2.0 Linear Time-invariant Systems

2.1 Discrete-time Systems: the Convolution Sum

- Representing an arbitrary signal as a sequence of unit impulses

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

an unit impulse located at n = k on the index n

See Fig. 2.1, p.76 of text

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$ a special case

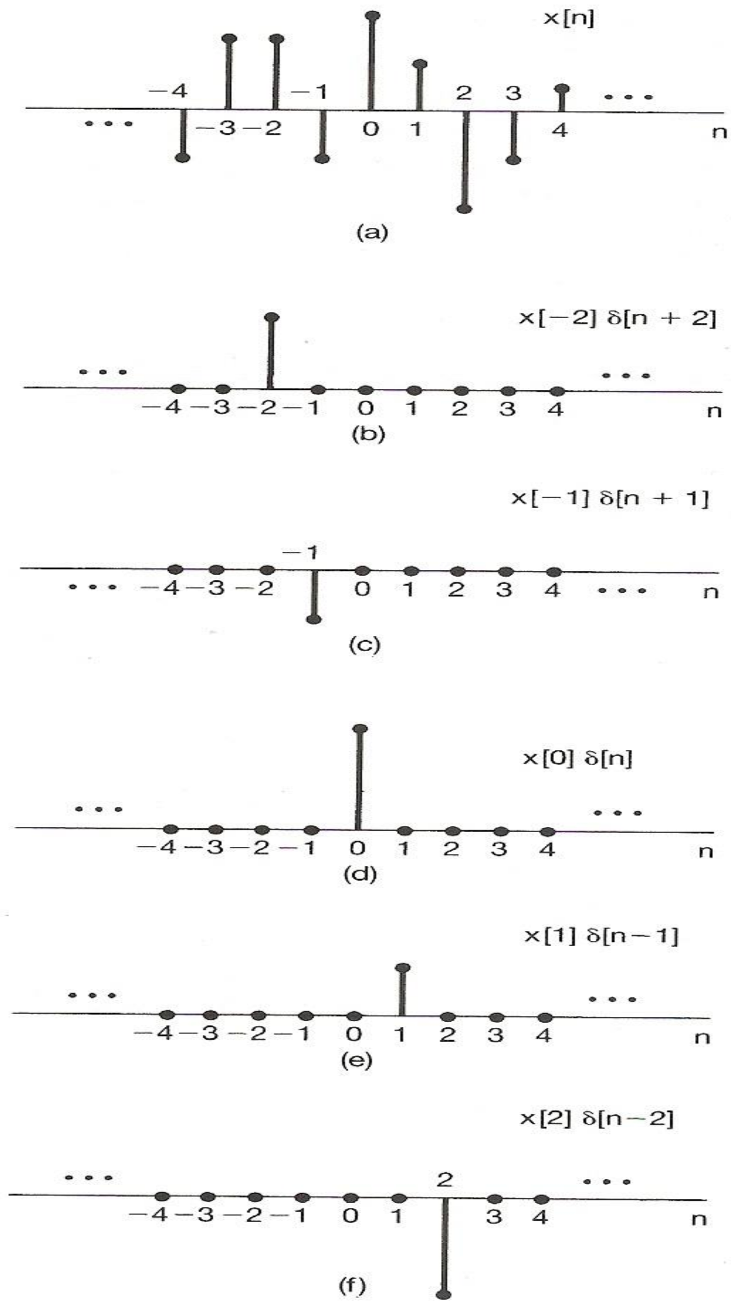


Figure 2.1 Decomposition of a discrete-time signal into a weighted sum of shifted impulses.

Vector Space Representation of Discrete-time Signals **(P.47 of 1.0)**

- n-dim

$$\vec{a} = (a_1, a_2, \cdots a_n)$$

$$\vec{b} = (b_1, b_2, \cdots b_n)$$

$$\widehat{v_1} = (1, 0, 0, \cdots 0)$$

$$\widehat{v_2} = (0,1, 0, \cdots\ 0)$$

$$\vdots$$

$$\vec{a} \cdot \vec{b} = \sum_i a_i\, b_i$$

$$\Rightarrow$$

$$\vec{x} = (x_1, x_2, \cdots x_n)$$

$$\vec{x} = \sum_i x_i\, \hat{v}_i \quad \text{合成}$$

$$x_j = \vec{x} \cdot \hat{v}_j \quad \text{分析}$$

Vector Space Representation of Discrete-time Signals (P.48 of 1.0)

- n extended to $\pm\infty$

$$\vec{x} = (\cdots, x_{-2}, x_{-1}, x_0, x_1, x_2, x_3, \cdots)$$

$$\widehat{v_0} = (\cdots, \quad 0, \quad 0, \; 1, \; 0, \; 0, \; 0, \cdots) = \delta[n]$$

$$\widehat{v_1} = (\cdots, \quad 0, \quad 0, \; 0, \; 1, \; 0, \; 0, \cdots) = \delta[n-1]$$

$$\widehat{v_k} = (\cdots, \quad 0, \quad 0, \; 0, \cdots, \; 0, \; 1, \cdots) = \delta[n-k]$$

$$\{\delta[n-k], \; k: \text{inter}\} \Longrightarrow \vec{x} = \sum_i x_i \widehat{v_i} \quad \text{合成}$$

$$x_j = \vec{x} \cdot \widehat{v_j} \quad \text{分析}$$

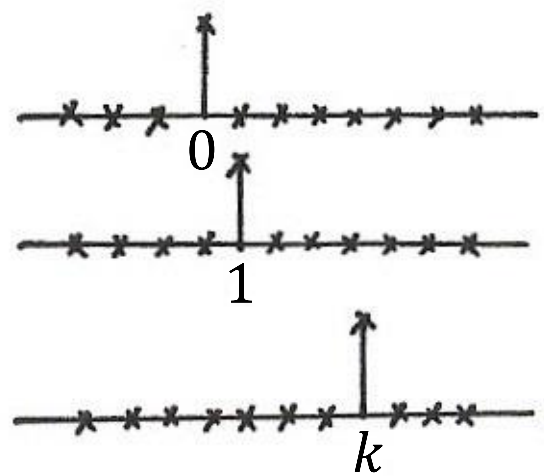
Vector Space Interpretation

$$\hat{v}_0 = (\cdots, 0, 1, 0, 0, 0, \cdots)$$

$$\hat{v}_1 = (\cdots, 0, 0, 1, 0, 0, \cdots)$$

$$\vdots$$

$$\hat{v}_k = (\cdots\cdots, 0, 0, 0, \underset{\substack{\uparrow \\ k}}{1}, \cdots)$$



$\delta[\,n\,]$

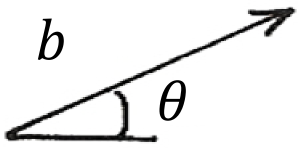
$\delta[\,n - 1\,]$

$\delta[\,n - k\,]$

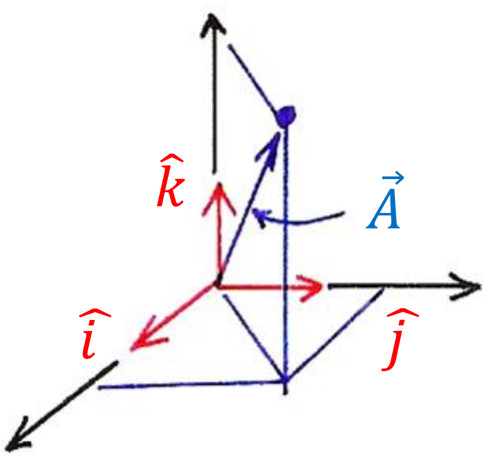
$$\{\delta[n-k], k = \cdots, -1, 0, 1, 2, \cdots\}$$

Vector Space (P.30 of 1.0)

$$V = \{v \mid \cdots\}$$
$$av$$
$$v_1 + v_2$$



3-dim Vector Space



$$(\vec{A}) \cdot \hat{j} = (a\hat{i} + b\hat{j} + c\hat{k}) \cdot \hat{j} \text{ (合成)}$$

$$b = \vec{A} \cdot \hat{j} \quad \text{(分析)}$$

N-dim Vector Space (P.31 of 1.0)

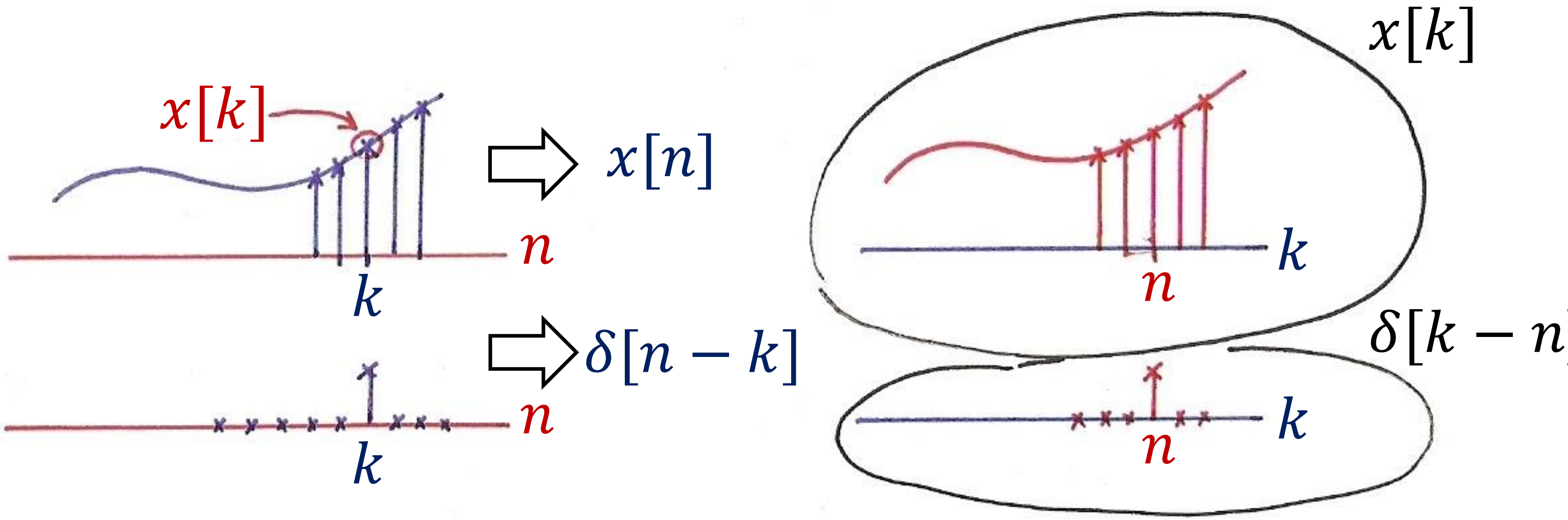
$$\vec{A} = \sum_{k=1}^{N} a_k \; \hat{v}_k$$ (合成)

$$a_j = \vec{A} \cdot \hat{v}_j$$ (分析)

$$\hat{v}_i \cdot \hat{v}_j = \delta_{ij}$$

Vector Space Interpretation

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta\,[n-k] \quad \text{(是合成也是分析)}$$



$$\vec{A} = \sum_k a_k\ \hat{v}_k \ \text{(合成)} \qquad a_j = \vec{A}\cdot\hat{v}_j \text{(分析)}$$

$$a_n = \vec{A}\cdot\hat{v}_n \left(\vec{A}\cdot\vec{B} = \sum_k a_k\, b_k\right)$$

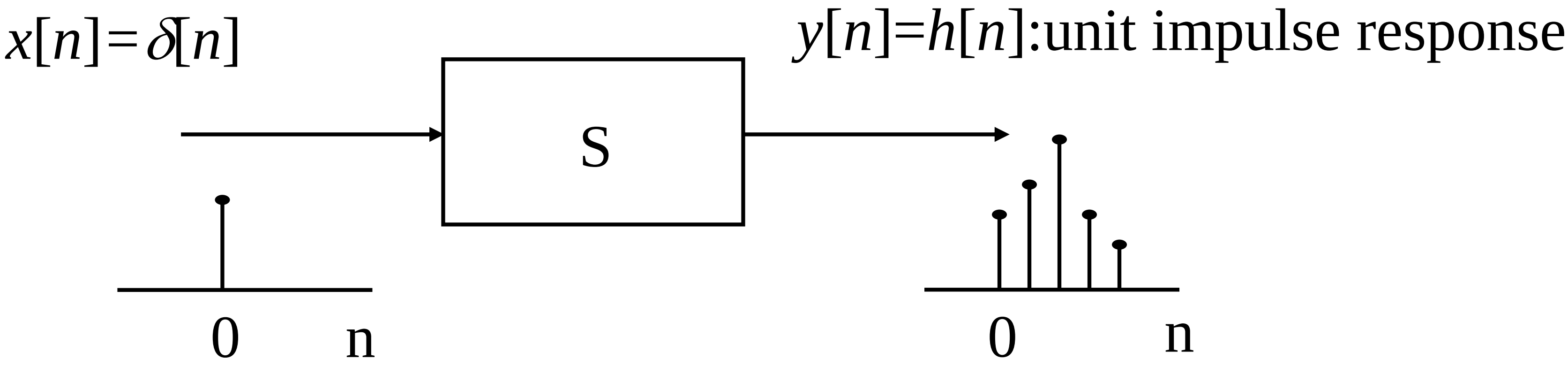
- Representing an arbitrary signal as a sequence of unit impulses

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \qquad (合成)$$

an unit impulse located at n = k on the index n

- Sifting property of the unit impulse:looked at on the index k, $\delta[n\text{-}k]$ is nonzero only at $k = n$, which "sifts" the value $x[n]$ out of the function $x[k]$. (分析)

- Defining the output for an unit impulse input as the Unit Impulse Response



- By Linearity (Superposition Property)
  - The output for an arbitrary input signal is the superposition of a series of "shifted, scaled unit impulse response"

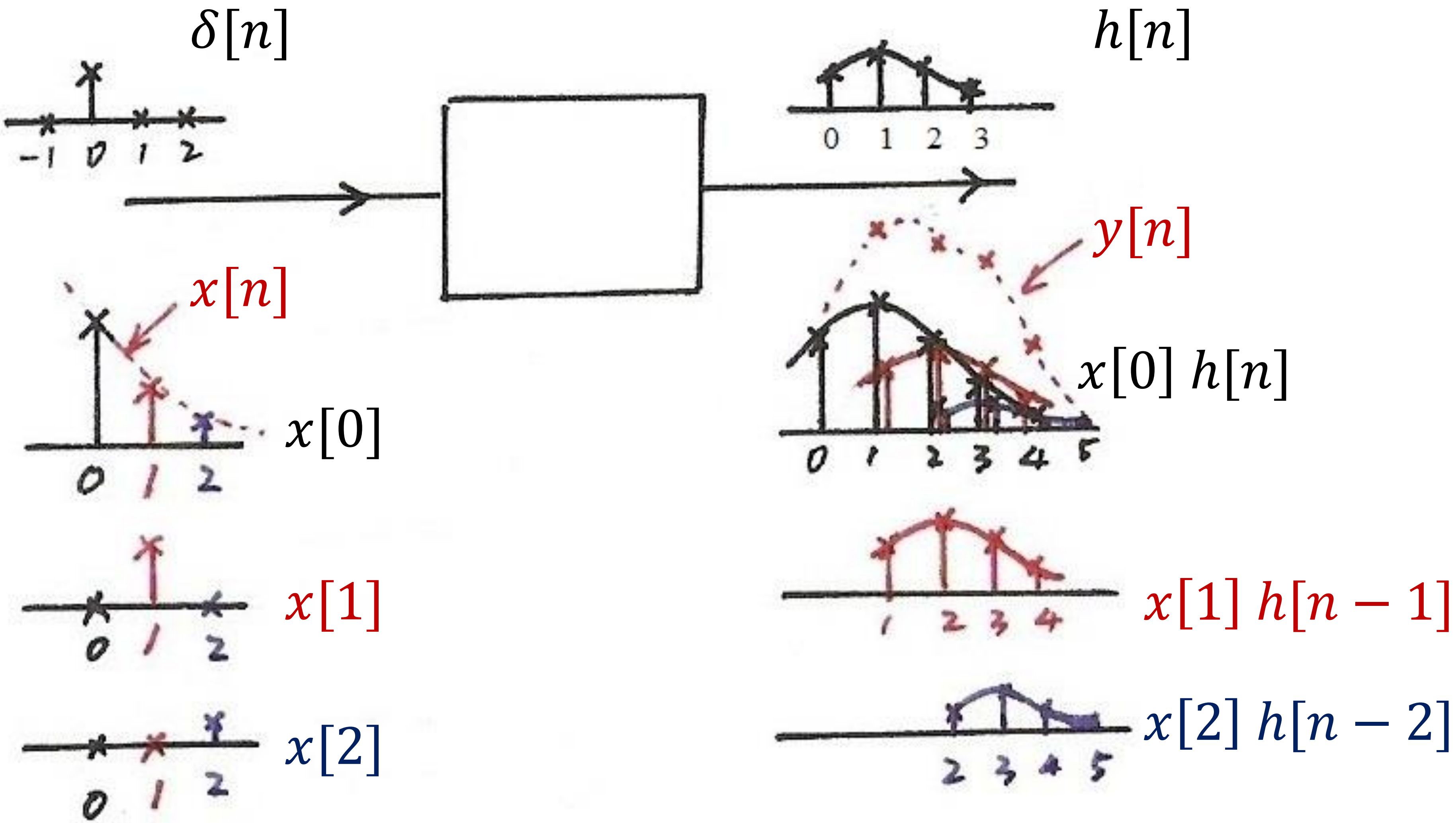
$$\sum_k \underset{\substack{\uparrow \\ x[k]}}{a_k}\, \underset{\substack{\uparrow \\ \delta[n-k]}}{x_k[n]} \longrightarrow \sum_k \underset{\substack{\uparrow \\ x[k]}}{a_k}\, \underset{\substack{\uparrow \\ h[n-k]}}{y_k[n]}$$

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv x[n] * h[n]}$$

Convolution Sum

See Fig2.3,p.80 of text

Input/Output Relation in Every Dimension



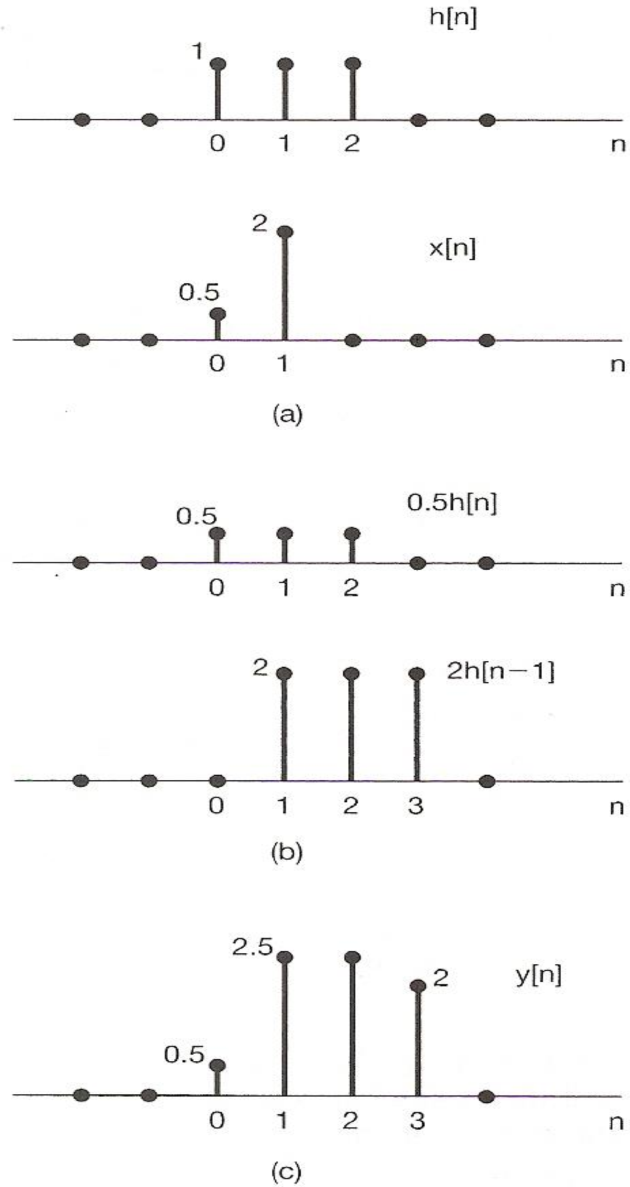


Figure 2.3 (a) The impulse response $h[n]$ of an LTI system and an input $x[n]$ to the system; (b) the responses or "echoes," $0.5h[n]$ and $2h[n-1]$, to the nonzero values of the input, namely, $x[0] = 0.5$ and $x[1] = 2$; (c) the overall response $y[n]$, which is the sum of the echos in (b).

