$x[n]r_1^{-n} = F^{-1}\left\{X\left(r_1e^{j\omega}\right)\right\} = \frac{1}{2\pi}\int_{2\pi} X\left(r_1e^{j\omega}\right)e^{j\omega n}d\omega$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X\left(r_1e^{j\omega}\right)\left(r_1e^{j\omega}\right)^n d\omega,$$

$$z = r_1e^{j\omega}$$

$$x[n] = \frac{1}{2\pi j}\oint X(z)\, z^{n-1}dz$$

$$dz = jr_1e^{j\omega}d\omega$$

$$= jzd\omega$$

– integration along a circle counterclockwise, $\{z \mid |z| = r_1\} \in \text{ROC}$, for a fixed r_1

Laplace Transform (p.28 of 9.0)

(ba~~s~~is?)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma_1 + j\omega)\, \boxed{e^{(\sigma_1 + j\omega)t}}\, d\omega$$ (合~~成~~ ?)

$$X(\sigma_1 + j\omega) = \int_{-\infty}^{\infty} x(t)\, \boxed{e^{-(\sigma_1 + j\omega)t}}\, dt$$ (分~~析~~ ?)

$$\neq$$

$\neq \vec{A} \cdot \vec{v}$ (分析) $\left[e^{(\sigma_1 + j\omega)t}\right]^* = e^{\sigma_1 t} \cdot e^{-j\omega t}$

basis

$$x(t)e^{-\sigma_1 t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma_1 + j\omega)\, \boxed{e^{j\omega t}}\, dt$$ (合成)

$$X(\sigma_1 + j\omega) = \int_{-\infty}^{\infty} \left[x(t)e^{-\sigma_1 t}\right] \cdot e^{-j\omega t} dt$$ (分析)

Z-Transform

basis? (X)

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X\left(r_1 e^{j\omega}\right)\left(r_1 e^{j\omega}\right)^{n} d\omega$$

(合成?) (X)

$$X(r_1 e^{j\omega}) = \sum_n x[n]\left(r_1 e^{j\omega}\right)^{-n}$$

(分析?) (X)

$\neq$

$\neq \vec{A}\cdot\vec{v}$ (分析)

$$[(r_1 e^{j\omega})^n]^* = r_1^n \cdot e^{j\omega n}$$

basis

$$x[n] r_1^{-n} = \frac{1}{2\pi}\int_{2\pi} X\left(r_1 e^{j\omega}\right)\left(e^{j\omega n}\right) d\omega$$

(合成)

$$X(r_1 e^{j\omega}) = \sum_n (x[n] r_1^{-n}) e^{-j\omega n}$$

(分析)

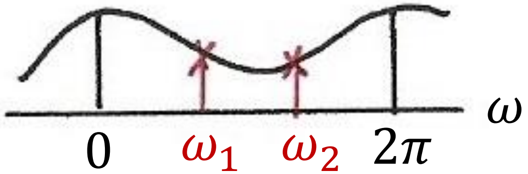
Z-Transform (p.13 of 10.0)

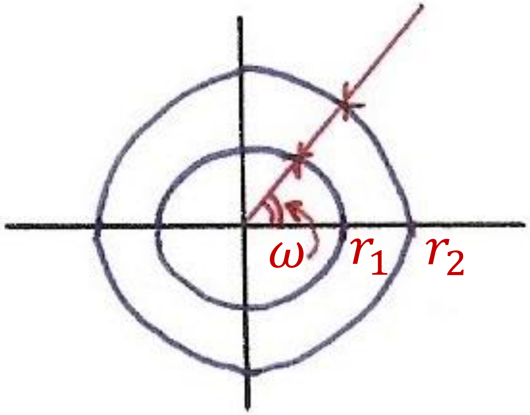
$$L[x(t)]$$

$$= \int_{-\infty}^{\infty} [\sum_n x[n]\,\delta(t-n)]\, e^{-st} dt$$

$$= \sum_n x[n] \boxed{e^{-s}}^{n} = \sum_n x[n] z^{-n} = Z[x[n]]$$

$$\boxed{e^{-s}} = z^{-1}$$





$$(e^{j\omega_1 n}) \perp (e^{j\omega_2 n})$$

orthogonal

$$(r_1 e^{j\omega})^n \not\perp (r_2 e^{j\omega})^n$$

Not orthogonal

Inverse Z-Transform

- Partial-fraction expansion practically useful:

$$X(z) = \sum_{i=1}^{m} \frac{A_i}{1 - a_i z^{-1}}$$

for each term $\dfrac{A_i}{1 - a_i z^{-1}}$

– ROC outside the pole at $z = a_i \rightarrow A_i a_i^{\,n} u[n]$

ROC inside the pole at $z = a_i \rightarrow -A_i a_i^{\,n} u[-n-1]$

Inverse Z-Transform

- Partial-fraction expansion practically useful:
  - Example:

$$X(z)=\frac{3-\frac{5}{6}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}=\frac{1}{\left(1-\frac{1}{4}z^{-1}\right)}+\frac{2}{\left(1-\frac{1}{3}z^{-1}\right)}$$

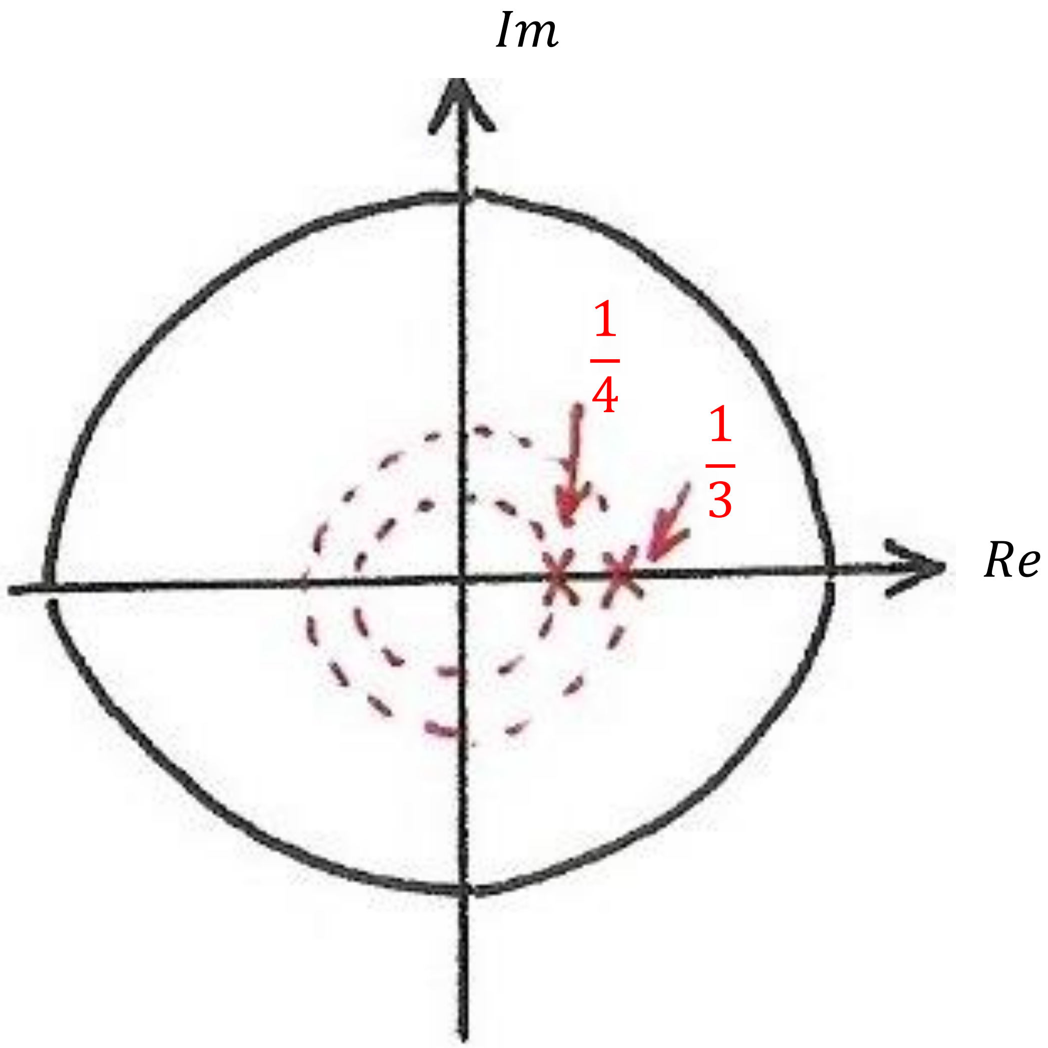
$$\mathrm{ROC}=\left\{z \,\middle|\, |z|>1/3\right\},\quad x[n]=\left(1/4\right)^n u[n]+2\left(1/3\right)^n u[n]$$

$$\mathrm{ROC}=\left\{z \,\middle|\, 1/3>|z|>1/4\right\},\quad x[n]=\left(1/4\right)^n u[n]-2\left(1/3\right)^n u[-n-1]$$

$$\mathrm{ROC}=\left\{z \,\middle|\, |z|<1/4\right\},\quad x[n]=-\left(1/4\right)^n u[-n-1]-2\left(1/3\right)^n u[-n-1]$$

See Example 10.9, 10.10, 10.11, p.758~760 of text

Example



Inverse Z-Transform

- Power-series expansion practically useful:

$$X(z)=\sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

  - right-sided or left-sided based on ROC

Inverse Z-Transform

- Power-series expansion practically useful:
  - Example:

$$X(z) = \frac{1}{1-az^{-1}}$$

$$\text{ROC} = \{z \,|\, |z| > |a|\}, \quad \frac{1}{1-az^{-1}} = 1 + az^{-1} + a^2 z^{-2} + \ldots\ldots$$

$$x[n] = a^n u[n]$$

$$\text{ROC} = \{z \,|\, |z| < |a|\}, \quad \frac{1}{1-az^{-1}} = -a^{-1}z - a^{-2}z^2 - \ldots\ldots$$

$$x[n] = -a^n u[-n-1]$$

See Example 10.12, 10.13, 10.14, p.761~763 of text

- Known pairs/properties practically helpful

10.2 Properties of Z-Transform

$$x[n] \overset{Z}{\longleftrightarrow} X(z), \ \text{ROC} = R$$

$$x_1[n] \overset{Z}{\longleftrightarrow} X_1(z), \ \text{ROC} = R_1$$

$$x_2[n] \overset{Z}{\longleftrightarrow} X_2(z), \ \text{ROC} = R_2$$

- Linearity

$$ax_1[n] + bx_2[n] \overset{Z}{\longleftrightarrow} aX_1(z) + bX_2(z), \text{ROC} \supset (R_1 \cap R_2)$$

– ROC = $R_1 \cap R_2$ if no pole-zero cancellation

Linearity & Time Shift

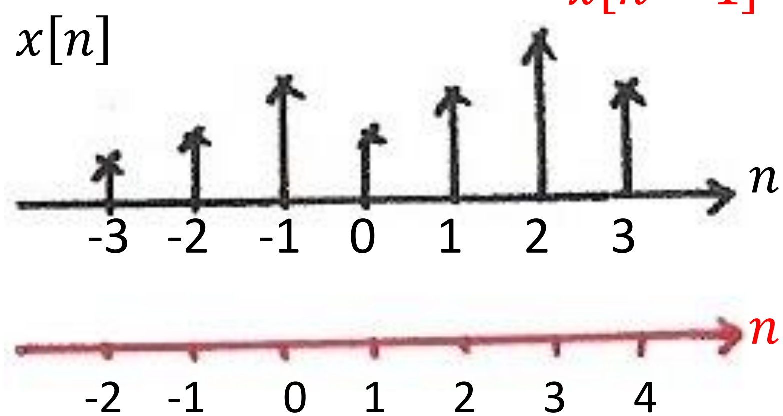
Linearity

$$F\{(ax_1[n] + bx_2[n])r^{-n}\}$$

$$= F\{ax_1[n]r^{-n}\} + F\{bx_2[n]r^{-n}\}$$

Time Shift

$x[n-1]$



z^3 z^2 z 1 z^{-1} z^{-2} z^{-3} ⇨ $X(z)$

z^2 z^1 1 z^{-1} z^{-2} z^{-3} z^{-4} ⇨ $z^{-1}X(z)$

- Time Shift

$$x[n-n_0] \stackrel{Z}{\longleftrightarrow} z^{-n_0} X(z),$$

ROC = R, except for possible addition or deletion of the origin or infinity

- $n_0 > 0$, poles introduced at $z = 0$ may cancel zeros at $z = 0$
- $n_0 < 0$, zeros introduced at $z = 0$ may cancel poles at $z = 0$
- Similarly for $z = \infty$

$$x[n-n_0] \leftrightarrow \boxed{e^{-j\omega}}^{n_0} X(e^{j\omega})$$

$$x(t-t_0) \leftrightarrow \boxed{e^{-s}}^{t_0} X(s)$$

- Scaling in z-domain

$$z_0^{\,n} x[n] \xleftrightarrow{Z} X\left(\frac{z}{z_0}\right), \ \text{ROC} = \left\{ |z_0| z \middle| z \in R \right\}$$

  – Pole/zero at $z = a$ → shifted to $z = z_0 a$
  – $z_0 = e^{j\omega_0}$

$$e^{j\omega_0 n} x[n] \xleftrightarrow{\;z\;} X\left(e^{-j\omega_0} z\right), \text{ROC} = R$$

rotation in z-plane by ω_0

See Fig. 10.15, p.769 of text

  – $z_0 = r_0 e^{j\omega_0}$

pole/zero rotation by ω_0 and scaled by r_0

Scaling in Z-domain

$$z_0^n x[n] \leftrightarrow \sum_n (z_0^n x[n]) z^{-n} = \sum_n x[n] \left(\frac{z}{z_0}\right)^{-n}$$

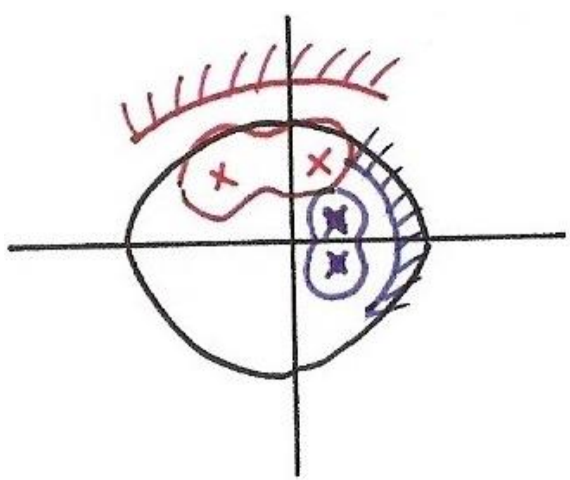
Shift in frequency domain

$$e^{j\omega_0 n} x[n] \leftrightarrow X\left(e^{j(\omega-\omega_0)}\right)$$

$$\frac{re^{j\omega}}{r_0 e^{j\omega_0}} = \frac{z}{z_0}$$

$$e^{s_0 t} x(t) \leftrightarrow X(s - s_0)$$

$$e^{(s-s_0)} = \frac{e^s}{e^{s_0}} = \frac{z}{z_0}$$



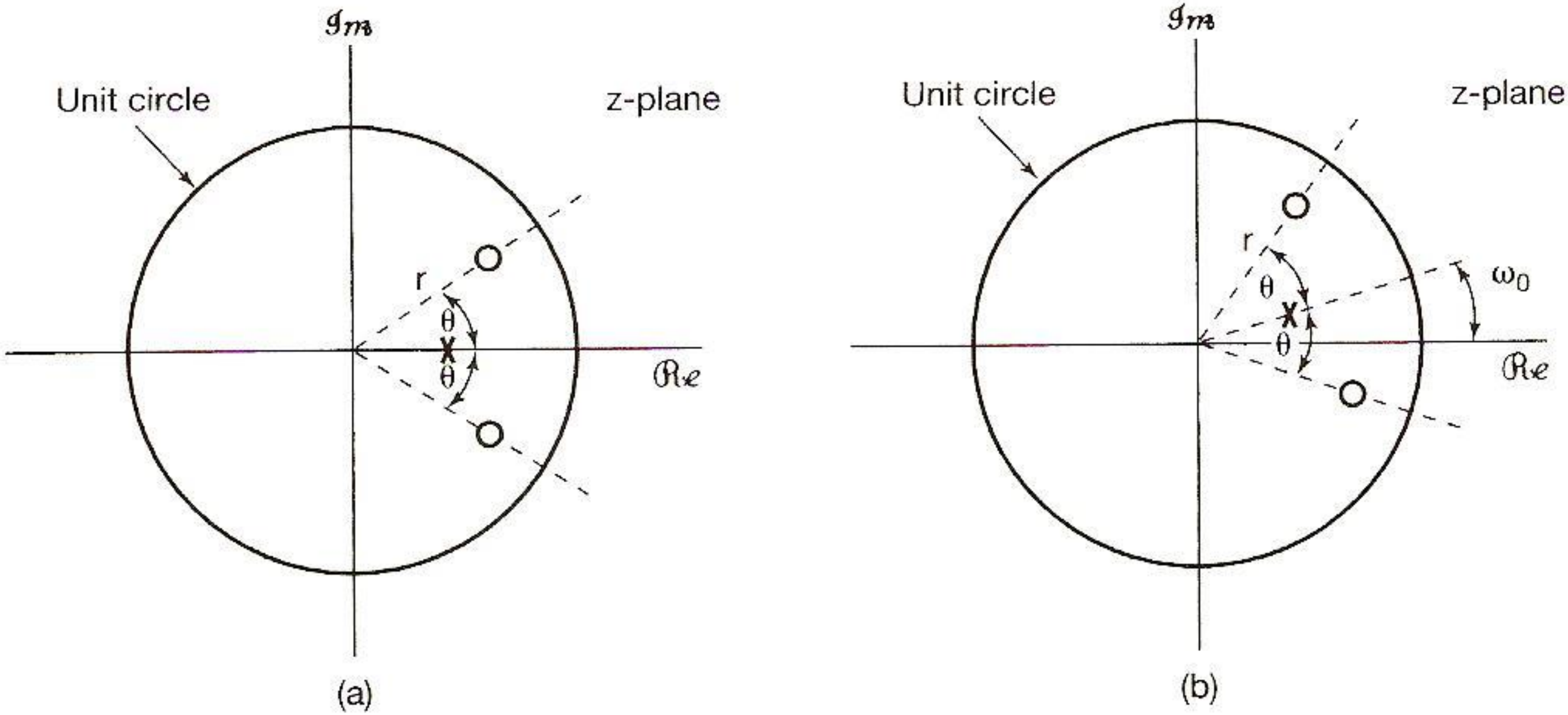
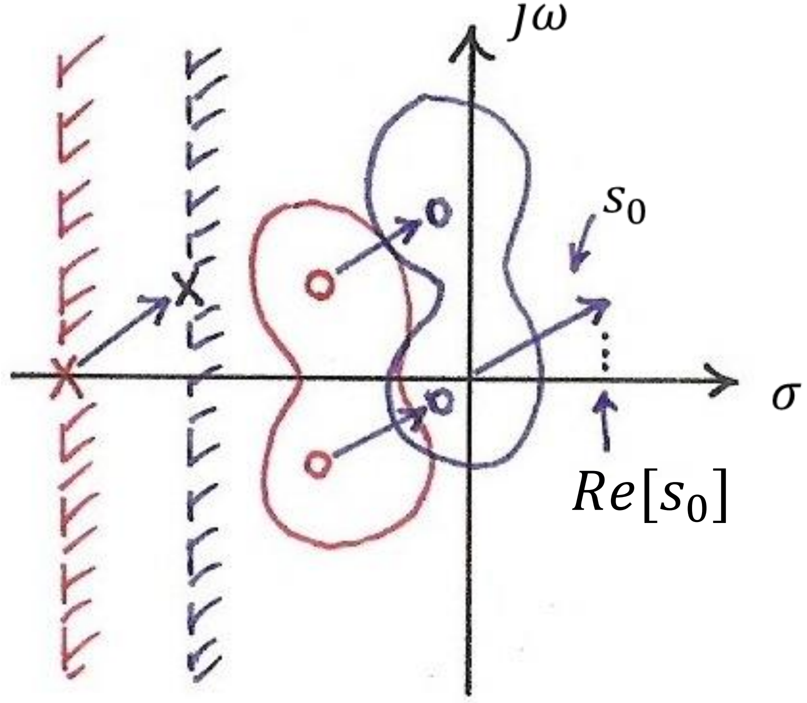
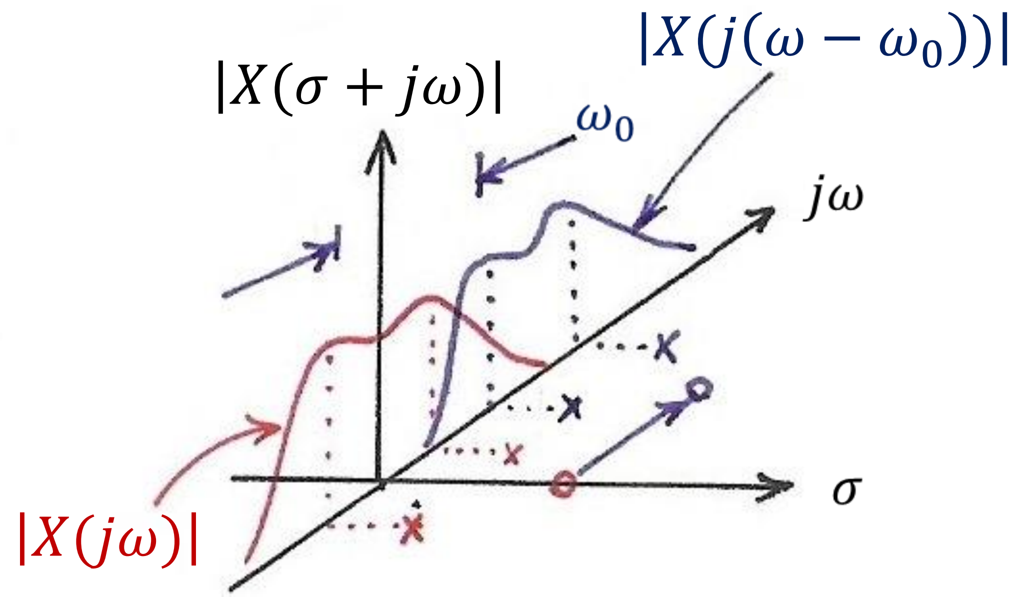