- A Different Way to visualize the convolution sum
  - looked at on the index *k*

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Contribution to the output signal at time n

input signal

reflected-over version of $h[k]$ located at $k = n$

  - on the dummy index k, $h[k]$ is reflected over and shifted to $k=n$, weighted by $x[k]$ and summed to produce an output sample $y[n]$ at time n

See Figs 2.5, 2.6, 2.7, pp. 83-85 of text

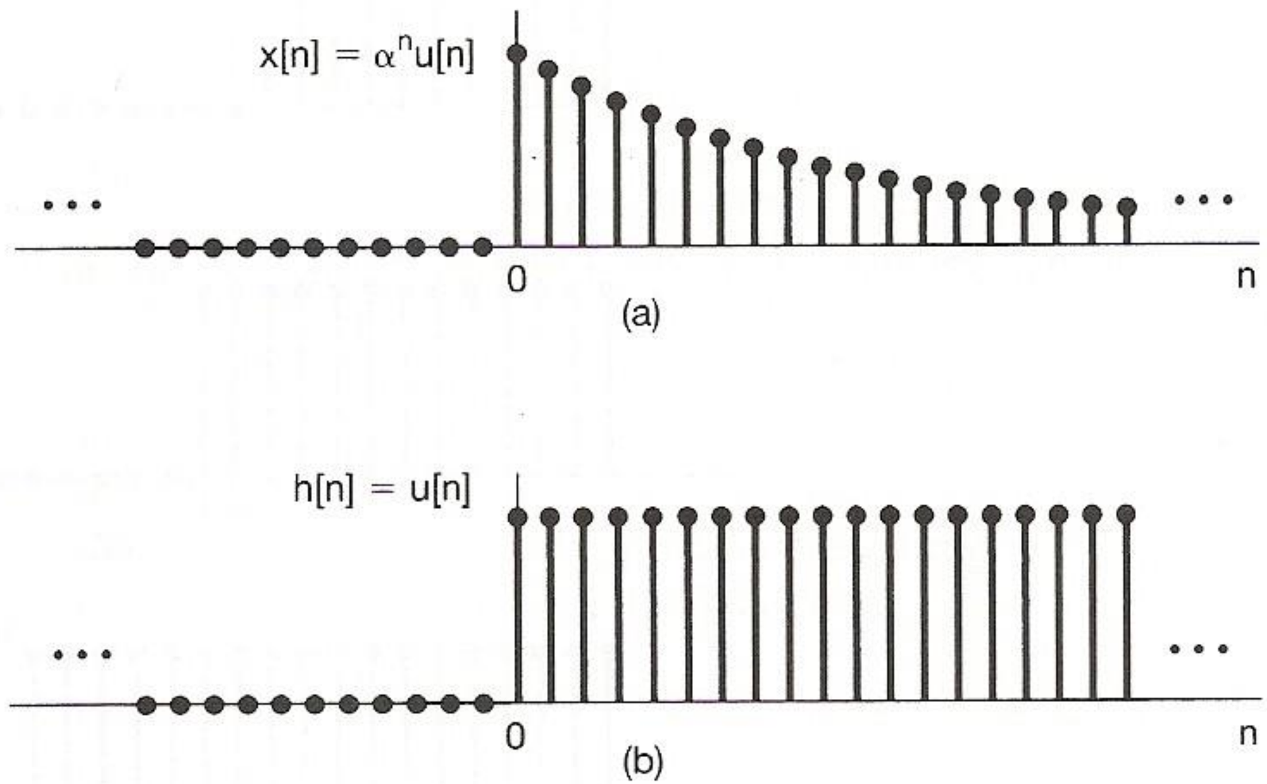
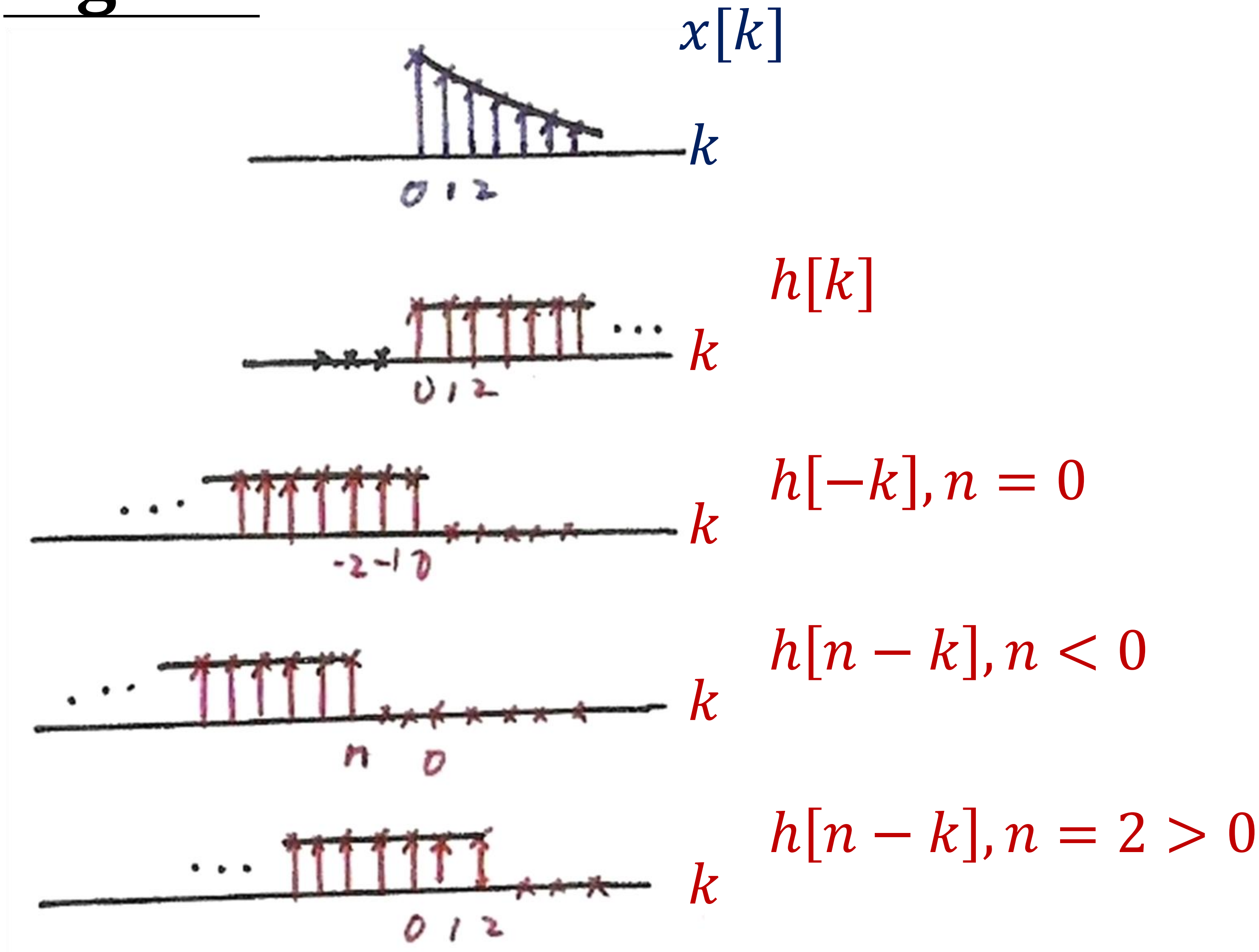


Figure 2.5 The signals $x[n]$ and $h[n]$ in Example 2.3.

Fig. 2.6



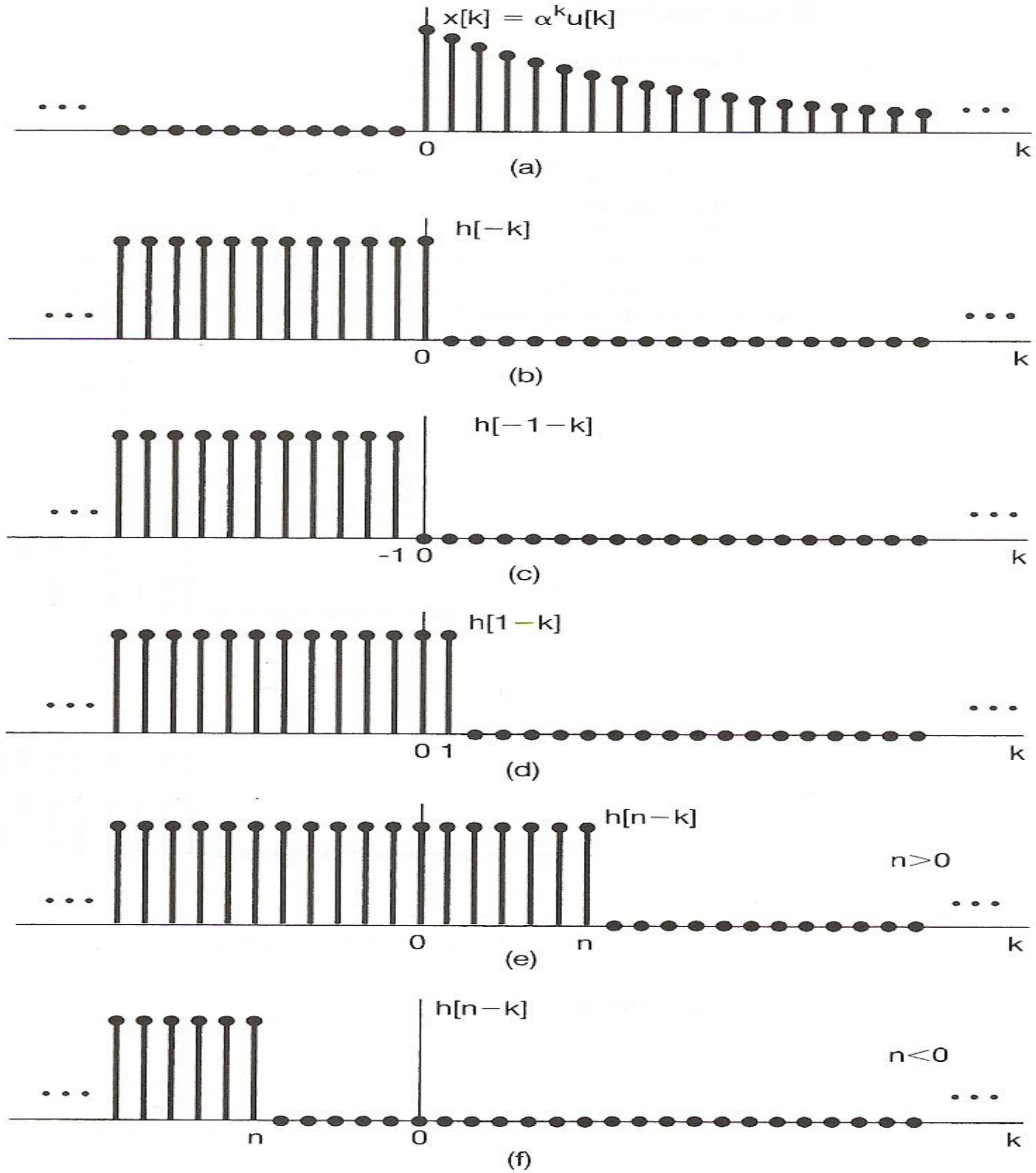


Figure 2.6 Graphical interpretation of the calculation of the convolution sum for Example 2.3.

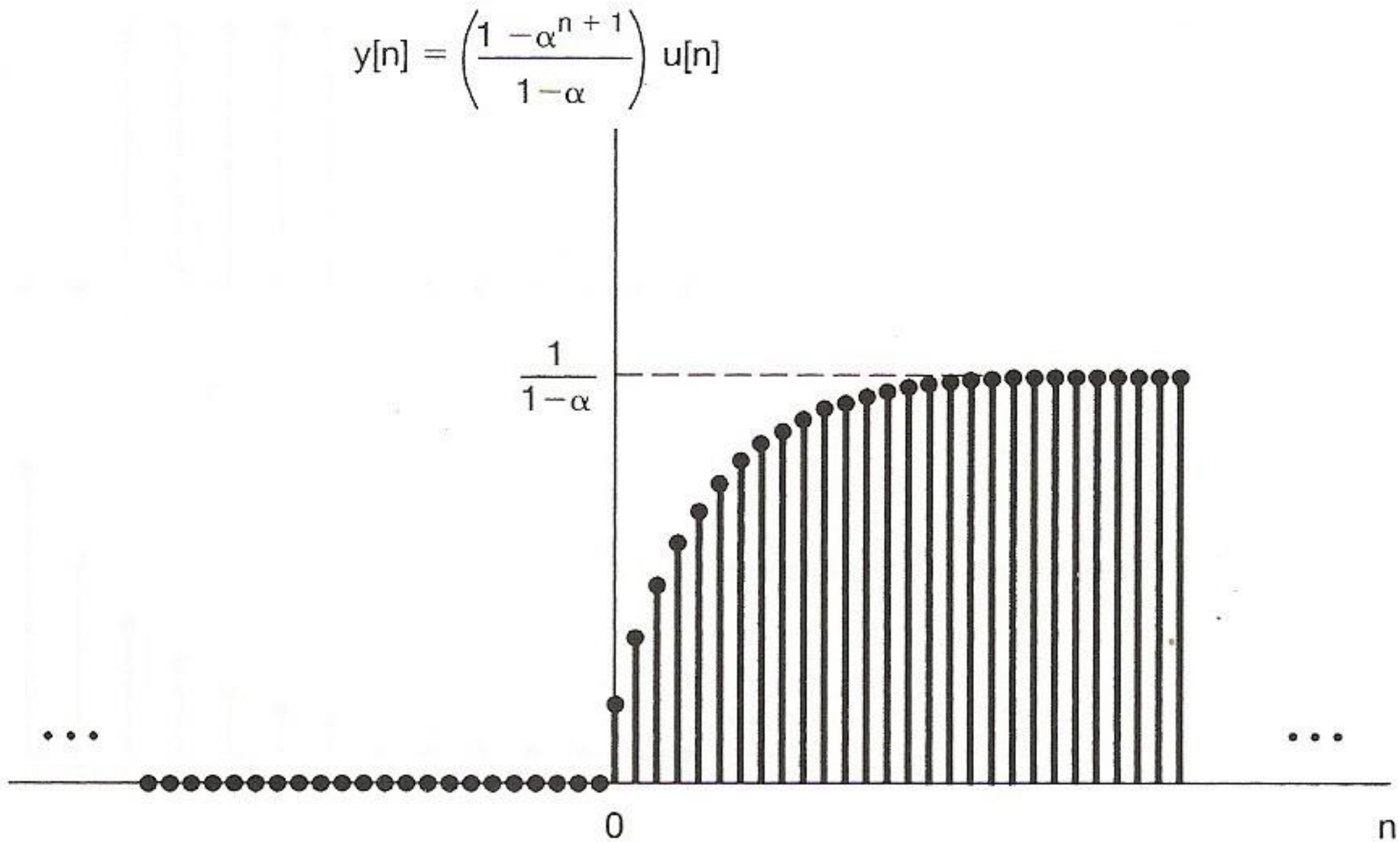


Figure 2.7 Output for Example 2.3.