Figure 10.15 Effect on the pole-zero plot of time-domain multiplication by a complex exponential sequence $e^{j\omega_0 n}$: (a) pole-zero pattern for the z-transform for a signal $x[n]$; (b) pole-zero pattern for the z-transform of $x[n]e^{j\omega_0 n}$.

Shift in *s*-plane (p.33 of 9.0)

$$s_0 = j\omega_0$$





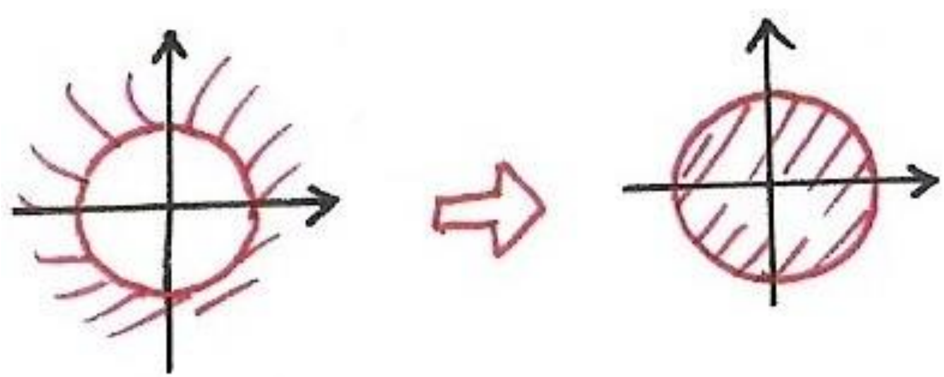
Time Reversal

$$x[-n] \stackrel{Z}{\longleftrightarrow} X\left(\frac{1}{z}\right), \quad \text{ROC} = \left\{ \frac{1}{z} \,\middle|\, z \in R \right\}$$

$x[-3]$ $x[-2]$ $x[-1]$ $x[0]$ $x[1]$ $x[2]$ $x[3]$ $\Rightarrow X(z^{-1})$

z^3 z^2 z^1 1 z^{-1} z^{-2} z^{-3} $\Rightarrow X(z)$

right-sided ↔ left-sided

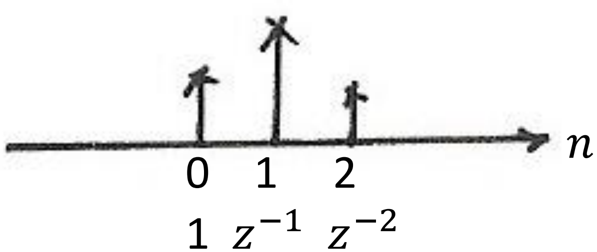


Time Expansion

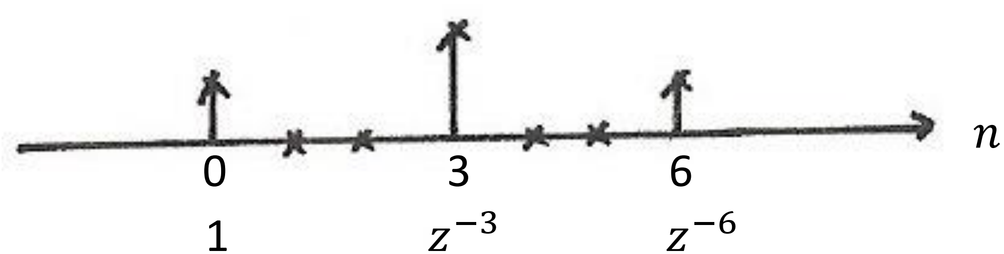
$$x_{(k)}[n] = \begin{cases} x[n/k], & n \text{ is a multiple of } k \\ 0 & , \text{ else} \end{cases}$$

$$x_{(k)}[n] \overset{Z}{\longleftrightarrow} X\left(z^{k}\right),\ \text{ROC} = \left\{ z^{1/k} \,\middle|\, z \in R \right\}$$