- A linear time-invariant system is completely characterized by its unit impulse response

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

2.2 Vector Space Interpretation for Discrete-time Systems

Vector Space Concept

- The Set of All Discrete-time Signals Defined in (N_1, N_2) Forms A Vector Space, with Each Signal Being A Vector

  $\{x[n], x[n] \text{ is a discrete-time signal defined in } (N_1, N_2)\}=V$

  – definitions and properties

  $$x[n] + y[n], \mathrm{a}x[n]$$

Vector Space Concept

- Definition of Inner-product
  - Example

$$\left\{A=\left(a_1, a_2, \ldots\ldots\ldots a_N\right), a_i \in R\right\}=V$$

$$A \cdot B=\sum_{i=1}^{N} a_i b_i$$

  - Simple Definition

$$v_1, v_2 \in V, \quad v_1 \cdot v_2 : V \times V \rightarrow R\,(\text{or}\;C)$$

Vector Space Concept

- Definition of Inner-product
  - Physical Properties: magnitude, similarity

    $$A \cdot B = |A||B| \cos\theta$$

  - Mathematical Properties

    (i) *Commutative*

    $$(x[n]) \cdot (y[n]) = [(y[n]) \cdot (x[n])]^*$$

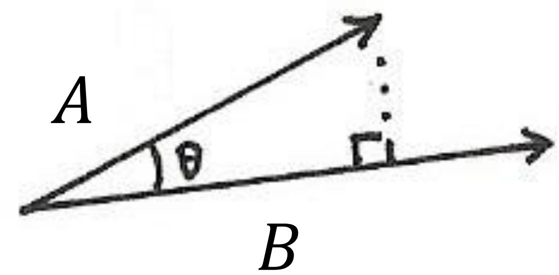
    (ii) *Linearity*

    $$\begin{aligned} &(x[n]) \cdot (a_1 y_1[n] + a_2 y_2[n]) \\ &= a_1 (x[n]) \cdot (y_1[n]) + a_2 (x[n]) \cdot (y_2[n]) \end{aligned}$$

    (iii) *Non-degeneracy*

    $(x[n]) \cdot (x[n]) \geq 0$ "=" iff $x[n] = 0$, all n

Inner Product

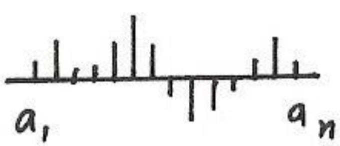


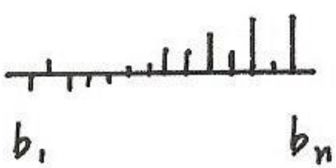
$$A \cdot B = \underbrace{|A||B|}_{\text{magnitude}} \; \underbrace{\cos\theta}_{\text{similarity}}$$

$|A| = (A \cdot A)^{1/2}$: magnitude, $\left(\sum_i a_i^2\right)^{1/2}$

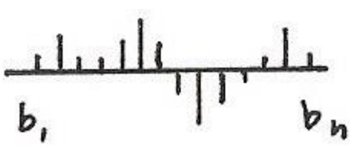
$\cos\theta = \frac{A \cdot B}{|A||B|}$: similarity, $-1 \leq \cos\theta \leq 1$

$A \cdot B = 0$: orthogonal

A: 

B_1: 

$A \cdot B_2 > A \cdot B_1$

B_2: 

Linearity in Inner Product

$$x \cdot (ay) = a(x \cdot y)$$

$$x \cdot (y_1 + y_2) = x \cdot y_1 + x \cdot y_2$$

magnitude
similarity

Commutative, Non-degeneracy

magnitude
similarity

Vector Space Concept

- Definition of Inner-product
  - Physical Properties: magnitude, similarity
  - For the vector space of signals defined in (N_1, N_2)

$$(x[n])\cdot(y[n]) : V \times V \to R(\text{or } C)$$

$$(x[n])\cdot(y[n]) = \sum_{n=N_1}^{N_2} x[n]y^*[n]$$

  - Inner-product Space
  - (N_1, N_2) extended to ($-\infty$, ∞)

Vector Space Concept

- Magnitude/Similarity

$$\|x[n]\| = [(x[n]) \cdot (x[n])]^{\frac{1}{2}}$$

$$S(x[n], y[n]) = \frac{(x[n]) \cdot (y[n])}{\|x[n]\| \, \|y[n]\|}$$

Vector Space Concept

- Orthonormal bases
  - orthogonal

$$(\Phi_i[n])\cdot(\Phi_j[n]) = \sum_{n=N_1}^{N_2}\Phi_i[n]\cdot\Phi_j^*[n] = 0$$

  - orthonormal

$$\{\Phi_k(n), k = 1, 2, ...\mathrm{M}\}$$

$$(\Phi_i[n])\cdot(\Phi_j[n]) = 0, \quad i \neq j$$

$$= 1, \quad i = j$$

  - orthonormal basis

    any vector (signal) in the vector space can be expanded by the set of orthonormal signals

$$x[n] = \sum_{k=1}^{M} x_k \Phi_k[n] \qquad (合成)$$

System Characterization

- Vector Space

  $$\{x[n],\ x[n]\ \text{defined for} -\infty < n < \infty\} = V$$

  - Orthonormal bases

    $$\{\ \Phi_k[n] = \delta[n-k],\quad \mathrm{k} = 0, \pm 1, \pm 2, \ldots\ldots\}$$

    time-shifted unit impulses, dim = ∞

  - A signal expanded by this set of orthonormal basis

    $$x[n] = \sum_{k=-\infty}^{\infty} \underbrace{x[k]}_{x_k}\, \underbrace{\delta[n-k]}_{\Phi_k[n]}$$

    (合成)

System Characterization

- A Different View of the Sifting Property
  - 3-dim vector space

$$A = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$a_2 = A \cdot \hat{j} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot \hat{j}$$

  - Sifting property

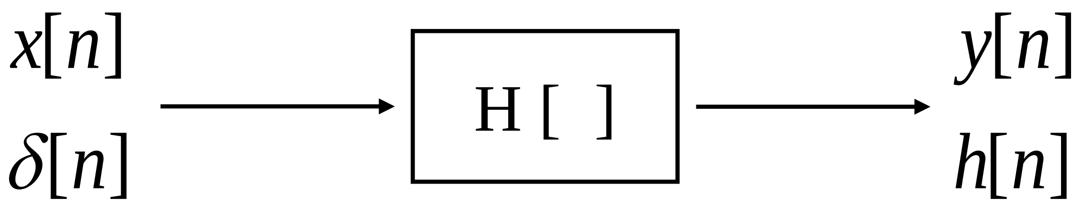
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

(分析)

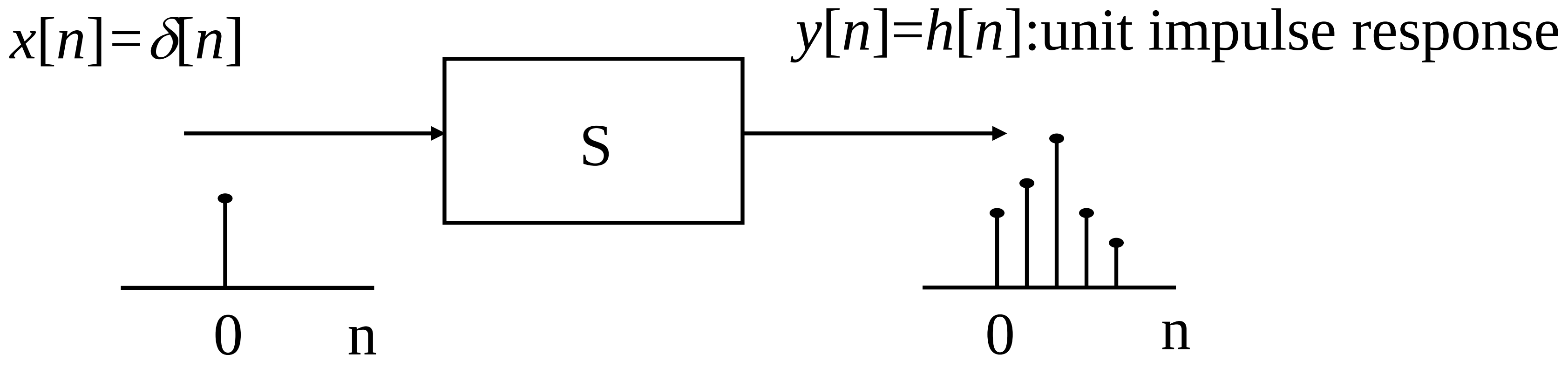
$$(x[k]) \cdot \left(\Phi_n[k]\right)$$

System Characterization

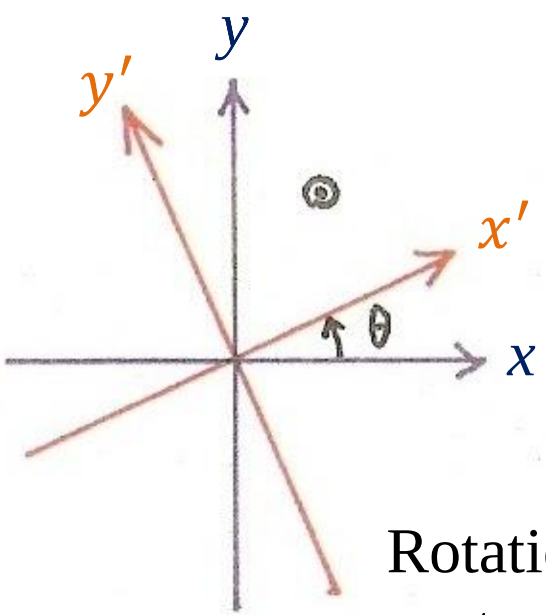
- Transformation of Vectors
  - A System is a Transformation which maps one vector $x[n]$ to another vector $y[n]$ (linear, time-invariant)



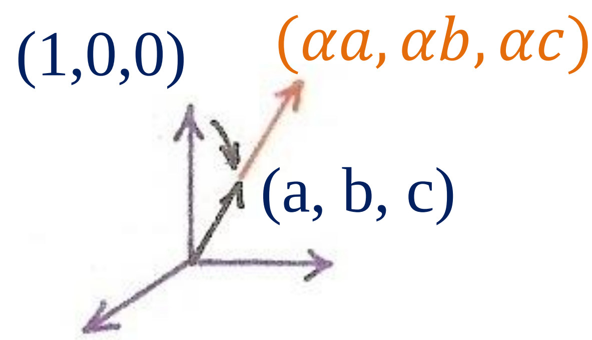
- Defining the output for an unit impulse input as the Unit Impulse Response **(P.10 of 2.0)**



- Rotation as a vector transformation



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

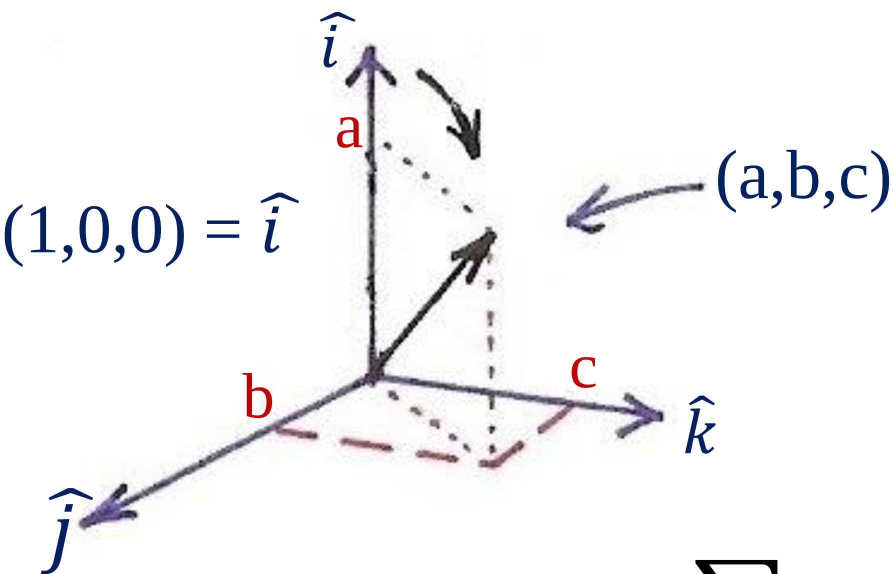


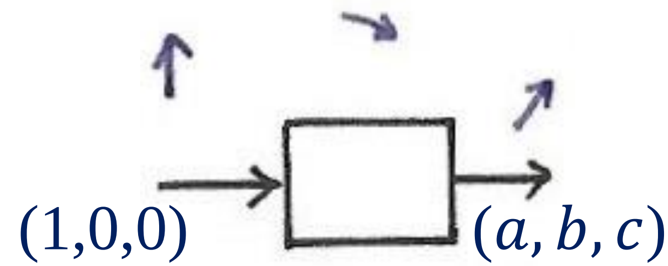
Rotation as a
vector transformation

Rotation plus
vector scaling

$(1, 0, 0) \rightarrow (0.9, 0.3, 0.3)$ $\quad (1, 0, 0, 0, \cdots) \rightarrow (0.9, 0.3, 0.2, \cdots)$
$(0, 1, 0) \rightarrow (0.3, 0.9, 0.3)$ $\quad (0, 1, 0, 0, \cdots) \rightarrow (\quad 0, 0.9, 0.3, 0.2, \cdots)$
$(0, 0, 1) \rightarrow (0.3, 0.3, 0.9)$ $\quad (0, 0, 1, 0, \cdots) \rightarrow (\quad 0, \quad 0, 0.9, 0.3, 02, \cdots)$

Splitting a single basis vector into a few more basis vectors





$$\delta[n] \rightarrow h[n]$$

$$\delta[n-k] \rightarrow h[n-k]$$

$$\sum_k x[k]\ \delta[n-k] \rightarrow \sum_k x[k]\ h[n-k]$$

– transforming a single basis vector enough to describe the transformation of any vectors

System Characterization

- Transformation of Vectors
  - unit impulse response is the transformation of $\delta[n]$. A basis vector is transformed to a set of other bases
  - superposition property for convolution sum

$$\sum_k \underset{\underset{x[k]}{\uparrow}}{a_k}\, \underset{\underset{\delta[n-k]}{\uparrow}}{x_k[n]} \rightarrow \sum_k \underset{\underset{x[k]}{\uparrow}}{a_k}\, \underset{\underset{h[n-k]}{\uparrow}}{y_k[n]}$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

System Characterization

- Linear, Time-invariant Transformation of vectors can be expressed by a matrix

$$x \longrightarrow \boxed{H = [h_{nk}]} \longrightarrow y$$

$$y = Hx$$

$$[y_n] = [h_{nk}][x_k], \quad y_n = \sum_k h_{nk} x_k$$

$$h_{nk} = h[n-k] \longrightarrow \text{Convolution Sum}$$

System Characterization

- Matrix Representation of Convolution Sum

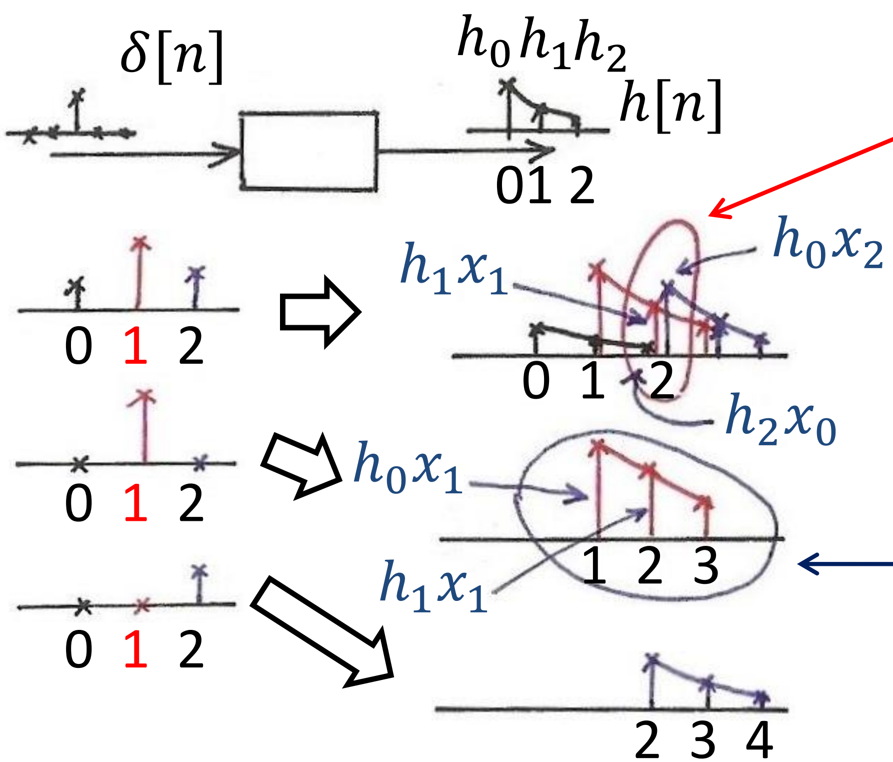
  example: $x[n]=\cdots 0,0,x_0,x_1,x_2,0,0,\cdots$

  $h[n]=\cdots 0,0,h_0,h_1,h_2,0,0,\cdots$

  $y[n]=\cdots 0,0,y_0,y_1,y_2,y_3,y_4,0,0,\cdots$

$$
\begin{bmatrix} \vdots \\ 0 \\ 0 \\ y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ 0 \\ 0 \\ \vdots \end{bmatrix}
=
\left[\begin{array}{cc|ccc|cc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots h_0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\
\cdots h_1 & h_0 & 0 & 0 & 0 & 0 & 0 \cdots \\
\cdots h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 \cdots \\
\cdots 0 & h_2 & h_1 & h_0 & 0 & 0 & 0 \cdots \\
\cdots 0 & 0 & h_2 & h_1 & h_0 & 0 & 0 \cdots \\
\cdots 0 & 0 & 0 & h_2 & h_1 & h_0 & 0 \cdots \\
\cdots 0 & 0 & 0 & 0 & h_2 & h_1 & h_0 \cdots \\
\cdots 0 & 0 & 0 & 0 & 0 & h_2 & h_1 \cdots \\
\cdots 0 & 0 & 0 & 0 & 0 & 0 & h_2 \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right]
\begin{bmatrix} \vdots \\ 0 \\ 0 \\ x_0 \\ x_1 \\ x_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}
$$

Matrix Representation



$$\begin{bmatrix} \vdots \\ y_2 \\ \vdots \end{bmatrix} = [h_2\ h_1\ h_0] \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \cdot & h_0 & \cdot \\ \cdot & h_1 & \cdot \\ \cdot & h_2 & \cdot \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} \vdots \\ y_1 \\ y_2 \\ y_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} \cdot & h_0 & \cdot & \cdot & \cdot \\ \cdot & h_1 & h_0 & & \cdot \\ \cdot & h_2 & h_1 & h_0 & \cdot \\ \cdot & & h_2 & h_1 & \cdot \\ \cdot & & \cdot & h_2 & \cdot \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix}$$

$$\begin{matrix} & \cdot & \cdot & \cdot & \\ \cdot & h_0 & 0 & 0 & \cdot \\ \cdot & h_1 & h_0 & 0 & \cdot \\ \cdot & h_2 & h_1 & h_0 & \cdot \\ & \cdot & \cdot & \cdot & \end{matrix} \Longleftarrow \begin{matrix} & \cdot & \cdot & \cdot & \\ \cdot & h_{11} & h_{12} & h_{13} & \cdot \\ \cdot & h_{21} & h_{22} & h_{23} & \cdot \\ \cdot & h_{31} & h_{32} & h_{33} & \cdot \end{matrix}$$

System Characterization

- Matrix Representation of Convolution Sum
  - each component of the input vector is transformed to a set of other components of the output vector, and all these are superpositioned (superposition property)
  - each component of the output vector is contributed respectively by the transformed component of each input vector component (reflected, shifted, weighted sum)

2.3 Continuous-time System : the Convolution Integral

- Representing an arbitrary signal as an integral of impulses

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$$

See Fig 2.12 , p.91 of text

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

an impulse located at $t = \tau$ whose value is $x(\tau)$ (合成)

$$u(t) = \int_{0}^{\infty} \delta(t-\tau)d\tau$$ a special case

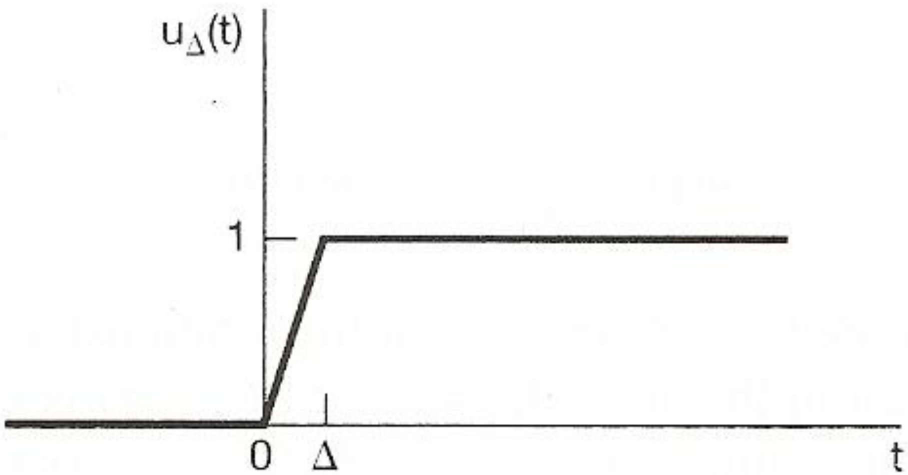


Figure 1.33 Continuous approximation to the unit step, $u_\Delta(t)$.