- pole/zero at $z = a \rightarrow$ shifted to $z = a^{1/k}$



⇨ $X(z)$



⇨ $X(z^k)$

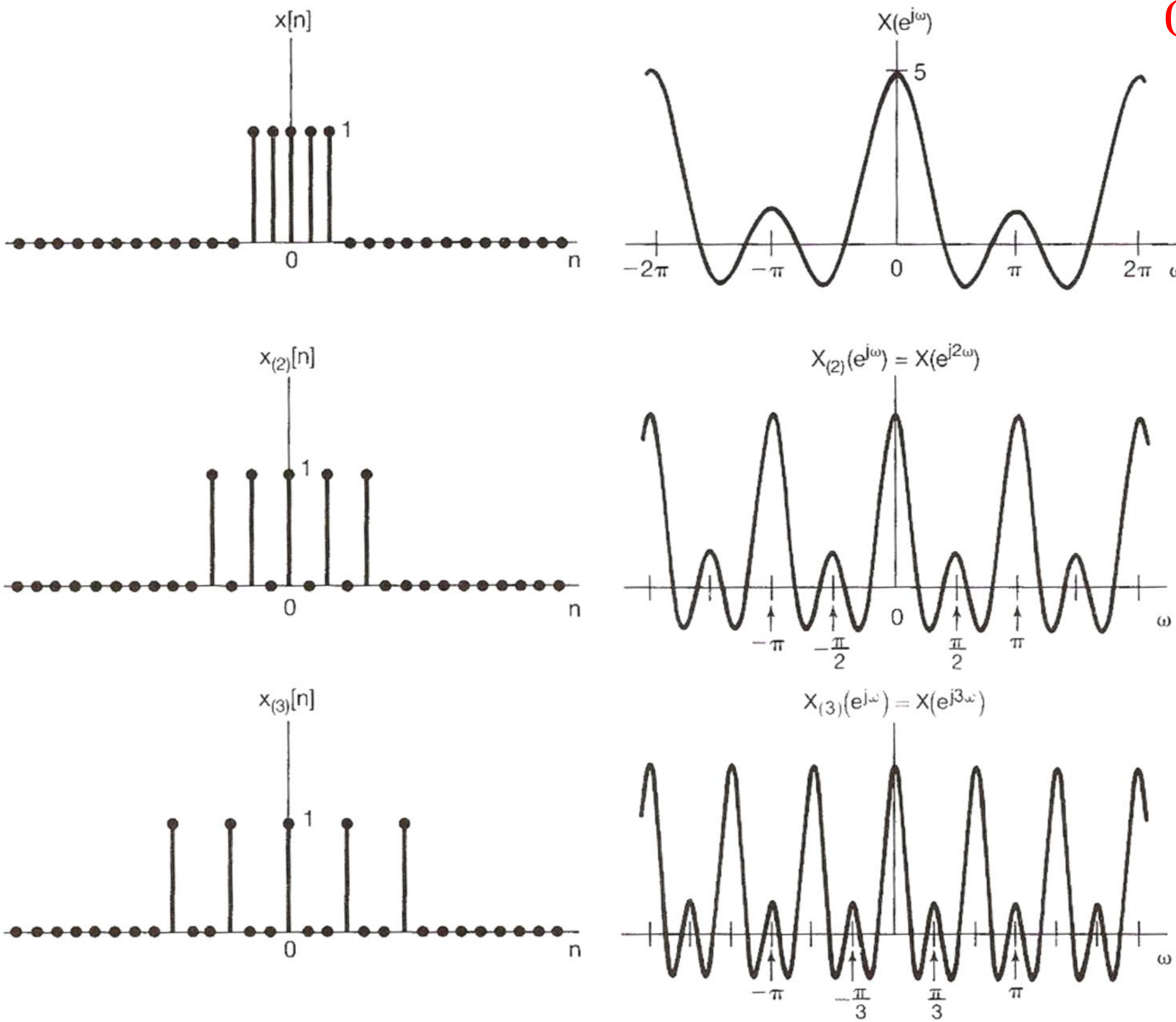
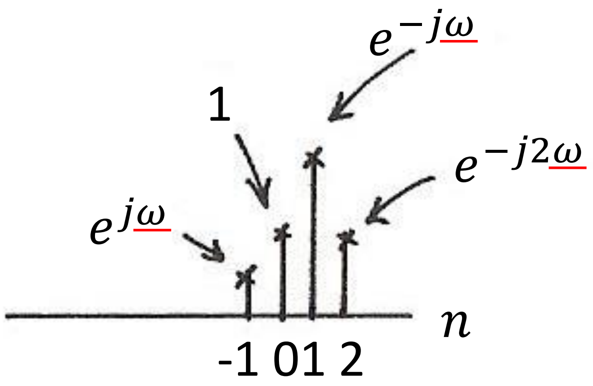
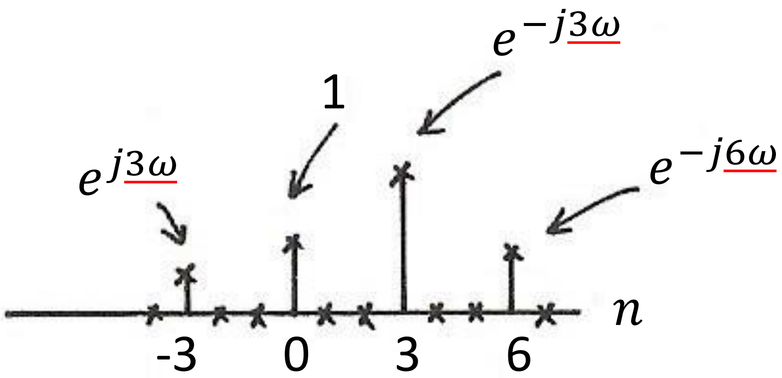


Figure 5.14 Inverse relationship between the time and frequency domains: As k increases, $x_{(k)}[n]$ spreads out while its transform is compressed.

Time Expansion (p.40 of 5.0)





$(k\omega)3\omega$ $(k\omega)3\omega$

$$X\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}x[n]\,e^{-j\omega n}$$

Conjugation

$$x^*[n] \xleftrightarrow{Z} X^*(z^*), \quad \text{ROC} = R$$

– if $x[n]$ is real

$$X(z) = X^*(z^*)$$

a pole/zero at $z = z_0$ → a pole/zero at $z = z_0^*$

$$x^*[n] \overset{F}{\leftrightarrow} X^*(e^{-j\omega})$$

$$x^*(t) \overset{L}{\leftrightarrow} X^*(s^*)$$

$$x^*[n] \overset{Z}{\leftrightarrow} X^*(z^*)$$

$$e^{s^*} = e^{\sigma - j\omega} = e^{\sigma} e^{-j\omega} = z^*$$

$$x^*[n] \leftrightarrow \sum_n x^*[n]\, z^{-n}$$

$$= [\sum_n x[n]\, (z^*)^{-n}]^*$$

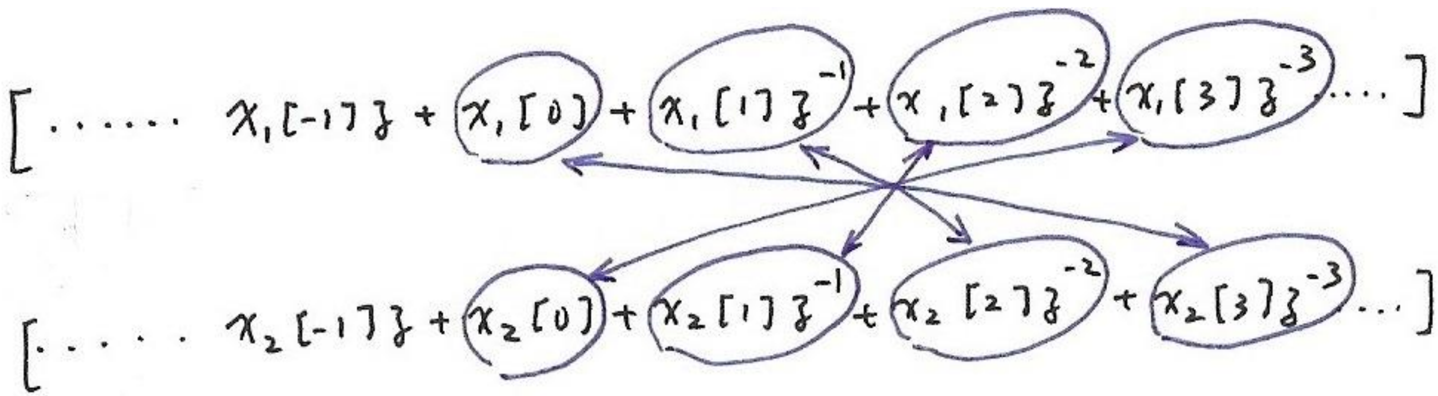
$$= X^*(z^*)$$

Convolution

$$x_1[n] * x_2[n] \overset{Z}{\longleftrightarrow} X_1(z)X_2(z), \ \text{ROC} \supset (R_1 \cap R_2)$$

ROC may be larger if pole/zero cancellation occurs

– power series expansion interpretation



$X_1(z)$

$X_2(z)$

$$\left[\cdots x_1[0]x_2[3] + x_1[1]x_2[2] + x_1[2]x_2[1] + x_1[3]x_2[0] \cdots \right] z^{-3}$$

$$y[3] = \sum_k x_1[k]x_2[3-k]$$

$$y[n] = x_1[n] * x_2[n]$$

$$Y(z) = X_1(z) \cdot X_2(z)$$

- Multiplication **(p.33 of 3.0)**

$$x(t) \xleftrightarrow{FS} a_k, \quad y(t) \xleftrightarrow{FS} b_k$$

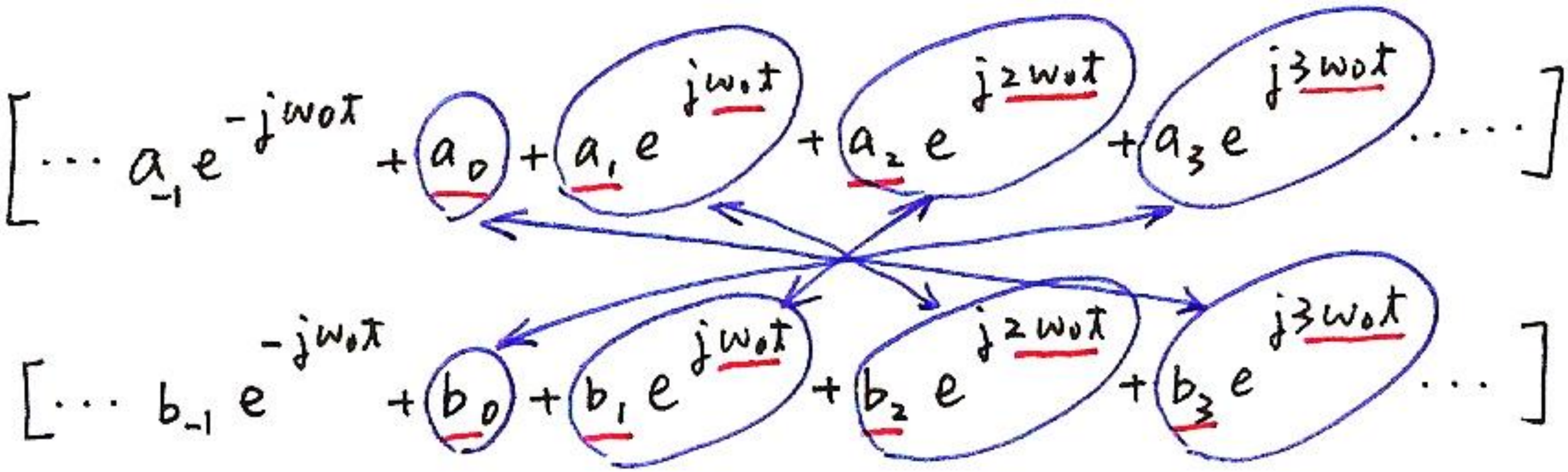
$$x(t)y(t) \xleftrightarrow{FS} d_k = \sum_{j=-\infty}^{\infty} a_j b_{k-j} = a_k * b_k$$

- Conjugation

$$x^*(t) \xleftrightarrow{FS} a_{-k}^*$$

$$a_{-k} = a_k^*, \text{ if } \ x(t) \text{ real}$$

Multiplication (p.31 of 3.0)



$$\underbrace{(\cdots, a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0, \cdots)}_{d_3 = \sum_j a_j b_{3-j}} e^{j3\omega_0 t}$$