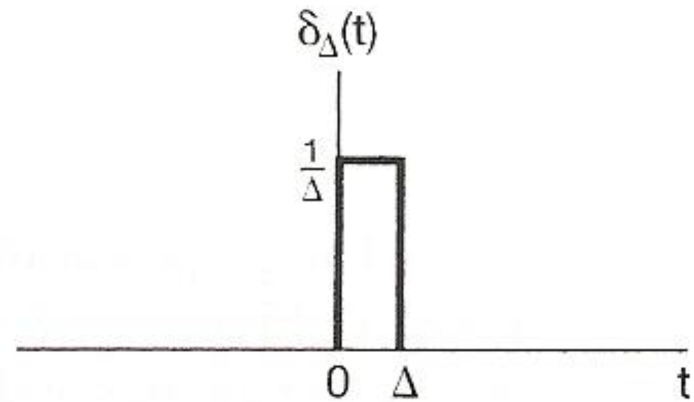


Figure 1.34 Derivative of $u_\Delta(t)$.

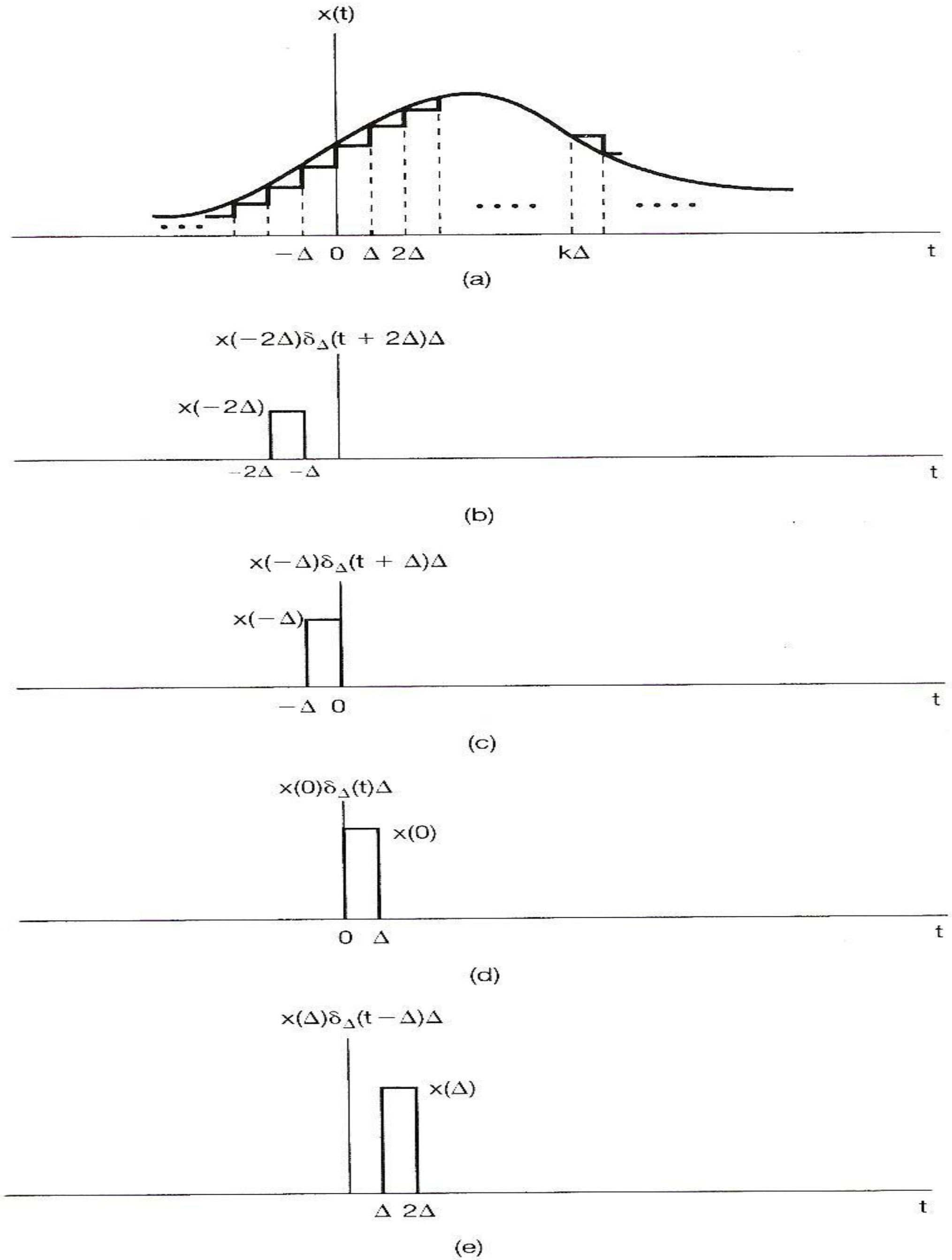


Figure 2.12 Staircase approximation to a continuous-time signal.

Vector Space Representation of Discrete-time Signals (P.48 of 1.0)

- n extended to $\pm\infty$

$$\vec{x} = (\cdots, x_{-2}, x_{-1}, x_0, x_1, x_2, x_3, \cdots)$$

$$\widehat{v_0} = (\cdots, \quad 0, \quad 0, \ 1, \ 0, \ 0, \ 0, \cdots) = \delta[n]$$

$$\widehat{v_1} = (\cdots, \quad 0, \quad 0, \ 0, \ 1, \ 0, \ 0, \cdots) = \delta[n-1]$$

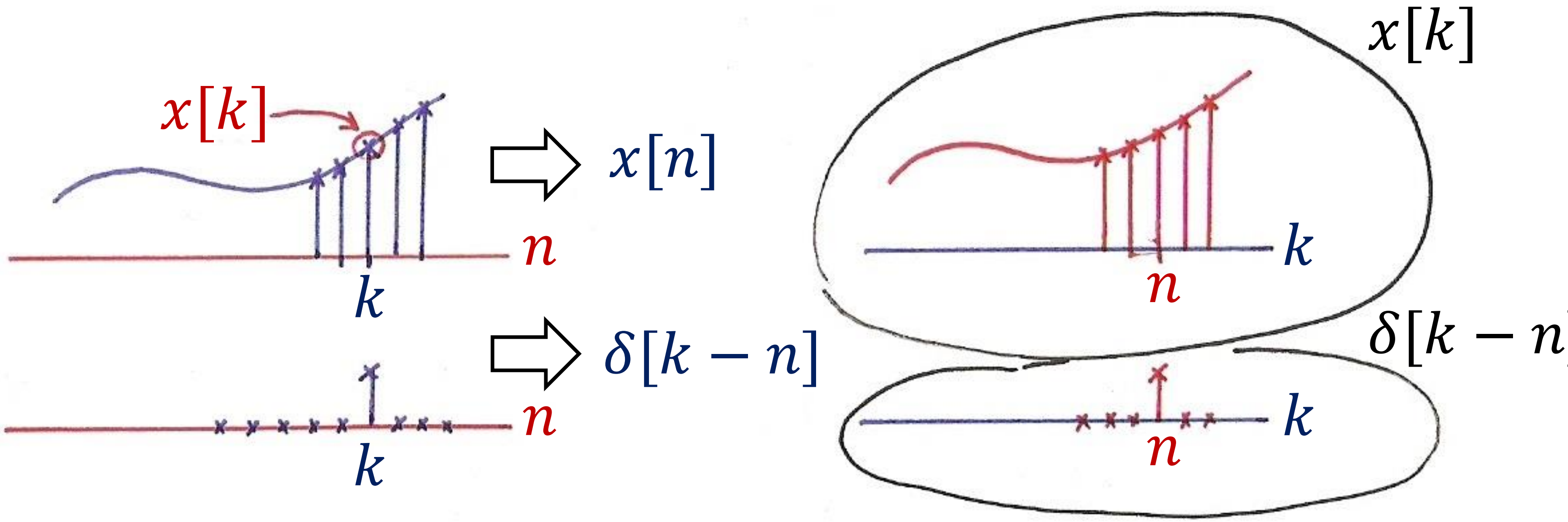
$$\widehat{v_k} = (\cdots, \quad 0, \quad 0, \ 0, \cdots, \ 0, \ 1, \cdots) = \delta[n-k]$$

$$\{\delta[n-k], \ k\text{: inter}\} \Longrightarrow \vec{x} = \sum_i x_i \widehat{v}_i \quad \text{合成}$$

$$x_j = \vec{x} \cdot \widehat{v}_j \quad \text{分析}$$

<u>Vector Space Interpretation</u> (P.8 of 2.0)

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta\,[n-k] \quad (\text{是合成也是分析})$$



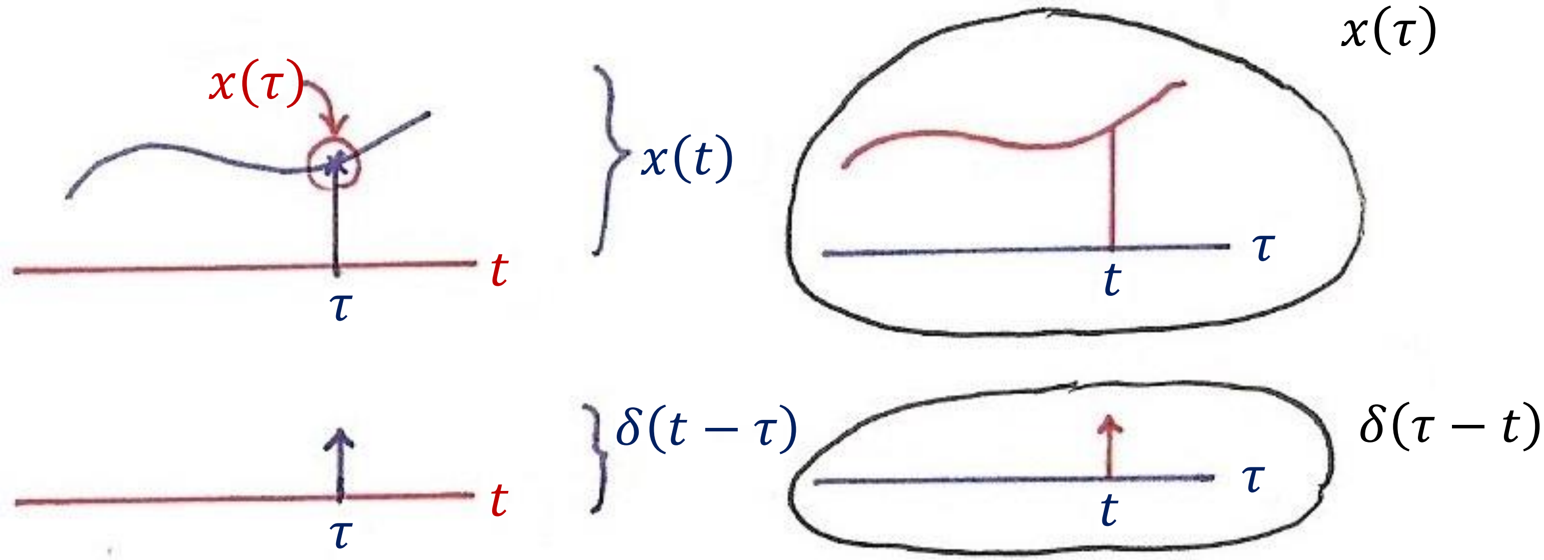
$$\vec{A} = \sum_{k} a_k\,\widehat{v_k} \quad (\text{合成})$$

$$a_j = \vec{A}\cdot\widehat{v_j} \quad (\text{分析})$$

$$a_n = \vec{A}\cdot\widehat{v_n} \; (\vec{A}\cdot\vec{B} = \sum_k a_k\, b_k)$$

Vector Space Representation

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)d\tau \text{(是合成也是分析)}$$



$$\vec{A} = \sum_k a_k\,\widehat{v_k} \text{ (合成)}$$

$$a_j = \vec{A}\cdot\widehat{v_j} \text{ (分析)}$$

discrete-time: $\vec{A}\cdot\vec{B} = \sum_n a[n]\,b^*[n]$

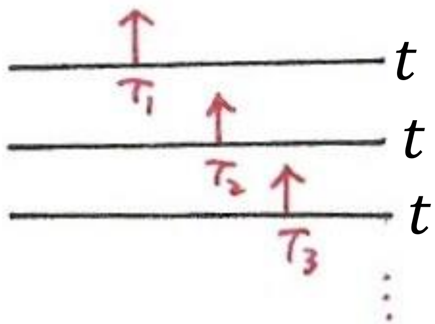
continuous-time: $\vec{A}\cdot\vec{B} = \int_{-\infty}^{\infty} a(t)\,b^*(t)\,dt$

Continuous-time Vector Space

$\{\, x(t) | \tau_2 \leq t \leq \tau_1 \,\}$ or $\{\, x(t) | -\infty < t < \infty \,\}$

$ax(t), x_1(t) + x_2(t)$

$\{\, \delta(t-\tau), -\infty < \tau < \infty \,\}$ is a set of basis



$\delta(t-\tau_1)$

$\delta(t-\tau_2)$

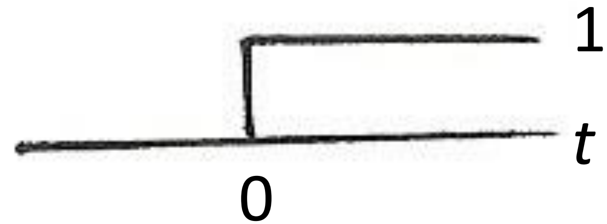
$\delta(t-\tau_3)$

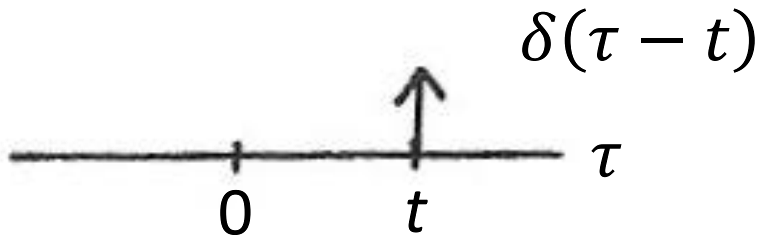
$$[x(t)] \cdot [y(t)] = \int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt$$

$$|x(t)| = \left| \int_{-\infty}^{\infty} |x(t)|^2\, dt \right|^{1/2}$$

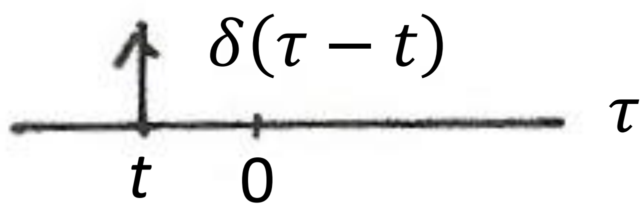
Unit Step

$$u(t) = \int_0^{\infty} \delta(t-\tau)\, d\tau$$





$t > 0$



$t < 0$

– running integral

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau, \tau = t - \tau'$$

- Representing an arbitrary signal as an integral of impulses
  - Sifting Property : (分析)

    Looked at on the index τ, $\delta(t-\tau)$ located at $\tau = t$ shifts the value $x(t)$ out of the function $x(\tau)$

    See Fig 2.14 ,p.94 of text

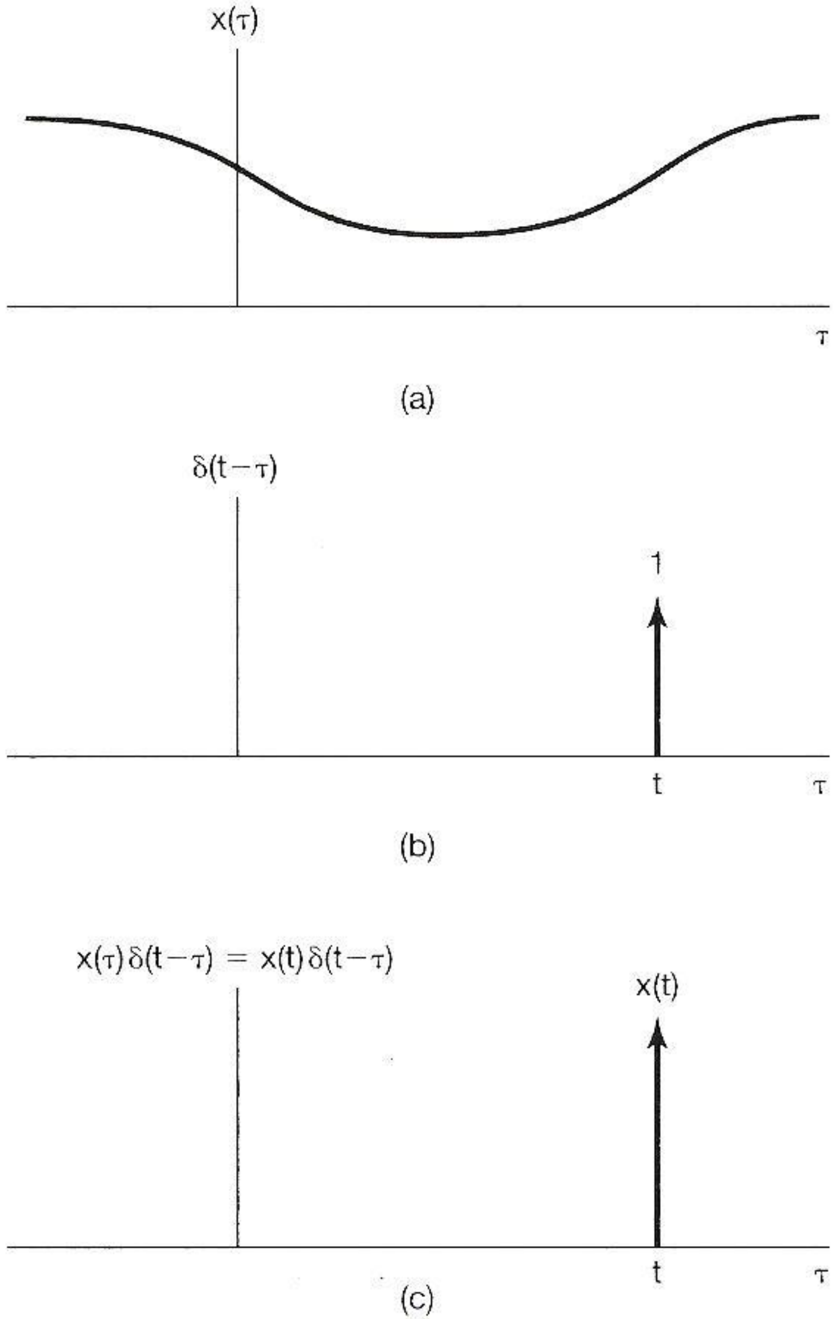
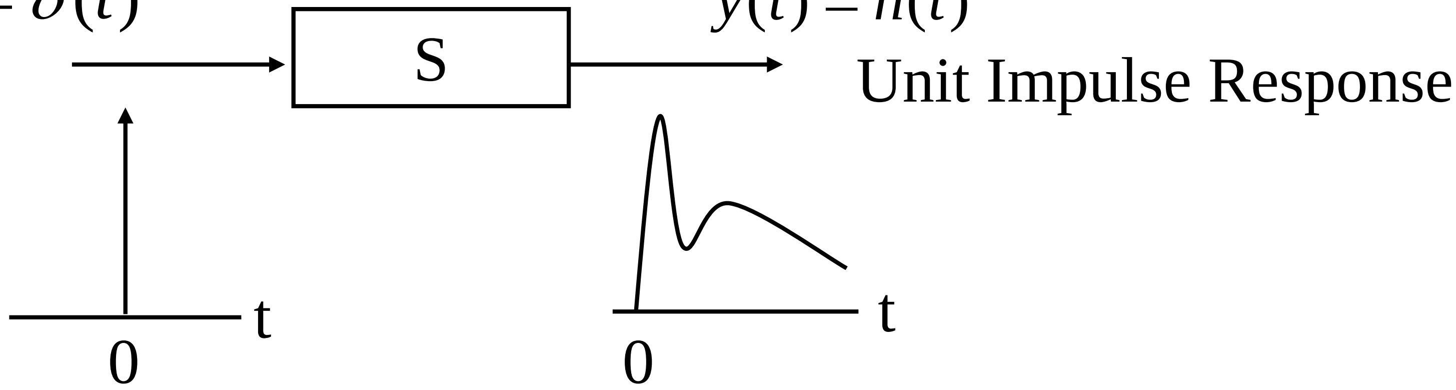


Figure 2.14 (a) Arbitrary signal $x(\tau)$; (b) impulse $\delta(t-\tau)$ as a func of τ with t fixed; (c) product of thes two signals.

- Defining the output for an unit impulse input as the unit impulse response

$x(t) = \delta(t)$ $\quad$ $y(t) = h(t)$



- By Linearity (Superposition Property)
  - The output for an arbitrary input signal is the superposition of a series of "shifted, scaled unit impulse response"

$$\sum_k a_k x_k(t) \to \sum_k a_k y_k(t)$$

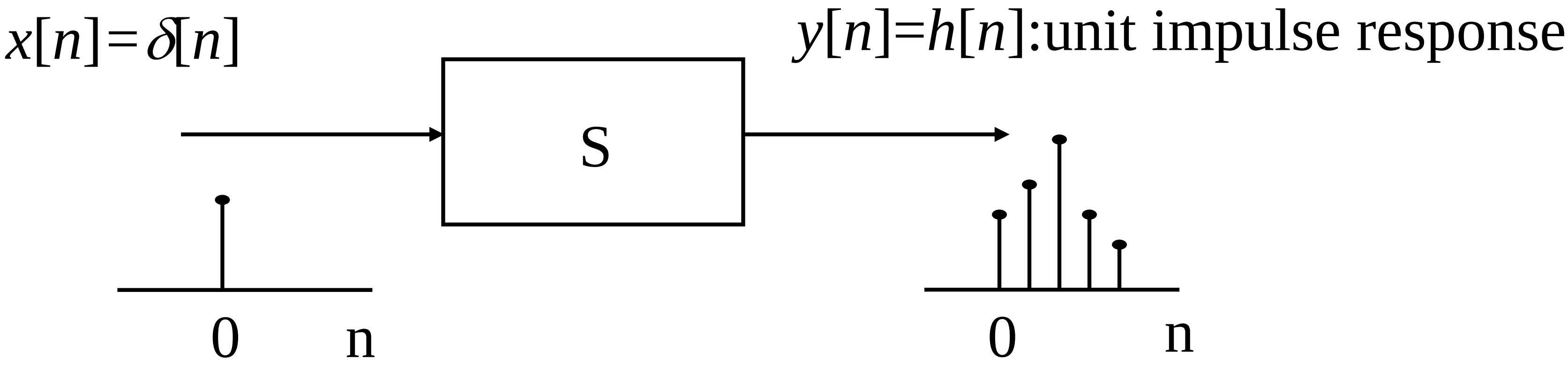
$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \equiv x(t) * h(t)}$$

Convolution Integral

- Defining the output for an unit impulse input as the Unit Impulse Response **(P.10 of 2.0)**



- By Linearity (Superposition Property) **(P.11 of 2.0)**
  - The output for an arbitrary input signal is the superposition of a series of "shifted, scaled unit impulse response"

$$\sum_k \underset{\substack{\uparrow \\ x[k]}}{a_k}\, \underset{\substack{\uparrow \\ \delta[n-k]}}{x_k[n]} \longrightarrow \sum_k \underset{\substack{\uparrow \\ x[k]}}{a_k}\, \underset{\substack{\uparrow \\ h[n-k]}}{y_k[n]}$$

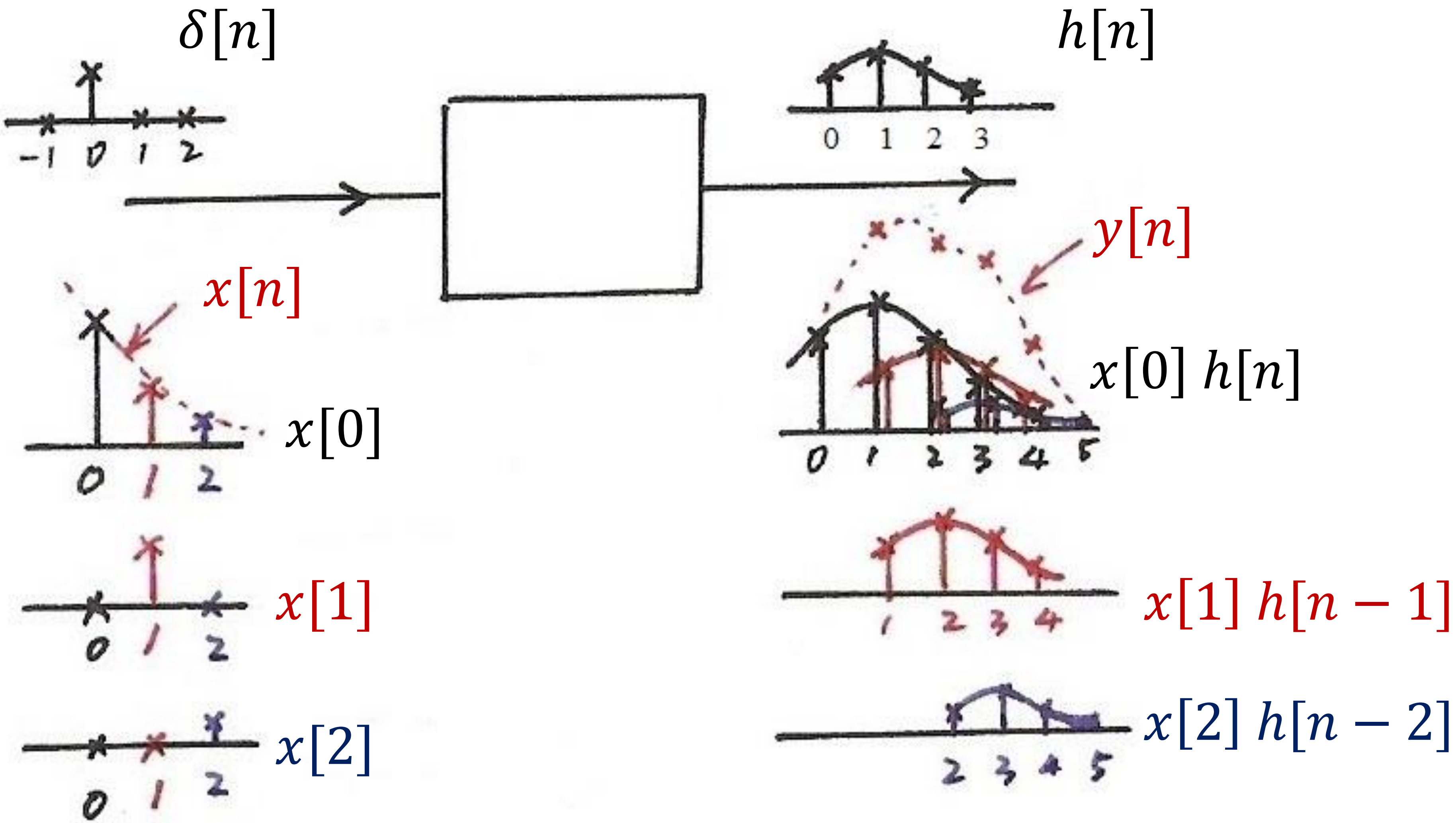
$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv x[n] * h[n]}$$

Convolution Sum

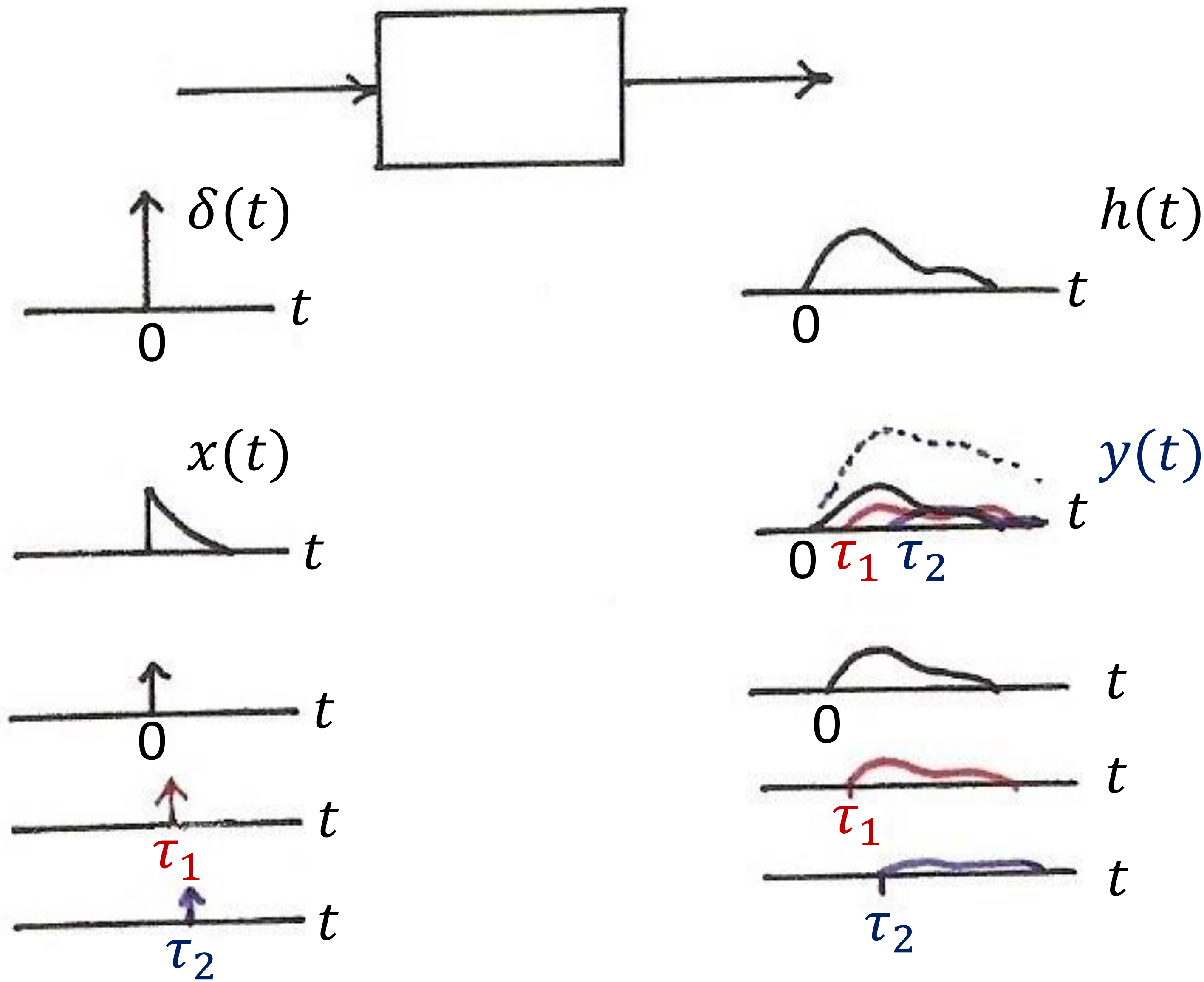
See Fig2.3,p.80 of text

Input/Output Relation in Every Dimension

(P.12 of 2.0)



Input/Output Relation in Every Dimension



- A Different Way to visualize the convolution integral
  - Look on the index τ

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

input signal

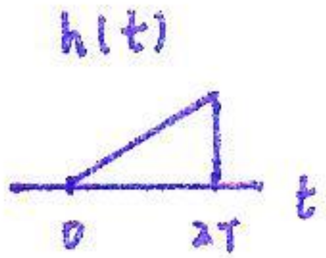
output signal at time t

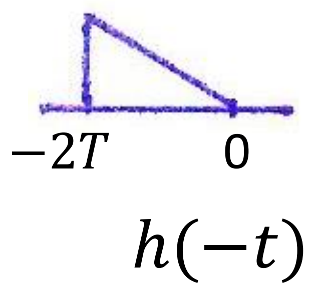
reflected-over version of $h(t)$located at $\tau = t$

  - On the dummy index τ, $h(t)$ is reflected over and shifted to $\tau = t$, weighted by $x(t)$ and integrated to produce the output value at time t, $y(t)$

    see Figs.2.19,2.20,2.21,pp.100-101 of text

- A linear time-invariant system is completely characterized by its unit impulse response





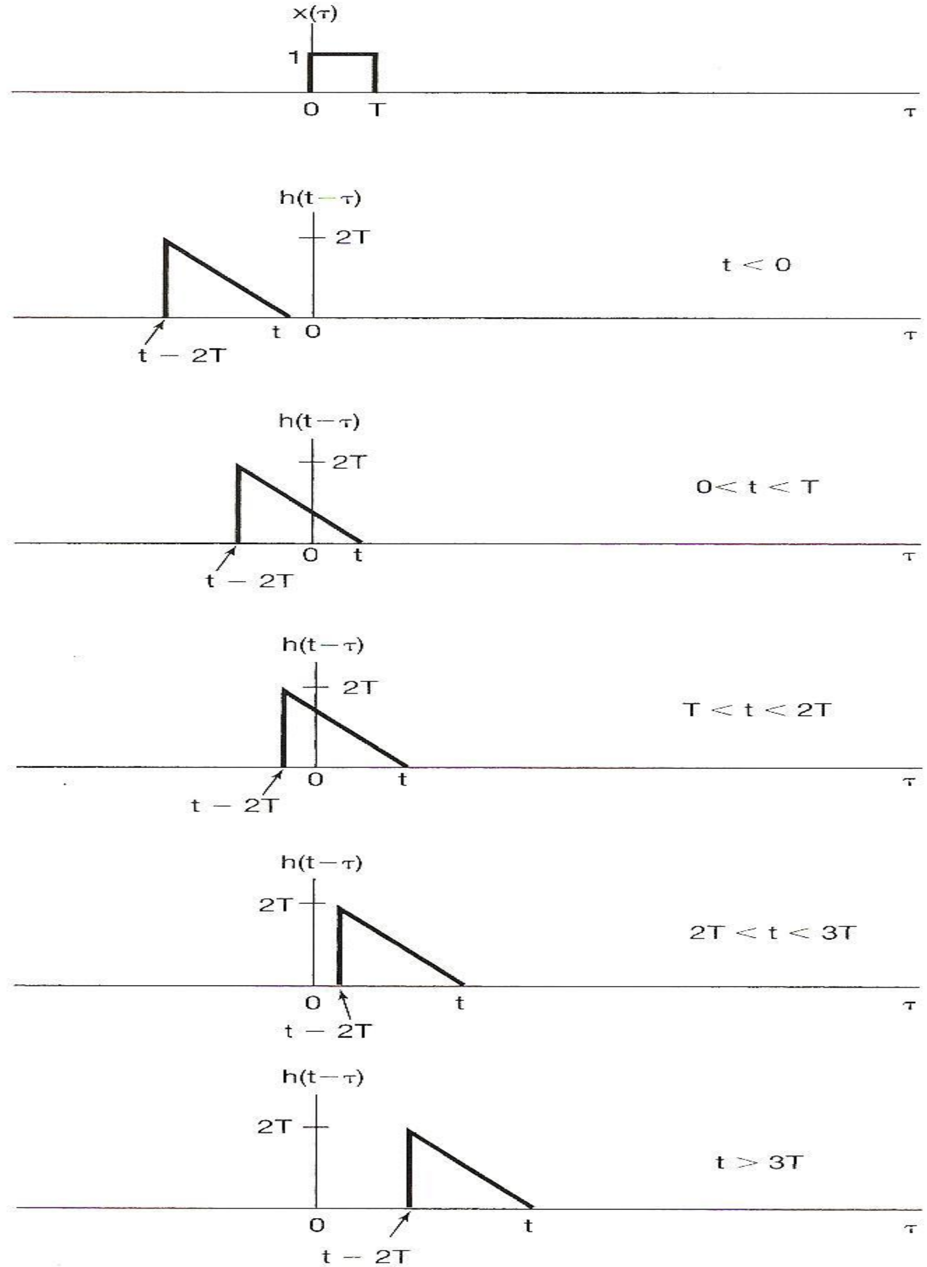


Figure 2.19 Signals $x(\tau)$ and $h(t-\tau)$ for different values of t for Example 2.7.