- First Difference/Accumulation

$$x[n]-x[n-1] \overset{Z}{\longleftrightarrow} \left(1-z^{-1}\right)X(z)$$

ROC = R with possible deletion of $z = 0$

and/or addition of $z = 1$

$1-z^{-1}$: ROC= $\{z||z|>0\}$

a zero at $z=1$

a pole at $z=0$

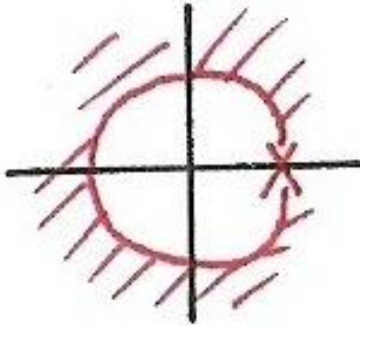
First Difference/Accumulation

$$\sum_{k=-\infty}^{n} x[k] \overset{z}{\longleftrightarrow} \frac{1}{1-z^{-1}} X(z)$$

$$\text{ROC} \supset R \cap \left\{ z \,\middle|\, |z| > 1 \right\}$$

$$\sum_{k=-\infty}^{n} x[k] = x[n] * u[n]$$

$$u[n] \overset{z}{\leftrightarrow} U(z) = \frac{1}{1-z^{-1}}$$, ROC$= \{|z| > 1\}$



- Differentiation in z-domain

$$nx[n] \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz}, \quad \text{ROC} = R$$

- Initial-value Theorem

$$\text{if} \quad x[n] = 0, \ \ n < 0$$

$$x[0] = \lim_{z \to \infty} X(z)$$

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

- Summary of Properties/Known Pairs

See Tables 10.1, 10.2, p.775, 776 of text

TABLE 10.1 PROPERTIES OF THE z-TRANSFORM

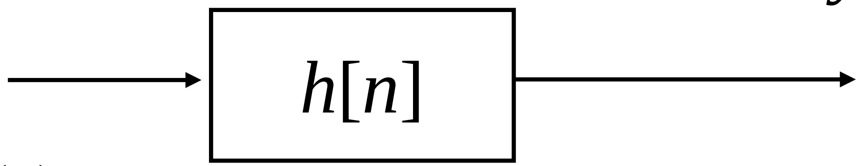
Section	Property	Signal	z-Transform	ROC
		$x[n]$	$X(z)$	R
		$x_1[n]$	$X_1(z)$	R_1
		$x_2[n]$	$X_2(z)$	R_2
10.5.1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
10.5.2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the origin
10.5.3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
		$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
		$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $\|a\|R$ = the set of points $\{\|a\|z\}$ for z in R)
10.5.4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1}, where z is in R)
10.5.5	Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
10.5.6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
10.5.7	Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
10.5.7	First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $\|z\| > 0$
10.5.7	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - z^{-1}}X(z)$	At least the intersection of R and $\|z\| > 1$
10.5.8	Differentiation in the z-domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R
10.5.9	Initial Value Theorem If $x[n] = 0$ for $n < 0$, then $x[0] = \lim_{z \to \infty} X(z)$			

TABLE 10.2 SOME COMMON z-TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1 - z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1 - z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z, except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\frac{1}{1 - \alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
6. $-\alpha^n u[-n-1]$	$\frac{1}{1 - \alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
7. $n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
8. $-n\alpha^n u[-n-1]$	$\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
9. $[\cos \omega_0 n]u[n]$	$\frac{1 - [\cos \omega_0]z^{-1}}{1 - [2\cos \omega_0]z^{-1} + z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin \omega_0 n]u[n]$	$\frac{[\sin \omega_0]z^{-1}}{1 - [2\cos \omega_0]z^{-1} + z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n \cos \omega_0 n]u[n]$	$\frac{1 - [r\cos \omega_0]z^{-1}}{1 - [2r\cos \omega_0]z^{-1} + r^2 z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n \sin \omega_0 n]u[n]$	$\frac{[r\sin \omega_0]z^{-1}}{1 - [2r\cos \omega_0]z^{-1} + r^2 z^{-2}}$	$\lvert z\rvert > r$

10.3 System Characterization with Z-Transform

$x[n]$ $\qquad$ $y[n]=x[n]*h[n]$



$X(z)$ $\qquad$ $H(z)$ $\qquad$ $Y(z)=X(z)H(z)$

system function, transfer function

- Causality
  - A system is causal if and only if the ROC of $H(z)$ is the exterior of a circle including infinity (may extend to include the origin in some cases)

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}, \text{ right-sided}$$

$$\text{if } \ h[n_0] \neq 0, \ n_0 < 0$$

$$H(z) \text{ includes } \ x[n_0] z^{-n_0}$$

  - A system with rational $H(z)$ is causal if and only

    if (1)ROC is the exterior of a circle outside the outermost pole including infinity

    or (2)order of N(z) ≤ order of D(z)
    H(z) finite for z → ∞

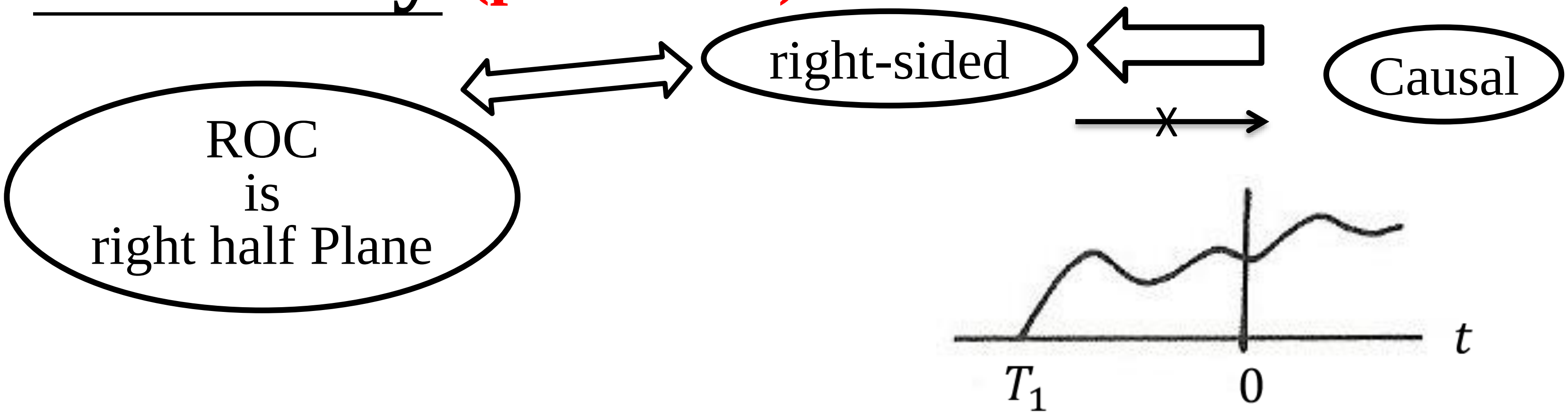
- Causality **(p.44 of 9.0)**
  - A causal system has an $H(s)$ whose ROC is a right-half plane

    $h(t)$ is right-sided
  - For a system with a rational $H(s)$, causality is equivalent to its ROC being the right-half plane to the right of the rightmost pole
  - Anticausality

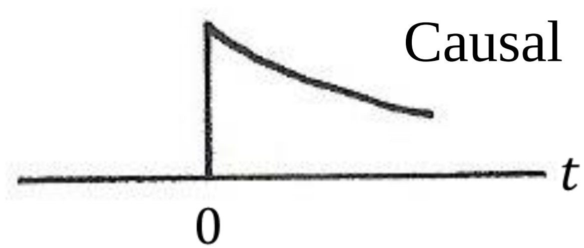
    a system is anticausal if $h(t) = 0, t > 0$

    an anticausal system has an $H(s)$ whose ROC is a left-half plane, etc.

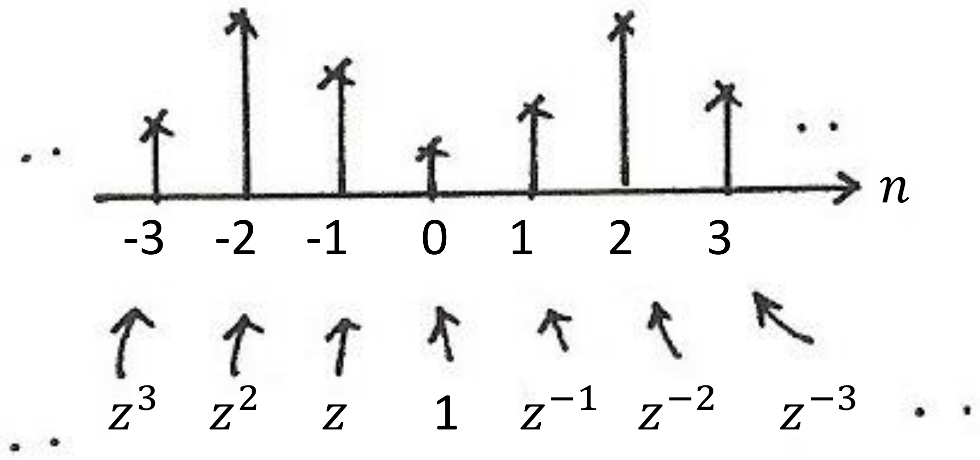
Causality (p.45 of 9.0)



$$X(s) = \sum_i \frac{A_i}{s + a_i} \;, \qquad \frac{A_i}{s + a_i} \rightarrow A_i e^{-a_i t} u(t)$$



Z-Transform (p.4 of 10.0)



$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$x[n] \overset{Z}{\longleftrightarrow} X(z)$$

- Stability
  - A system is stable if and only if ROC of $H(z)$ includes the unit circle

    Fourier Transform converges, or absolutely summable

  - A causal system with a rational $H(z)$ is stable if and only if all poles lie inside the unit circle

      ROC is outside the outermost pole

- Systems Characterized by Linear Difference Equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