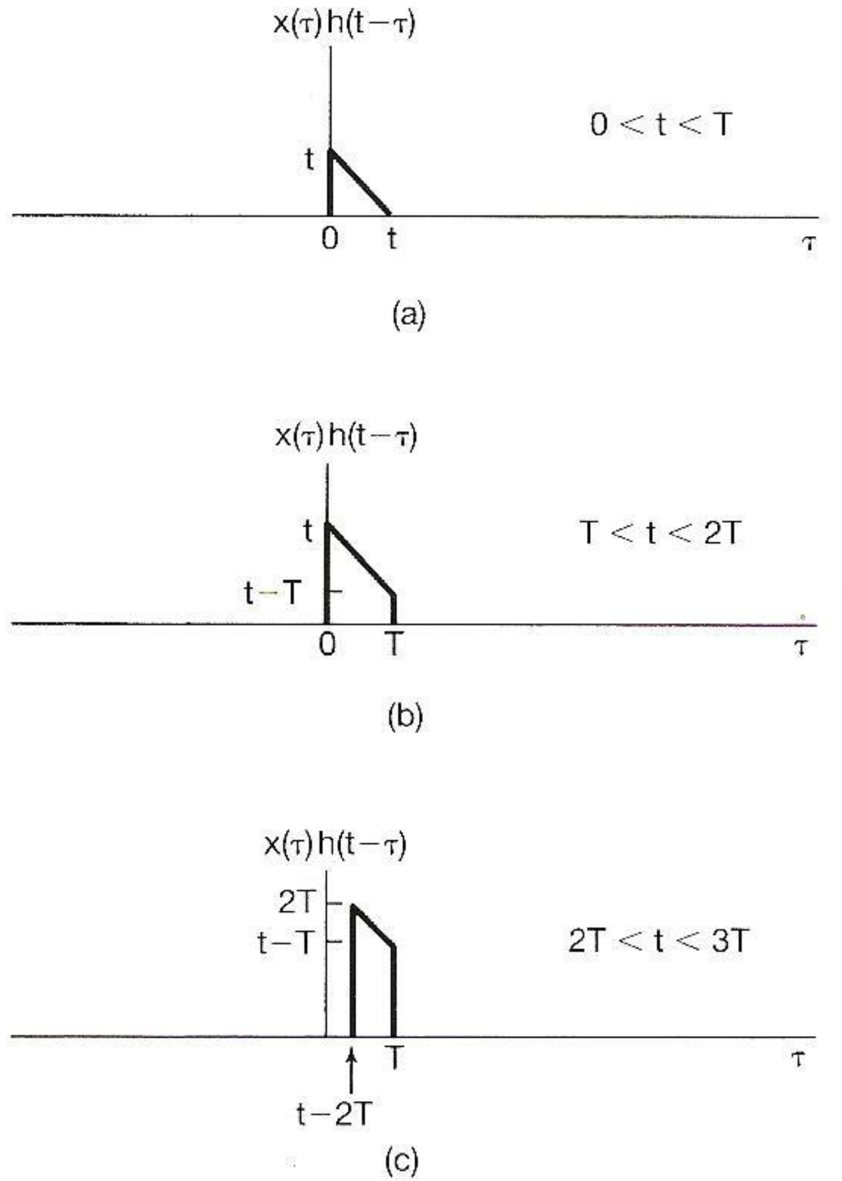


Figure 2.20 Product $x(\tau)h(t-\tau)$ for Example 2.7 for the three ranges of values of t for which this product is not identically zero. (See Figure 2.19.)

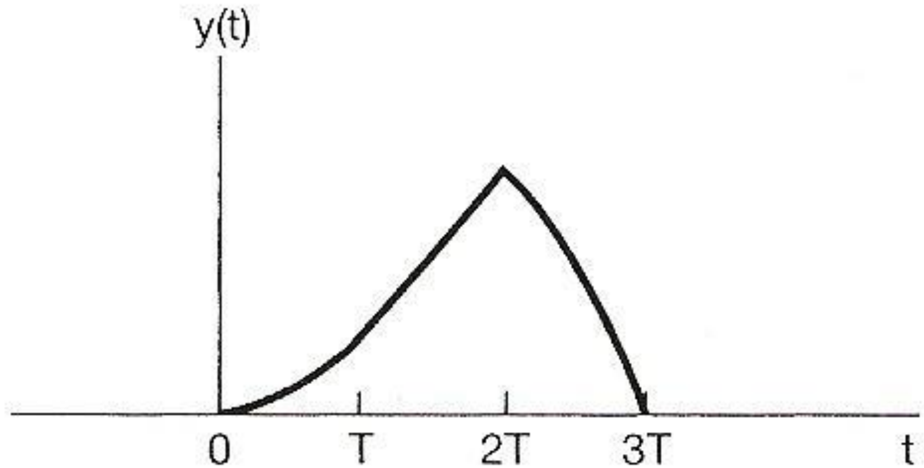


Figure 2.21 Signal $y(t) = x(t) * h(t)$ for Example 2.7.

2.4 Properties of Linear Time-invariant Systems

- Commutative Property

$$x[n]*h[n]=h[n]*x[n]$$

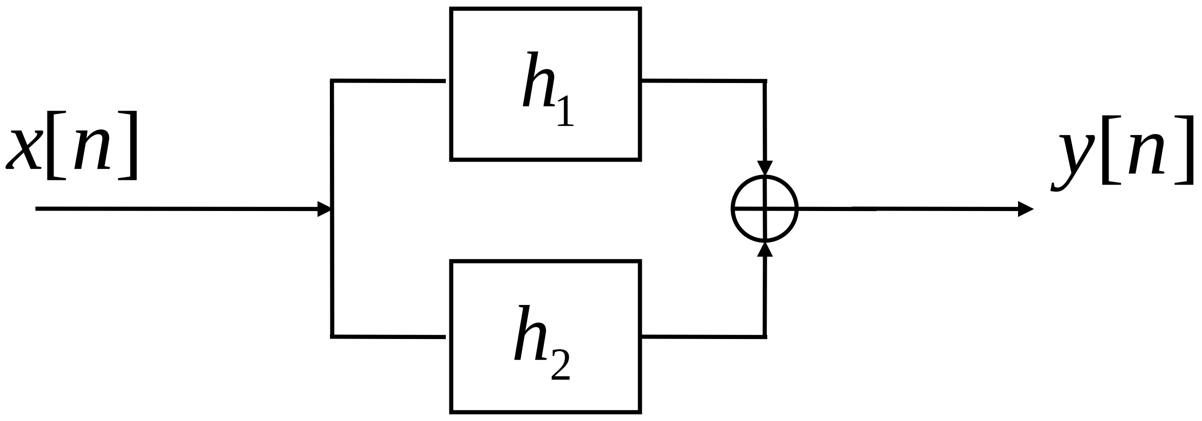
$$x(t)*h(t)=h(t)*x(t)$$

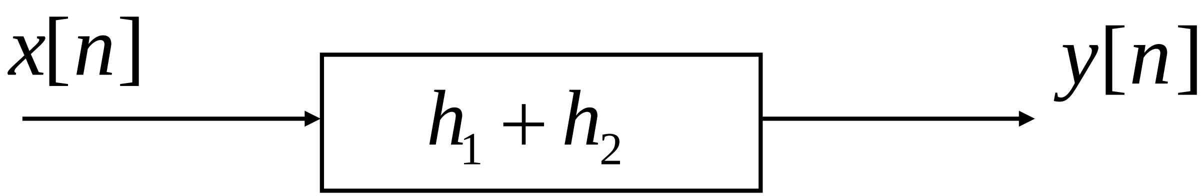
  – the role of input signal and unit impulse response is interchangeable, giving the same output signal
  – In evaluating the convolution sum or integral, the input signal can be reflected over and weighted by the unit impulse response

- Distributive Property

$$x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n]+x[n]*h_2[n]$$

$$x(t)*[h_1(t)+h_2(t)] = x(t)*h_1(t)+x(t)*h_2(t)$$



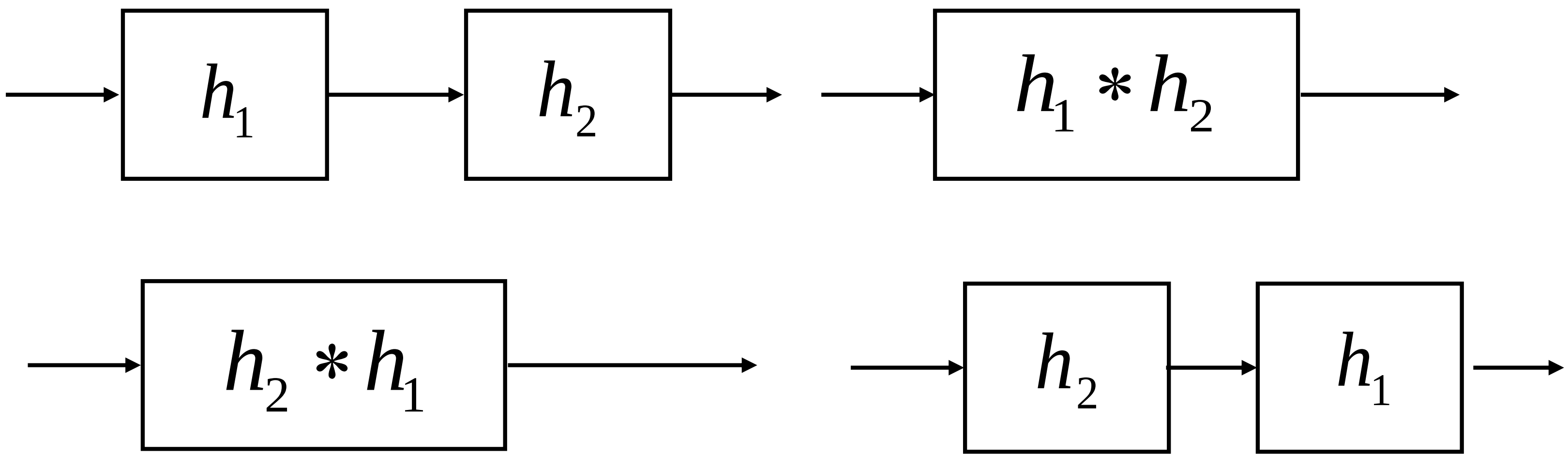


$$(x_1[n]+x_2[n])*h[n] = x_1[n]*h[n]+x_2[n]*h[n]$$

additive (linear) property

- Associative Property

$$x[n]*(h_1[n]*h_2[n]) = (x[n]*h_1[n])*h_2[n]$$



 – Cascade of two systems gives an unit impulse response which is the convolution of the unit impulse responses of the two individual systems
 – The behavior of a cascade of two systems is independent of the order in which the two systems are cascaded

- Causality
  - causal if $y[n]$ dose not depend on $x[k]$ for $k > n$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

  - Causal iff $h[n]=0$, $n < 0$

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

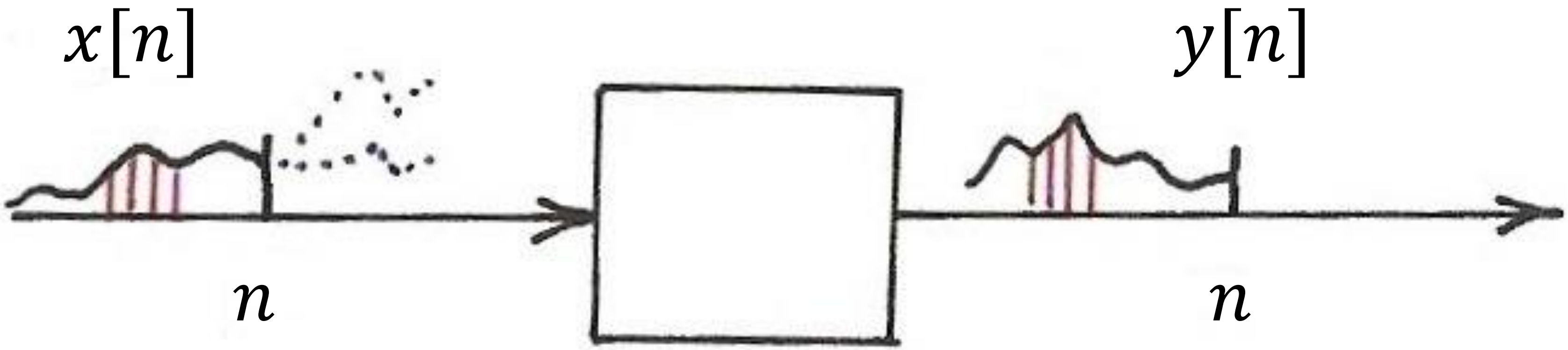
- Causality
  - continuous-time

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

  - Causal iff h(t)=0,t<0

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau$$

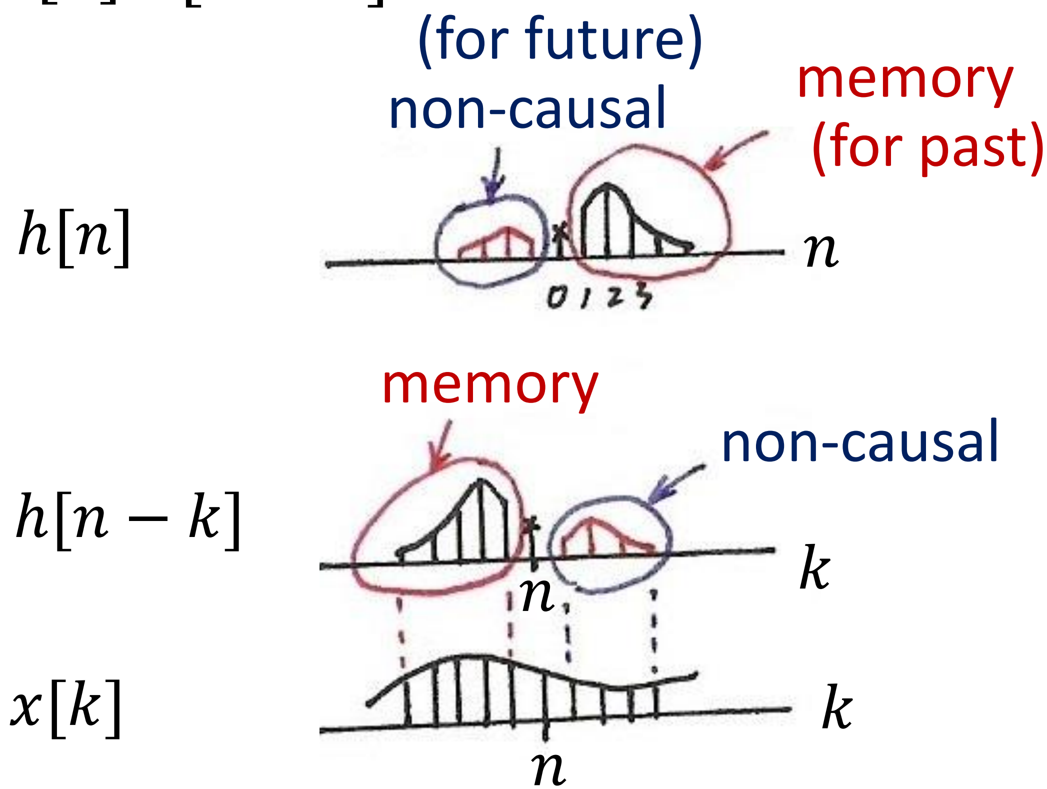
<u>Causality</u> (P.56 of 1.0)



$$y[n] = \sum_{k=-\infty}^{n+m} x[k]$$

Causality & Memory

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$



$$(\, h[n] = 0,\ n < 0 \,) \Leftrightarrow (\, h[n-k] = 0,\ k > n \,)\text{: Causality}$$

- Memoryless / with Memory
  - A linear, time-invariant, causal system is memoryless only

    if $h[n] = K\delta[n]$ $\quad h(t) = K\delta(t)$

    $y[n] = Kx[n]$ $\quad y(t) = Kx(t)$

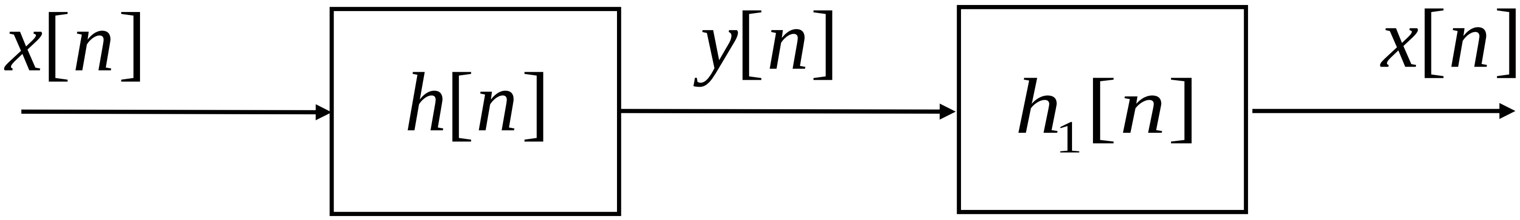
    if k=1 further, they are identity systems

$$y[n] = x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = x[n] * \delta[n]$$

$$y(t) = x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t) * \delta(t)$$

  - identity for convolution , i.e., convolution sum (or integral) with an unit impulse function gives the original signal

- Invertibility / Inverse system



$$h[n] * h_1[n] = \delta[n] \qquad h(t) * h_1(t) = \delta(t)$$

- Stability

stable if bounded input gives bounded output

$|x[n]| < B$, all n

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right| \le \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]| \le B \cdot \sum_{k=-\infty}^{\infty} |h[k]|$$

– Stable iff the impulse response is absolutely summable,

$$\sum_{k=-\infty}^{\infty} | h[k] | < \infty$$

or absolutely integrable,

$$\int_{-\infty}^{\infty} | h(t) | \, dt < \infty$$

the necessary condition can be proved

- Unit step response

output for an unit step function input

$$s[n] = u[n] * h[n] = \sum_{k=-\infty}^{n} h[k]$$ Running sum

$$h[n] = s[n] - s[n-1]$$ First difference

similarly

$$s(t) = \int_{-\infty}^{t} h(\tau) d\tau$$ Running integral

$$h(t) = \frac{ds(t)}{dt}$$ First derivative

2.5 Systems Described by Differential/Difference Equations

Continuous-time

- Differential Equation Specification for Input/Output Relationships

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

  - derived by physical phenomena and relationships
  - very often auxiliary conditions are needed to completely specify the system

Continuous-time

- Differential Equation Specification for Input/Output Relationships
  - the response $y(t)$ to an input $x(t)$ in general consists of two parts:

    (1) homogeneous solution (natural response): a solution for

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0$$

    (2) particular solution: to the complete differential equation

Continuous-time

- Initial Rest Condition for Causal Systems

$$x(t) = 0,\ t \le t_0 \rightarrow \quad y(t) = 0,\ t \le t_0$$

  - initial conditions

$$y(t_0) = \frac{dy(t_0)}{dt} = \frac{d^2 y(t_0)}{dt^2} = \ldots\ldots = \frac{d^N y(t_0)}{dt^N} = 0$$

Discrete-time

- Difference Equation Specification

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

  - derived by sequential behavior of different processes
  - auxiliary conditions needed
  - response $y(t)$ consists of two parts

    (1) homogeneous solution (natural response) for

$$\sum_{k=0}^{N} a_k y[n-k] = 0$$

    (2) particular solution

Discrete-time

- Initial Rest Condition for Causal Systems

$$x[n]=0,\ n\le n_0 \rightarrow \quad y[n]=0,\ n\le n_0$$

- Recursive Equation

$$y[n]=\frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k]-\sum_{k=1}^{N} a_k y[n-k]\right\}$$

  – output at time n expressed in terms of previous values of input/output

Discrete-time

- Recursive Equation

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right\}$$

  – N=0, reduced to a convolution sum

$$y[n] = \sum_{k=0}^{M} \left(\frac{b_k}{a_0}\right) x[n-k]$$

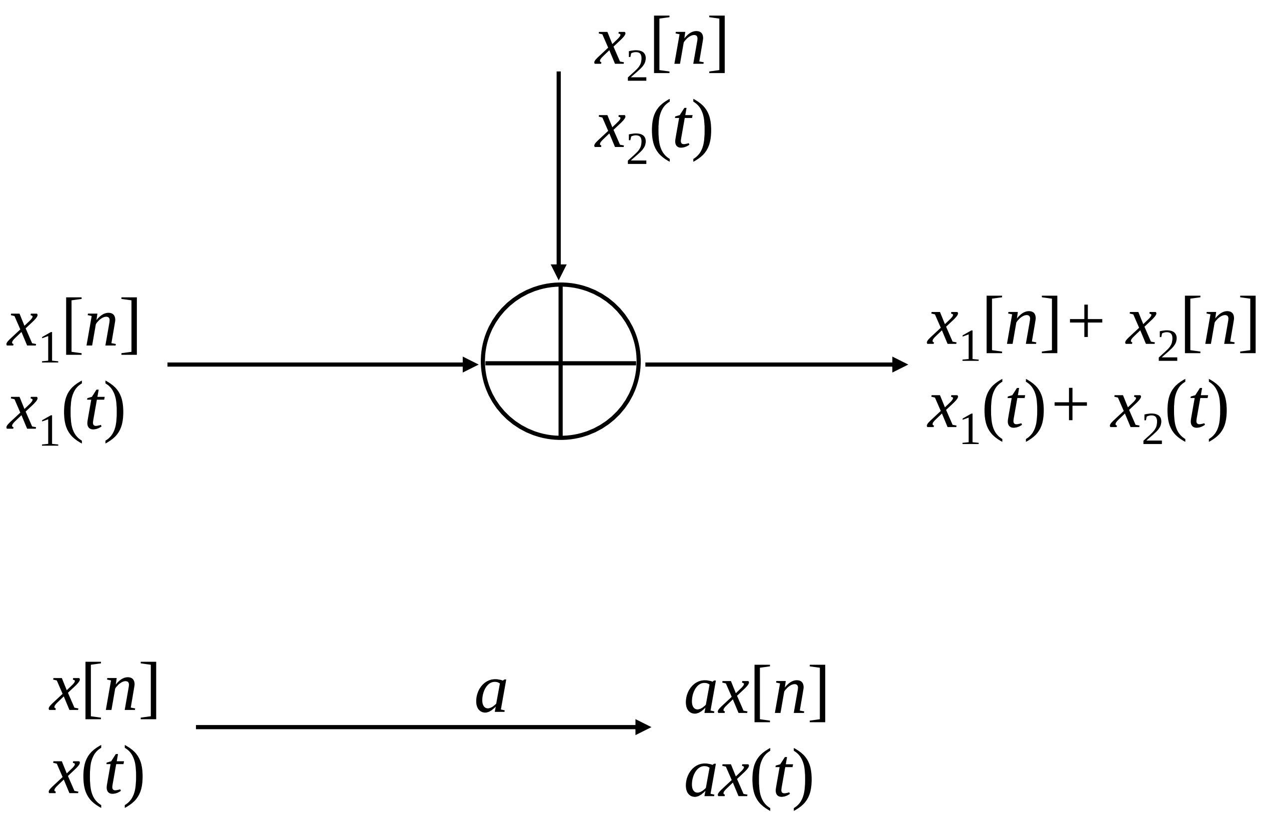
$$h[n] = \begin{cases} b_n/a_0, & 0 \le n \le M \\ 0, & \text{else} \end{cases}$$

Finite Impulse Response (FIR) systems

  – Infinite Impulse Response(IIR) Systems

Block Diagram Representation

- Elementary Operations



Block Diagram Representation

- Elementary Operations

$$x[n] \longrightarrow \boxed{\mathbf{D}} \longrightarrow x[n\text{-}1]$$

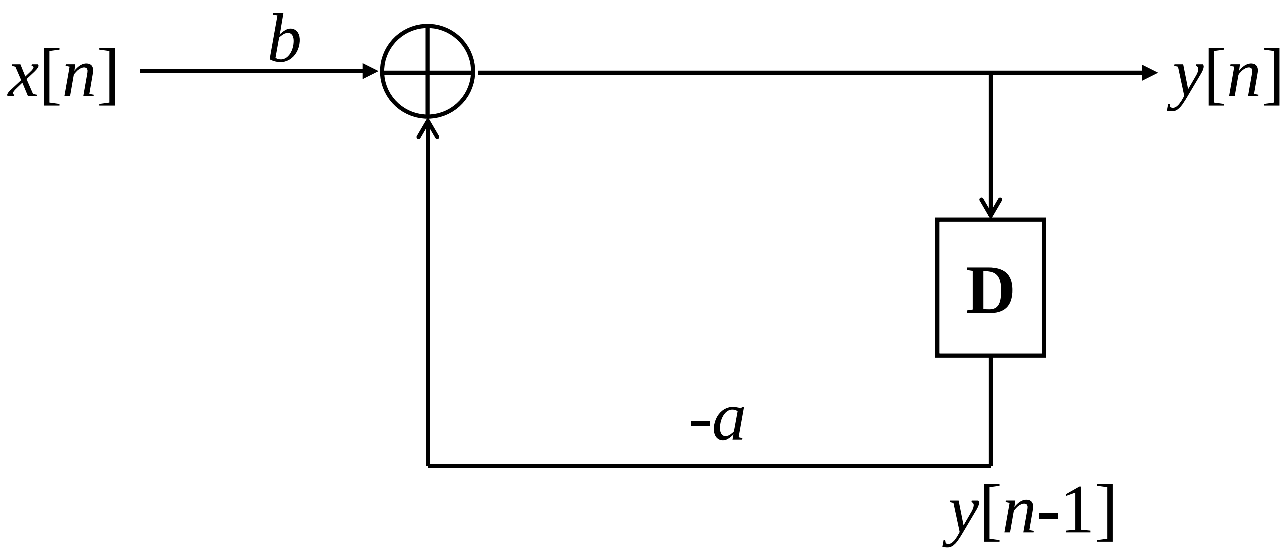
$$x(t) \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow \frac{dx(t)}{dt}$$

$$x(t) \longrightarrow \boxed{\int} \longrightarrow \int_{-\infty}^{t} x(\tau)\,d\tau$$

Block Diagram Representation

- An Example

$$y[n] + ay[n-1] = bx[n]$$

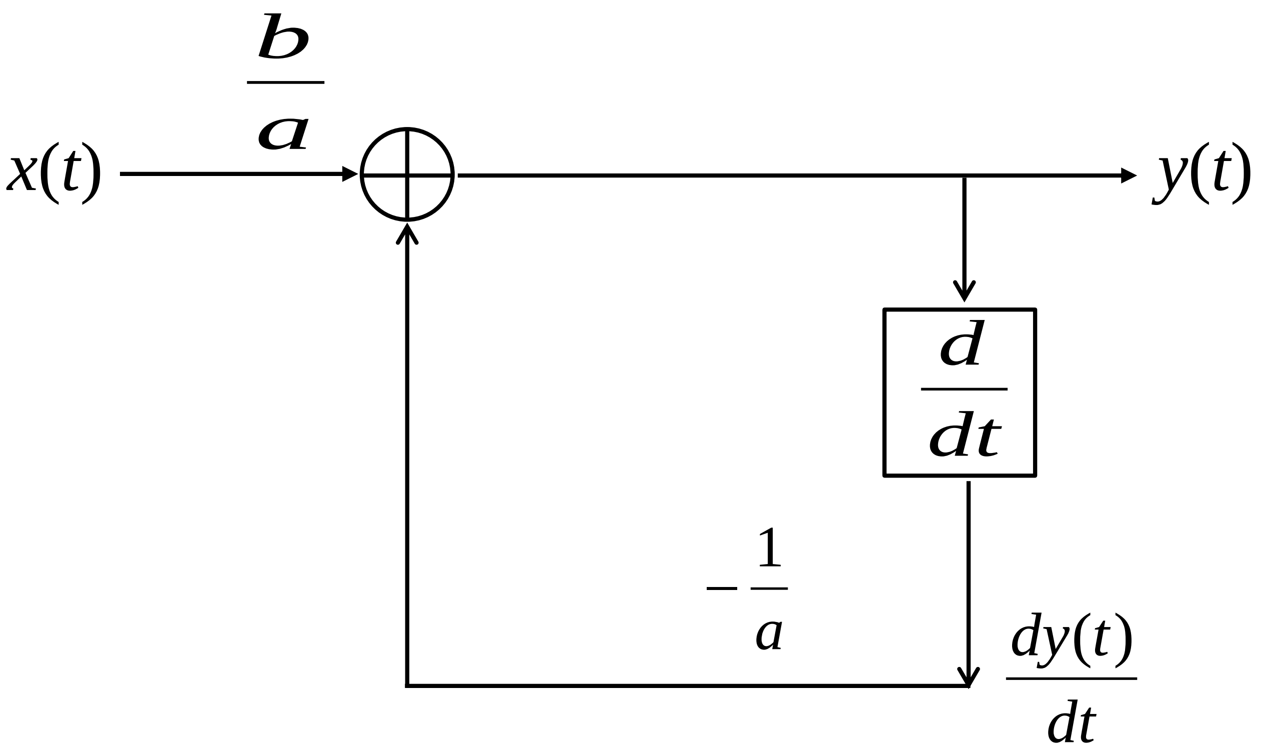


- Feedback, with memory, initial value of the memory element as the initial condition
- Initial rest condition: initial value in the memory element is zero

Block Diagram Representation

- Continuous-time Example

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$



Block Diagram Representation

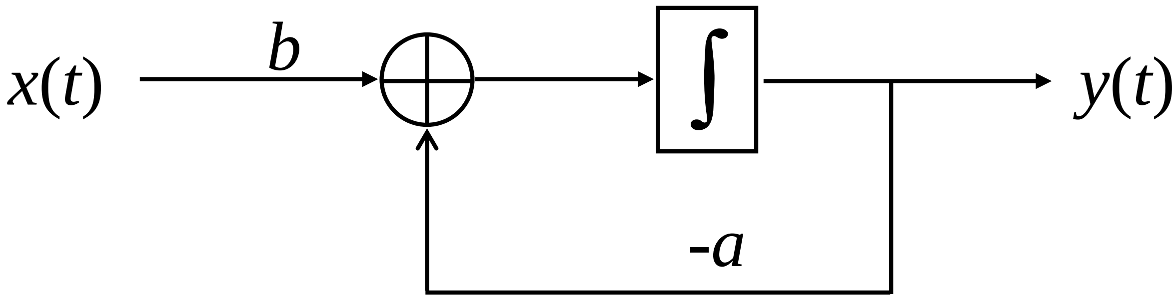
- Continuous-time Example

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

– Expressed by integrator, assuming initially at rest

$$y(t) = \int_{-\infty}^{t} [bx(\tau) - ay(\tau)]d\tau$$

$$y(t) = y(t_0) + \int_{t_0}^{t} [bx(\tau) - ay(\tau)]d\tau$$



– The integrator represents the memory element

2.6 The Unit Impulse for Continuous-time Cases

Many Different Functions Approaching *δ*(*t*) in the Limit

- $\delta_{\Delta}(t) \rightarrow \delta(t)$ as $\Delta \rightarrow 0$
- Identity for convolution

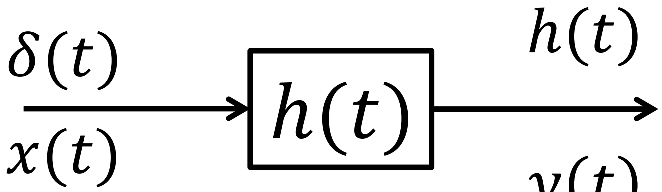
$$x(t) = x(t) * \delta(t), \quad x(t) = \delta(t)$$

$$\longrightarrow \delta(t) = \delta(t) * \delta(t)$$

$$\delta(t) = \delta(t) * \delta(t) * \delta(t)$$

$$\vdots$$

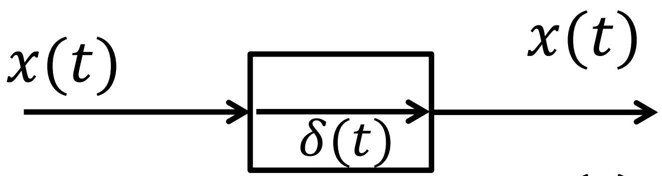
Unit Impulse



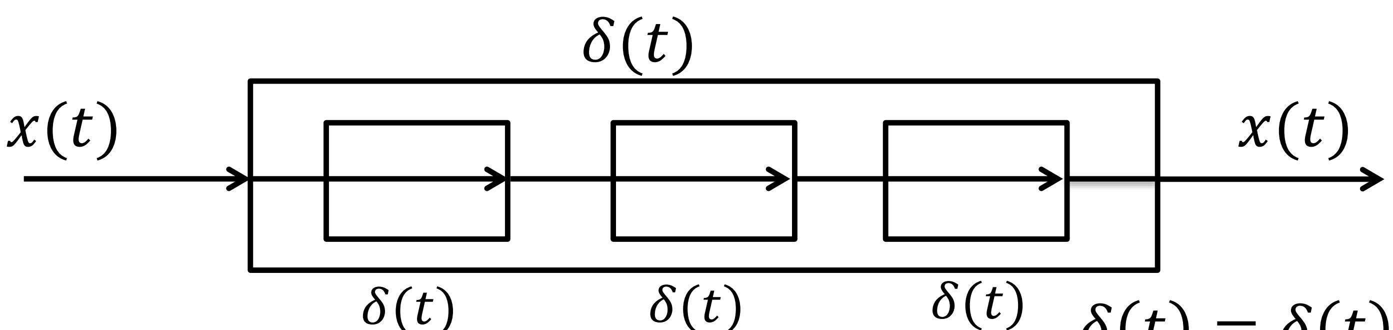
$$y(t) = x(t) * h(t)$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)d\tau = x(t) * \delta(t)$$

(是分析, 是合成, 是單位元素)



$$y(t) = x(t) * \delta(t) = x(t)$$



$$\delta(t) = \delta(t) * \delta(t) * \delta(t)$$

Many Different Functions Approaching δ(t) in the Limit

- Let $r_\Delta(t) = \delta_\Delta(t) * \delta_\Delta(t)$

  See Fig. 2.33, p. 128 of text

then $$\lim_{\Delta \to 0} r_\Delta(t) = \delta(t)$$

$$\lim_{\Delta \to 0} [r_\Delta(t) * \delta_\Delta(t)] = \delta(t)$$

$$\lim_{\Delta \to 0} [r_\Delta(t) * r_\Delta(t)] = \delta(t)$$

$$\vdots$$

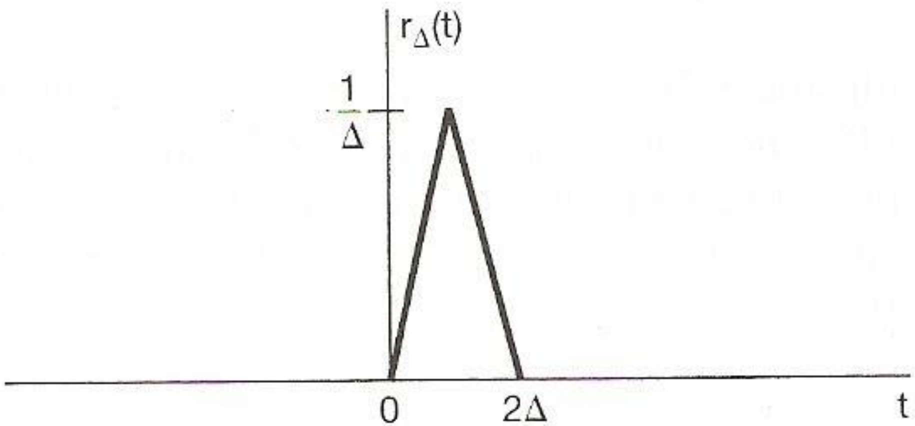


Figure 2.33 The signal $r_\Delta(t)$ defined in eq. (2.135).

Many Different Functions Approaching δ(t) in the Limit

- Let $r_\Delta(t) = \delta_\Delta(t) * \delta_\Delta(t)$
  - All of them ($r_\Delta(t)$, $r_\Delta(t) * \delta_\Delta(t)$, $r_\Delta(t) * r_\Delta(t)$……) do "behave like a unit impulse" in the limit, i.e., produce the unit impulse response of a system when entered as the input, if Δ is small enough.