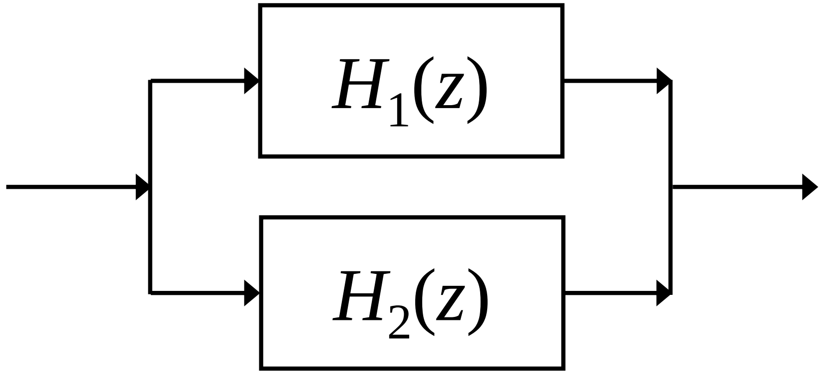
$$Y(z)\sum_{k=0}^{N} a_k z^{-k} = X(z)\sum_{k=0}^{M} b_k z^{-k}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \quad \begin{matrix} \longrightarrow \text{zeros} \\ \longrightarrow \text{poles} \end{matrix}$$

– difference equation doesn’t specify ROC

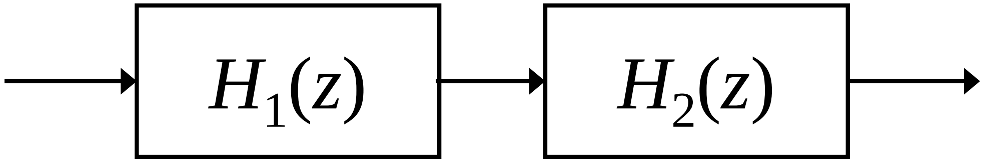
stability/causality helps to specify ROC

- Interconnections of Systems
  - Parallel



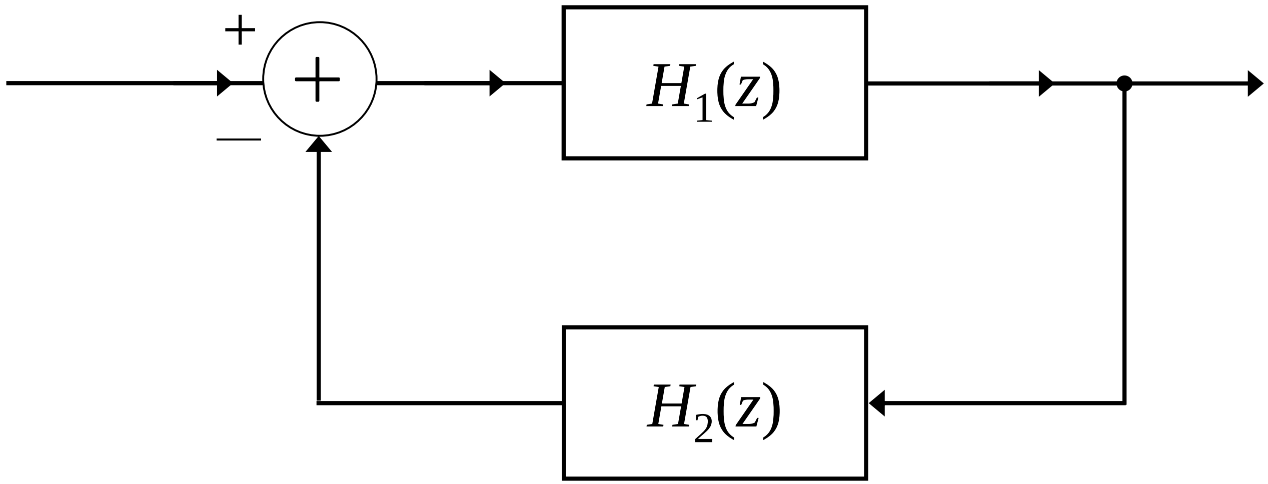
$$H(z)=H_1(z)+H_2(z)$$

  - Cascade



$$H(z)=H_1(z)H_2(z)$$

- Interconnections of Systems
  - Feedback



$$H(z) = \frac{H_1(z)}{1 + H_1(z)H_2(z)}$$

- Block Diagram Representation
  - Example:

$$y[n]+\frac{1}{4}y[n-1]-\frac{1}{8}y[n-2]=x[n]$$

$$H(z)=\frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}=\left(\frac{1}{1+\frac{1}{2}z^{-1}}\right)\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)$$

$$=\frac{2/3}{1+\frac{1}{2}z^{-1}}+\frac{1/3}{1-\frac{1}{4}z^{-1}}$$

  - direct form, cascade form, parallel form

    See Fig. 10.20, p.787 of text

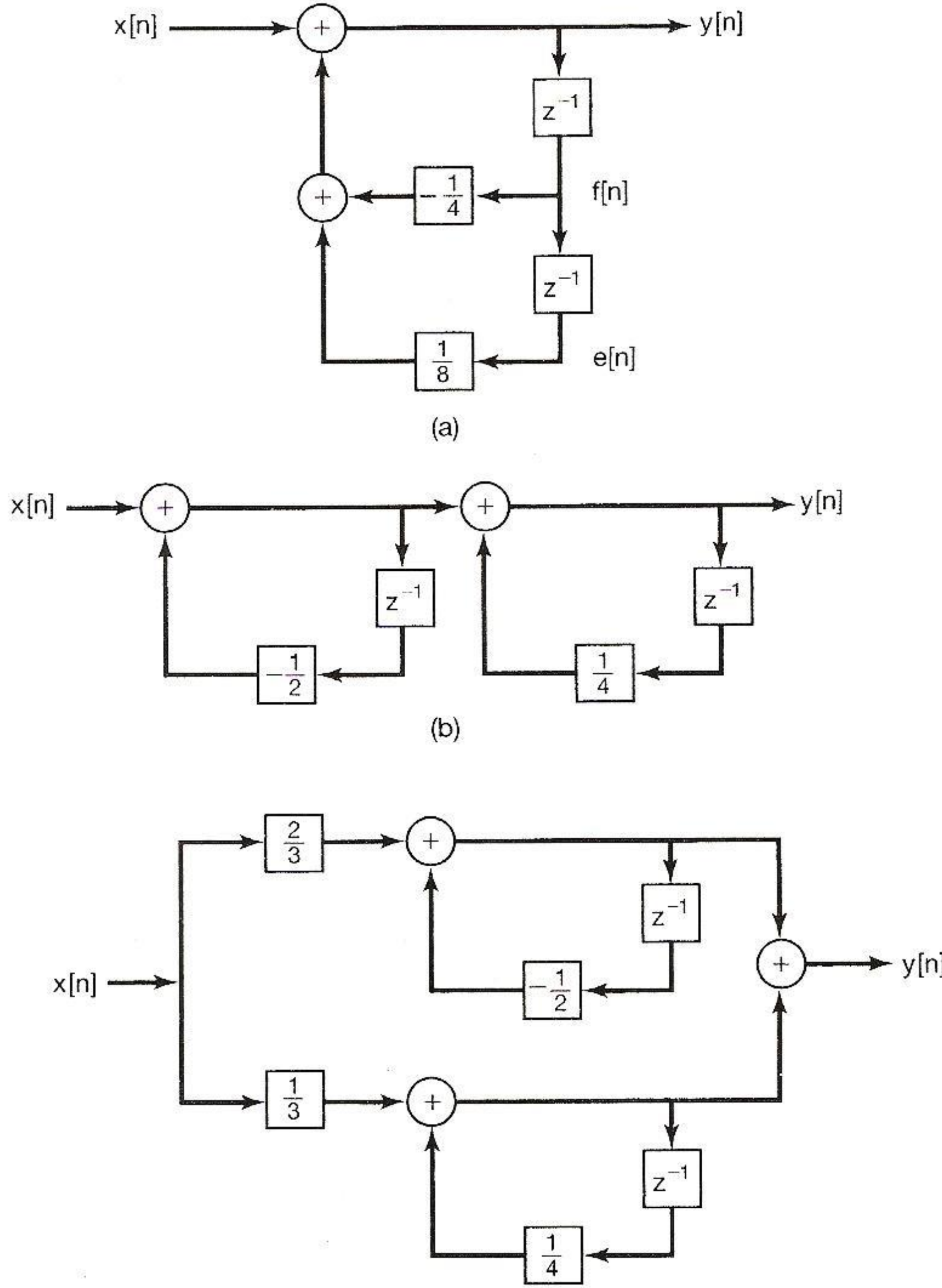


Figure 10.20 Block-diagram representations for the system in Example 10.30: (a) direct form; (b) cascade form; (c) parallel form.

10.4 Unilateral Z-Transform

$$X(z)_u = \sum_{n=0}^{\infty} x[n] z^{-n}$$ unilateral z - transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$ bilateral z - transform

– $Z\{x[n]u[n]\} = Z_u\{x[n]\}$

for $x[n] = 0,\ n < 0,\ X(z)_u = X(z)$

ROC of $X(z)_u$ is always the exterior of a circle including $z = \infty$

degree of $N(z) \le$ degree of $D(z)$ (converged for $z = \infty$)

Unilateral Z-Transform

- Time Delay Property (different from bilateral case)

$$x[n-1] \xleftrightarrow{Z_u} x[-1] + z^{-1}X(z)_u$$

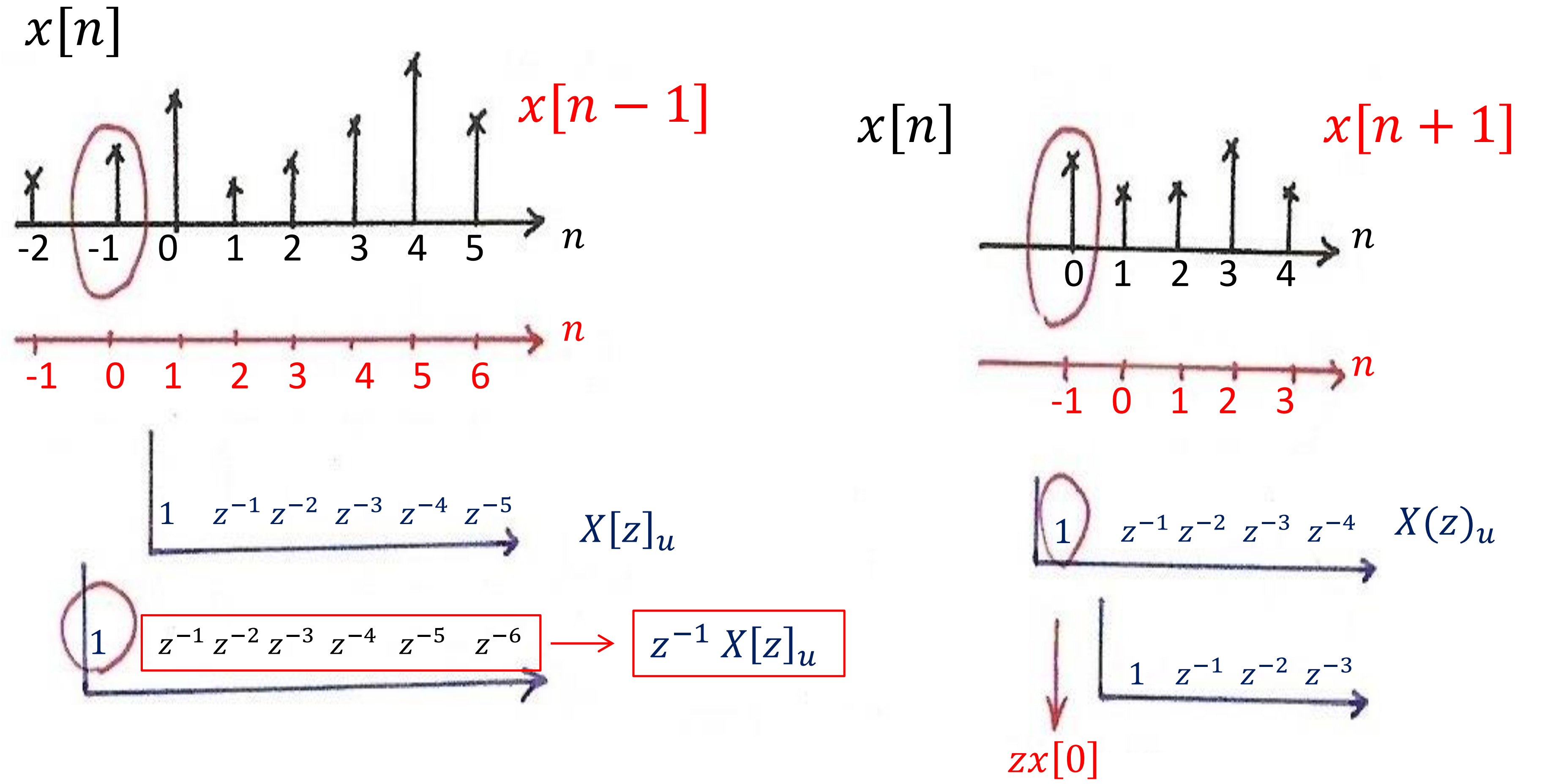
$$x[n-2] \xleftrightarrow{Z_u} x[-2] + x[-1]z^{-1} + z^{-2}X(z)_u$$

$$\sum_{n=0}^{\infty} x[n-1]z^{-n} = x[-1] + \sum_{n=1}^{\infty} x[n-1]z^{-n}$$

- Time Advance Property (different from bilateral case)

$$x[n+1] \xleftrightarrow{Z_u} zX(z)_u - zx[0]$$

Time Delay Property/Time Advance Property



Unilateral Z-Transform

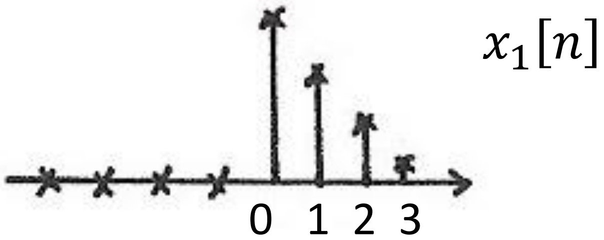
– Convolution Property

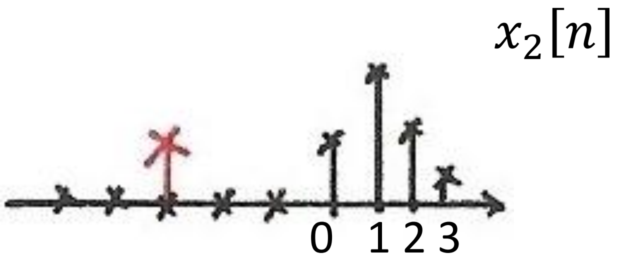
$$\text{if} \quad x_1[n] = x_2[n] = 0, \; n < 0$$

$$x_1[n] * x_2[n] \xleftrightarrow{Z_u} X_1(z)_u \, X_2(z)_u$$

this is not true if $x_1[n]$, $x_2[n]$ has nonzero values for $n < 0$

Convolution Property





$$x_1[n] = 0, n < 0$$

$$x_2[n] = 0, n < 0$$

$$\rightarrow x_1[n] * x_2[n] = 0, n < 0$$

$$\begin{array}{ccccc} x_1[n] & \overset{z}{\leftrightarrow} & X_1(z) & = & X_1(z)_u \\ * & & \cdot & & \cdot \\ x_2[n] & \overset{z}{\leftrightarrow} & X_2(z) & = & X_2(z)_u \\ \parallel & & \parallel & & \parallel \\ y[n] & \overset{z}{\leftrightarrow} & Y(z) & = & Y(z)_u \end{array}$$

Examples

- Example 10.4, p.747 of text

$$x[n] = \left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4} n\right) u[n]$$

$$= \frac{1}{2j}\left[\left(\frac{1}{3}\right)e^{j\frac{\pi}{4}}\right]^n u[n] - \frac{1}{2j}\left[\left(\frac{1}{3}\right)e^{-j\frac{\pi}{4}}\right]^n u[n]$$

$$\mathrm{X}(z) = \frac{3\frac{1}{\sqrt{2}}z}{(z-\frac{1}{3}e^{j\frac{\pi}{4}})(z-\frac{1}{3}e^{-j\frac{\pi}{4}})} \quad , \quad |z| > \frac{1}{3}$$

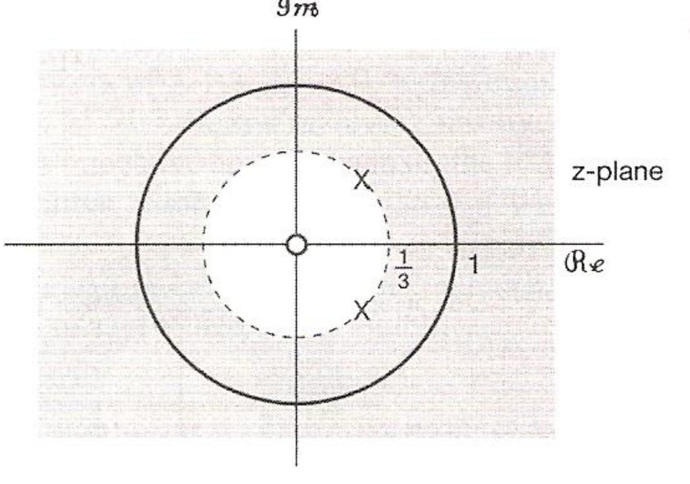


Figure 10.5 Pole-zero plot and ROC for the *z*-transform in Example 10.4.

Examples

- Example 10.6, p.752 of text

$$x[n] = \begin{cases} a^n, & 0 \le \mathrm{n} \le \mathrm{N} - 1,\ a > 0 \\ 0, & \text{else} \end{cases}$$

$$X(z) = \frac{1 - (az^{-1})^N}{1 - az^{-1}} = \left(\frac{1}{z^{N-1}}\right)\left(\frac{z^N - a^N}{z - a}\right)$$

$(N-1)$st order pole at orgin

N - 1 zeros at $z_k = ae^{j\left(\frac{2\pi k}{N}\right)},\ k = 1,2,\ldots N\text{-}1$

potential pole/zero canceled at $z = a$

$ROC = \{|z| > 0\}$

Examples

- Example 10.6, p.752 of text

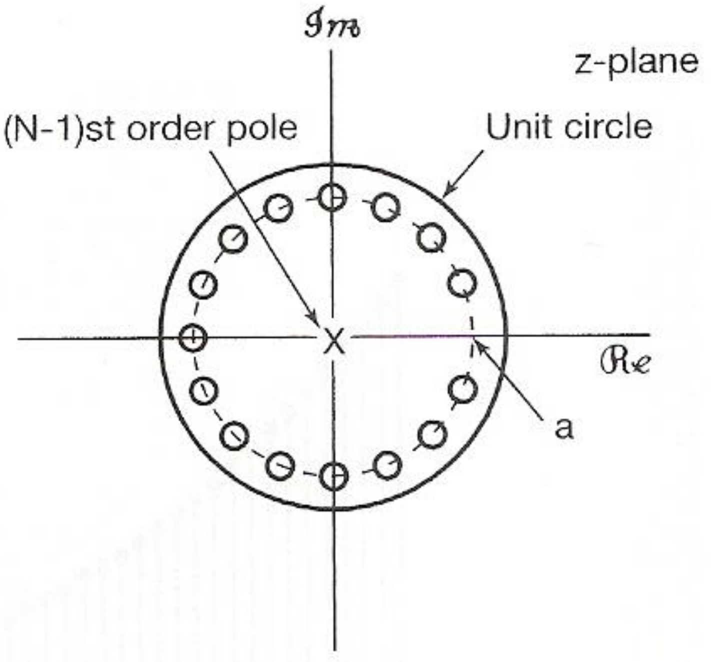


Figure 10.9 Pole-zero pattern for Example 10.6 with $N = 16$ and $0 < a < 1$. The region of convergence for this example consists of all values of z except $z = 0$.

Examples

- Example 10.17, p.772 of text

$$X(z) = \log\left(1 + az^{-1}\right), \ |z| > |a|$$

$$nx[n] \overset{Z}{\longleftrightarrow} -z\frac{dX(z)}{dz} = \frac{az^{-1}}{1 + az^{-1}}, \ |z| > |a|$$

$$\because \ a(-a)^n u[n] \overset{Z}{\longleftrightarrow} \frac{a}{1 + az^{-1}}, \ |z| > |a|$$

$$a(-a)^{n-1} u[n-1] \overset{Z}{\longleftrightarrow} \frac{az^{-1}}{1 + az^{-1}}, \ |z| > |a|, \ \text{time shift property}$$

$$x[n] = \frac{-(-a)^n}{n} u[n-1]$$

Examples

- Example 10.31, p.788 of text (Problem 10.38, P.805 of text)

$$H(z)=\frac{1-\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$$

$$=\left(\frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}\right)\left(1-\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}\right)$$

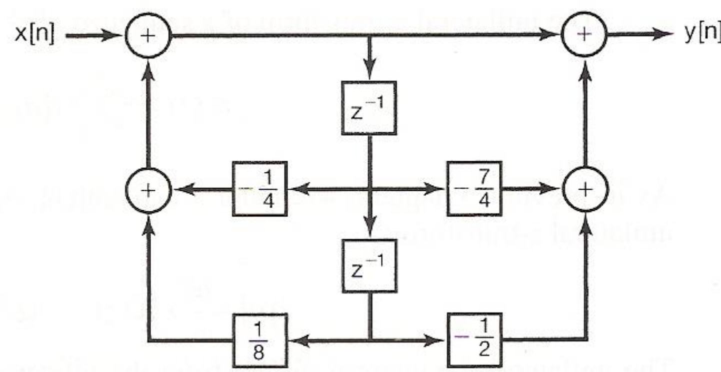


Figure 10.21 Direct-form representation for the system in Example 10.31.

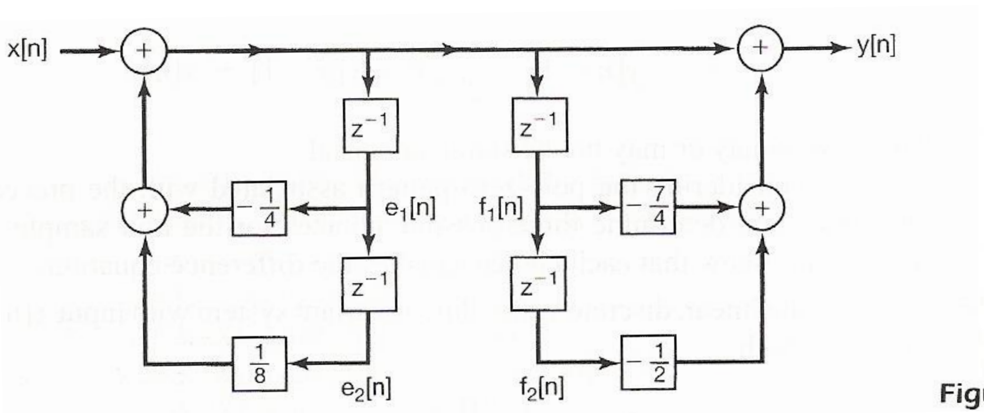


Figure P10.38

Direct form representation of the system

Problem 10.12, p.799 of text

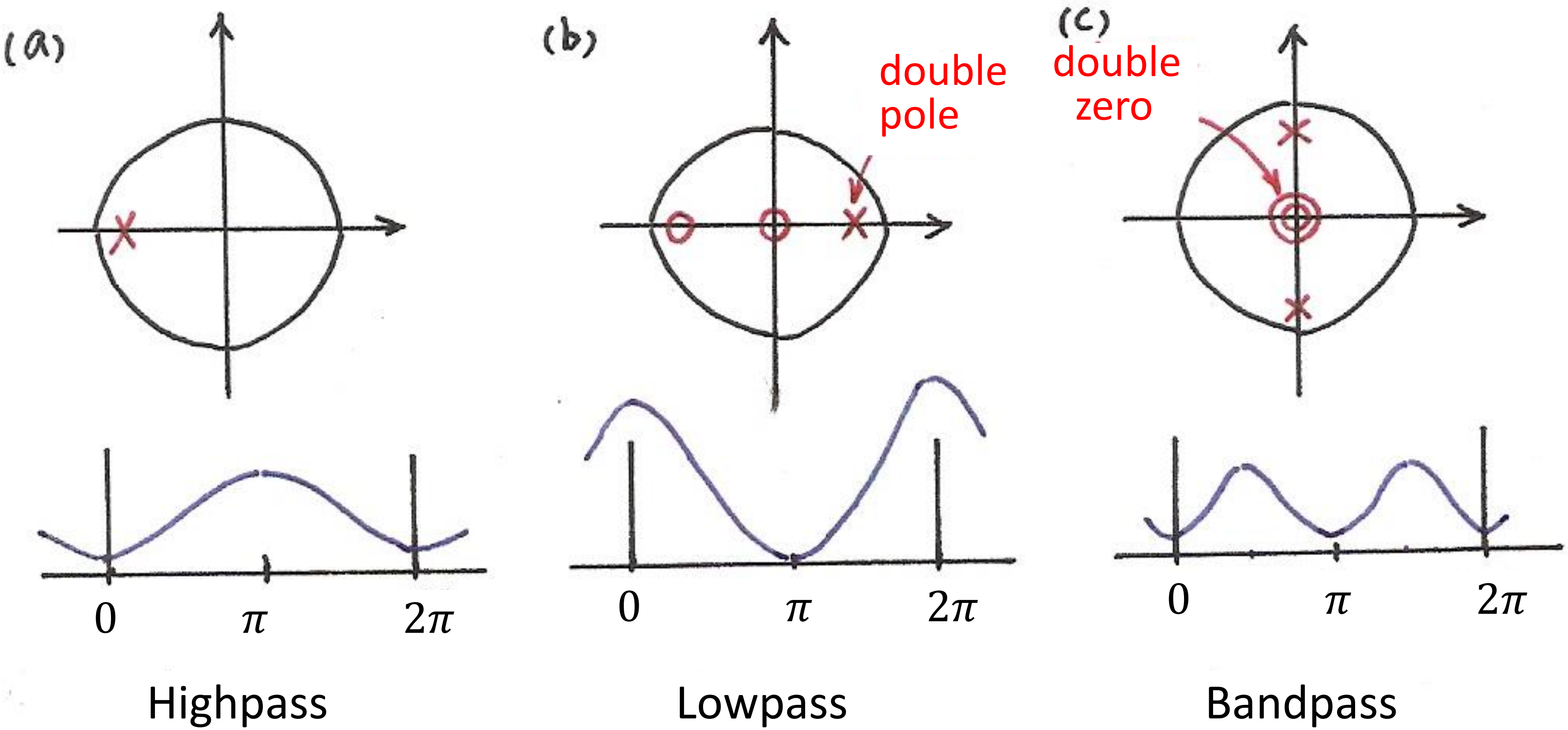
Which of the following system is approximately lowpass, highpass, or bandpass?

$$\text{(a) } H(z) = \frac{z^{-1}}{1 + \frac{8}{9} z^{-1}}, \quad |z| > \tfrac{8}{9}$$

$$\text{(b) } H(z) = \frac{1 + \frac{8}{9} z^{-1}}{1 - \frac{16}{9} z^{-1} + \frac{64}{81} z^{-2}}, \quad |z| > \tfrac{8}{9}$$

$$\text{(c) } H(z) = \frac{1}{1 + \frac{64}{81} z^{-2}}, \quad |z| > \tfrac{8}{9}$$

Problem 10.12, p.799 of text



<u>Problem 10.44</u>, p.808 of text

(c) $x_1[n] = x[2n]$

$$g[n] = \frac{1}{2}\left\{x[n] + (-1)^n x[n]\right\}$$

$$G(z) = \frac{1}{2}\left[X(z) + X(-z)\right]$$

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} g[2\text{n}] z^{-n}$$

$$= \sum_{\substack{n=-\infty \\ n:\ even}}^{\infty} g[\text{n}] z^{-\frac{n}{2}} = \text{G}\left(\text{z}^{\frac{1}{2}}\right)$$

$$= \frac{1}{2}\left[X\left(\text{z}^{\frac{1}{2}}\right) + X\left(-\text{z}^{\frac{1}{2}}\right)\right]$$

<u>Problem 10.46</u>, p.808 of text

$$y[n] = x[n] - e^{8\alpha} x[n-8], \quad e^{\alpha} < 1$$

$$H(z) = 1 - e^{8\alpha} z^{-8} = \frac{z^8 - e^{8\alpha}}{z^8}, \quad |z| > 0$$

8-th order pole at z=0 and 8 zeros

causal and stable

$$G(z) = \frac{1}{1 - e^{8\alpha} z^{-8}} = G'\left(z^8\right)$$

8-th order zero at z=0 and 8 poles

causal and stable

$$G'(z) = \frac{1}{1 - e^{8\alpha} z^{-1}}$$

$$g'[n] = e^{8\alpha n} u[n]$$

$$g[n] = g'_{(8)}[n] = \begin{cases} e^{8\alpha \frac{n}{8}} = e^{\alpha n}, & n = 0,8,16,\ldots. \\ 0, & \text{else} \end{cases}$$

Problem 10.46, p.808 of text