    See Fig. 2.34, p. 128-129 of text

  - "Δ is small enough" may mean differently in different cases

    See Fig. 2.35, pp. 130-131

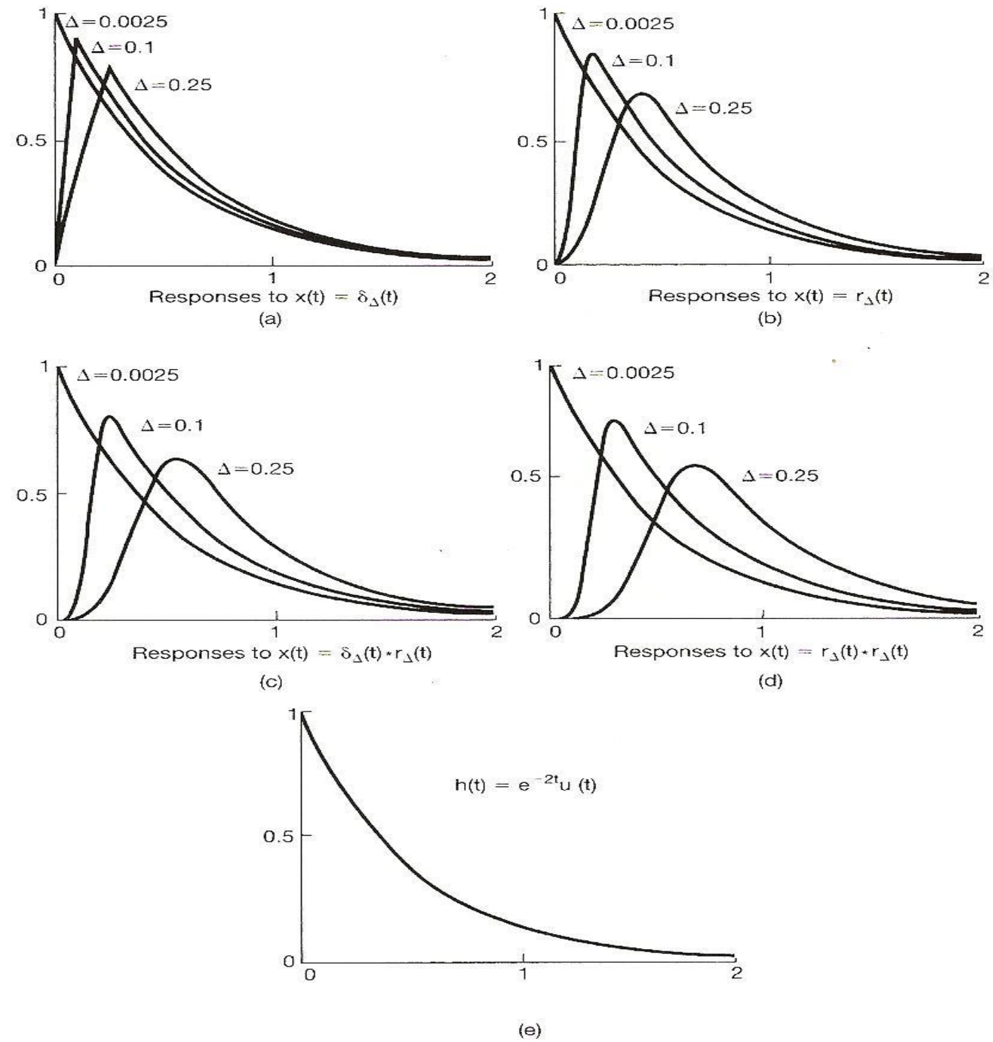


Figure 2.34 Interpretation of a unit impulse as the idealization of a pulse whose duration is "short enough" so that, as far as the response of an LTI system to this pulse is concerned, the pulse can be thought of as having been applied instantaneously: (a) responses of the causal LTI system described by eq. (2.136) to the input $\delta_\Delta(t)$ for $\Delta = 0.25$, 0.1, and 0.0025; (b) responses of the same system to $r_\Delta(t)$ for the same values of Δ; (c) responses to $\delta_\Delta(t) * r_\Delta(t)$; (d) responses to $r_\Delta(t) * r_\Delta(t)$; (e) the impulse response $h(t) = e^{-2t}u(t)$ for the system. Note that, for $\Delta = 0.25$, there are noticeable differences among the responses to these different signals; however, as Δ becomes smaller, the differences diminish, and all of the responses converge to the impulse response shown in (e).

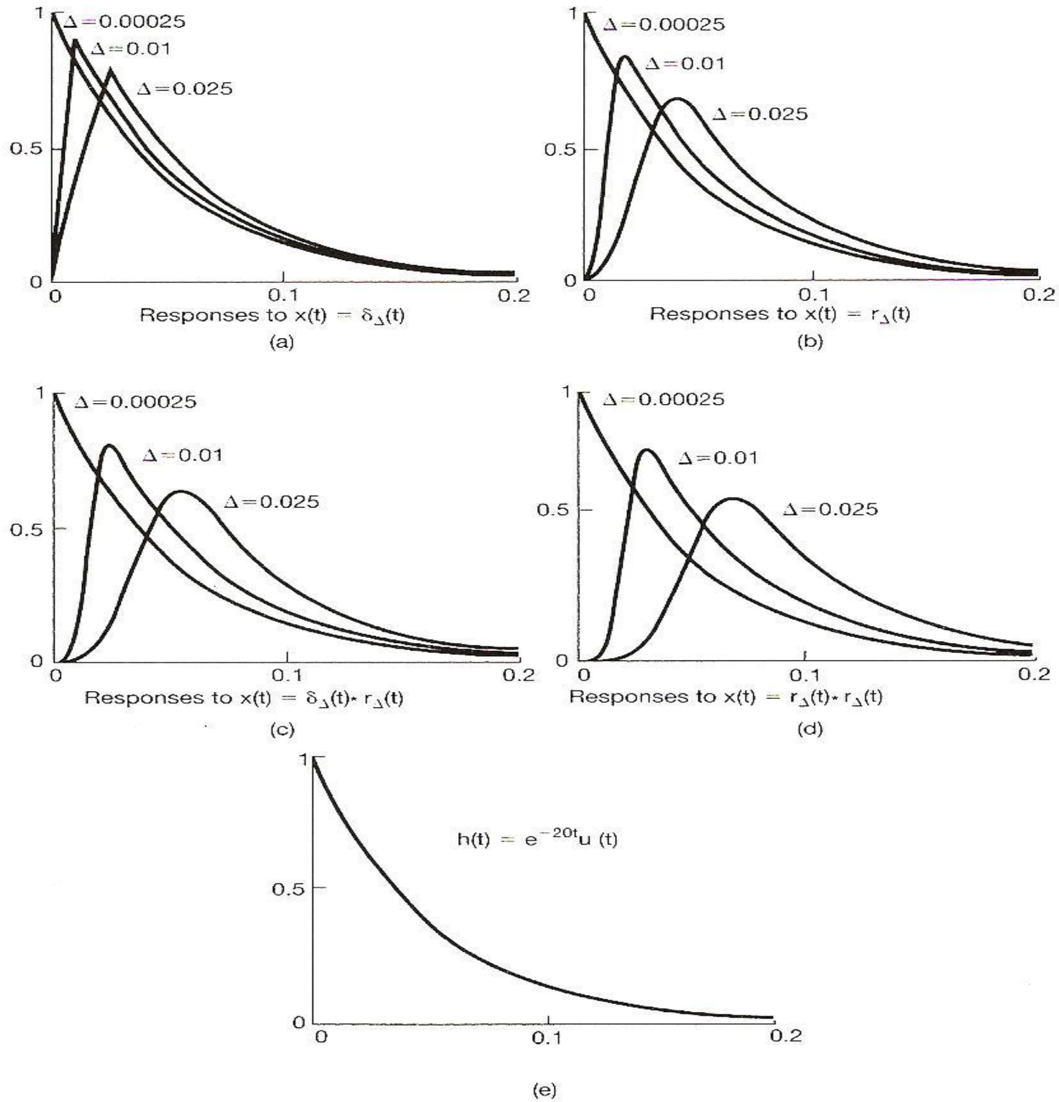


Figure 2.35 Finding a value of Δ that is "small enough" depends upon the system to which we are applying inputs: (a) responses of the causal LTI system described by eq. (2.137) to the input $\delta_\Delta(t)$ for $\Delta = 0.025$, 0.01, and 0.00025; (b) responses to $r_\Delta(t)$; (c) responses to $\delta_\Delta(t) * r_\Delta(t)$; (d) responses to $r_\Delta(t) * r_\Delta(t)$; (e) the impulse response $h(t) = e^{-20t}u(t)$ for the system. Comparing these responses to those in Figure 2.34, we see that we need to use a smaller value of Δ in this case before the duration and shape of the pulse are of no consequence.

Operational Definition of Unit Impulse

- A Function can be defined by
  - what it is at each value of the independent variable
  - what it does under some mathematical operation such as an integral or a convolution, or some mathematical constraints: Singularity Function

Operational Definition of Unit Impulse

- $\delta(t)$ can be defined as
  - $x(t) = x(t) * \delta(t)$ for any $x(t)$, $x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$
  - a signal which, when applied to a system, yields the impulse response $h(t) = h(t) * \delta(t)$
  - such definition leads to, or is equivalent to, other properties of $\delta(t)$,

    $$\int_{-\infty}^{\infty} \delta(\tau)d\tau = 1$$

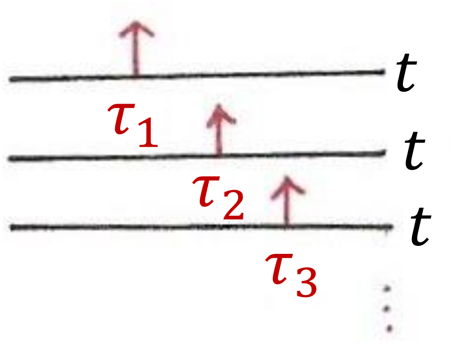
    $$\int_{-\infty}^{\infty} g(\tau)\delta(\tau)d\tau = g(0)$$

    they are also "operational definition" of $\delta(t)$
  - Such definition also leads to the sampling property $\mathrm{f}(t)\delta(t) = \mathrm{f}(0)\delta(t)$

Operational Definition of $\delta(t)$

- $\{\delta(t-\tau), -\infty < \tau < \infty\}$ is a set of basis

- $x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)d\tau = x(t) * \delta(t)$



是合成也是分析

是單位元素

- All are operational definitions

Differentiator, Integrator and Singularity Functions

- Differentiator

$$x(t) \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow y(t) = \frac{dx(t)}{dt}$$

unit impulse response

$$h(t) = \frac{d\delta(t)}{dt} = u_1(t) \qquad \text{unit doublet}$$

$$\frac{dx(t)}{dt} = x(t) * u_1(t) \qquad \text{for any } x(t)$$

operational definition of $u_1(t)$

Differentiator, Integrator and Singularity Functions

- Differentiator in terms of $\delta_\Delta(t)$ in the limit

$$\frac{d\delta_\Delta(t)}{dt} = \frac{1}{\Delta}[\delta(t) - \delta(t-\Delta)]$$

$$x(t) * \frac{d\delta_\Delta(t)}{dt} = \frac{1}{\Delta}[x(t) - x(t-\Delta)] \rightarrow \frac{dx(t)}{dt}$$

See Fig. 2.36, p. 134 of text

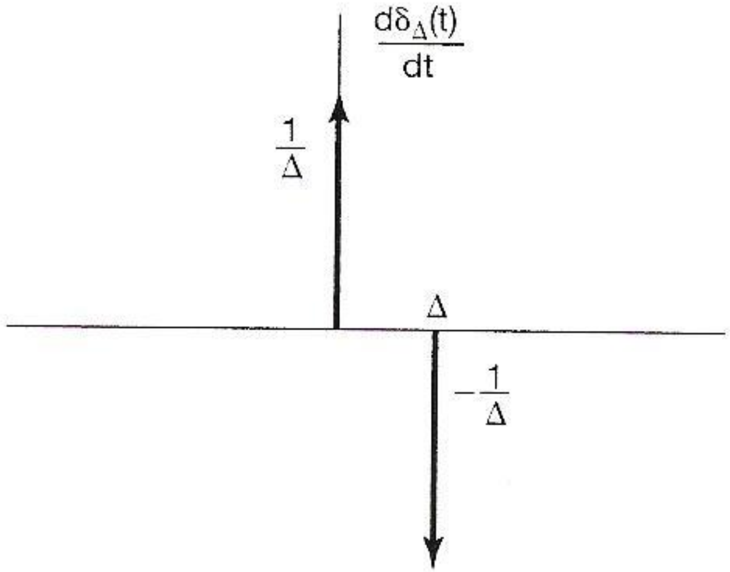


Figure 2.36 The derivative $d\delta_\Delta(t)/dt$ of the short rectangular pulse $\delta_\Delta(t)$ of Figure 1.34.

Differentiator, Integrator and Singularity Functions

- Cascade of Differentiators

$$\frac{d^2x(t)}{dt^2} = x(t)*u_1(t)*u_1(t) = x(t)*u_2(t)$$

operational definition of $u_2(t)$

$$u_2(t) = \frac{d^2\delta(t)}{dt^2}$$

$$u_2(t) = u_1(t)*u_1(t)$$

$$\frac{d^k\delta(t)}{dt^k} = u_k(t) = \underbrace{u_1(t)*u_1(t)\times........*u_1(t)}_{\text{k times}}, \ \text{all } k > 0$$

Differentiator, Integrator and Singularity Functions

- Integrator

$$x(t) \longrightarrow \boxed{\int} \longrightarrow y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

unit impulse response

$$h(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau = u(t) \text{ unit step function}$$

$$\therefore \int_{-\infty}^{t} x(\tau)\,d\tau = x(t) * u(t) \text{ for any } x(t)$$

operational definition of $u(t)$, but $u(t)$ well defined for each t

Differentiator, Integrator and Singularity Functions

- Cascade of Integrators

$$u_{-2}(t) = u(t) * u(t) = \int_{-\infty}^{t} u(\tau) d\tau = tu(t)$$

unit ramp function

$$x(t) * u_{-2}(t) = x(t) * u(t) * u(t) = \int_{-\infty}^{t} \left(\int_{-\infty}^{\tau} x(\sigma) d\sigma \right) d\tau$$

operational definition of $u_{-2}(t)$

Differentiator, Integrator and Singularity Functions

- Cascade of Integrators

$$u_{-k}(t) = \underbrace{u(t) * u(t) * \ldots\ldots\ldots * u(t)}_{\text{k times}} = \int_{-\infty}^{t} u_{-(k-1)}(\tau)\,d\tau$$

$$= \frac{t^{k-1}}{(k-1)!}u(t)$$

They are all well defined for each value of t

Differentiator, Integrator and Singularity Functions

- Unified Definition

$$\delta(t) \equiv u_0(t)$$

$$u(t) \equiv u_{-1}(t)$$

$$\cdots, \underbrace{u_{-k}(t), \cdots, u_{-2}(t), u_{-1}(t)}_{\text{integrators}}, \underset{\delta(t)}{\underset{\|}{u_0(t)}}, \underbrace{u_1(t), u_2(t), \cdots, u_k(t)}_{\text{differentiators}}, \cdots$$

$$u_1(t) * u_{-1}(t) = \delta(t) \quad \text{inverse systems}$$

$$u_2(t) * u_{-2}(t) = \delta(t)$$

$$u_k(t) * u_r(t) = u_{k+r}(t)$$

Differentiator, Integrator and Singularity Functions

- Unified Definition

$$\cdots, \underbrace{u_{-k}(t), \cdots, u_{-2}(t), u_{-1}(t)}_{\text{integrators}}, \underset{\delta(t)}{\overset{u_0(t)}{\|}}, \underbrace{u_1(t), u_2(t), \cdots, u_k(t)}_{\text{differentiators}}, \cdots$$

  - operational definitions with singularity functions
  - manipulate operations efficiently and easily

2.7 Vector Space Interpretation for Continuous-time Systems

Vector Space Concept

- The Set of All Continuous-time Signals Defined in (t_1, t_2) Forms A Vector Space

  $\{x(t)$, $x(t)$ is a continuous-time signal defined in $(t_1,t_2)\}=V$

  – definitions and properties

  $$x(t)+y(t), ax(t)$$

Vector Space Concept

- Definition of Inner-product

$$[x(t)]\cdot[y(t)]: V\times V\to R \text{ or } C$$

$$[x(t)]\cdot[y(t)]: \int_{t_2}^{t_1} x(t)y^*(t)dt$$

  – similar to

$$(x[n])\cdot(y[n]): \sum_{n=N_1}^{N_2} x[n]y^*[n]$$

$$A\cdot B=\sum_{n=1}^{N} a_i b_i$$

  – (t_1,t_2) extendable to $(-\infty,\infty)$

Vector Space Concept

- Magnitude/Similarity

$$\|x(t)\| = \left([x(t)] \cdot [x(t)]\right)^{\frac{1}{2}}$$

$$S[x(t), y(t)] = \frac{[x(t)] \cdot [y(t)]}{\|x(t)\| \, \|y(t)\|}$$

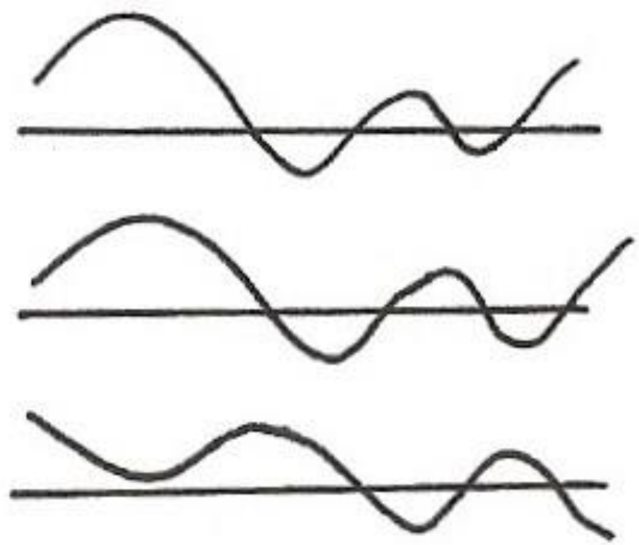
Inner Product for Continuous-time Signals

$$[x(t)] \cdot [y(t)] = \int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt$$

$x(t)$

$y_1(t)$

$y_2(t)$



$$[x(t)] \cdot [y_1(t)] > [x(t)] \cdot [y_2(t)]$$

Vector Space Concept

- Orthonormal Bases
  - orthogonal

$$\left[\phi_i(t)\right]\cdot\left[\phi_j(t)\right]=\int_{t_1}^{t_2}\phi_i(t)\phi_j^*(t)dt=0$$

  - orthonormal

$$\left\{\phi_k(t),\ \mathrm{k}=1,2,3,\ldots\mathrm{M}\right\}$$

$$\left[\phi_i(t)\right]\cdot\left[\phi_j(t)\right]=0,\quad \mathrm{i}\neq \mathrm{j}$$

$$=1,\quad \mathrm{i}=\mathrm{j}$$

Vector Space Concept

– a typical example of orthonormal signal set

$$\left\{\phi_k(t) = b\delta_\Delta(t - k\Delta), k = k_1, k_1 + 1, \ldots k_2\right\}$$

b: scaling factor

$$k_1\Delta = t_1, \quad (k_2 + 1)\Delta = t_2$$

– orthonormal basis

any vector in the vector space can be expanded by the set of orthonormal signals

$$x(t) = \sum_{k=1}^{M} x_k \phi_k(t)$$

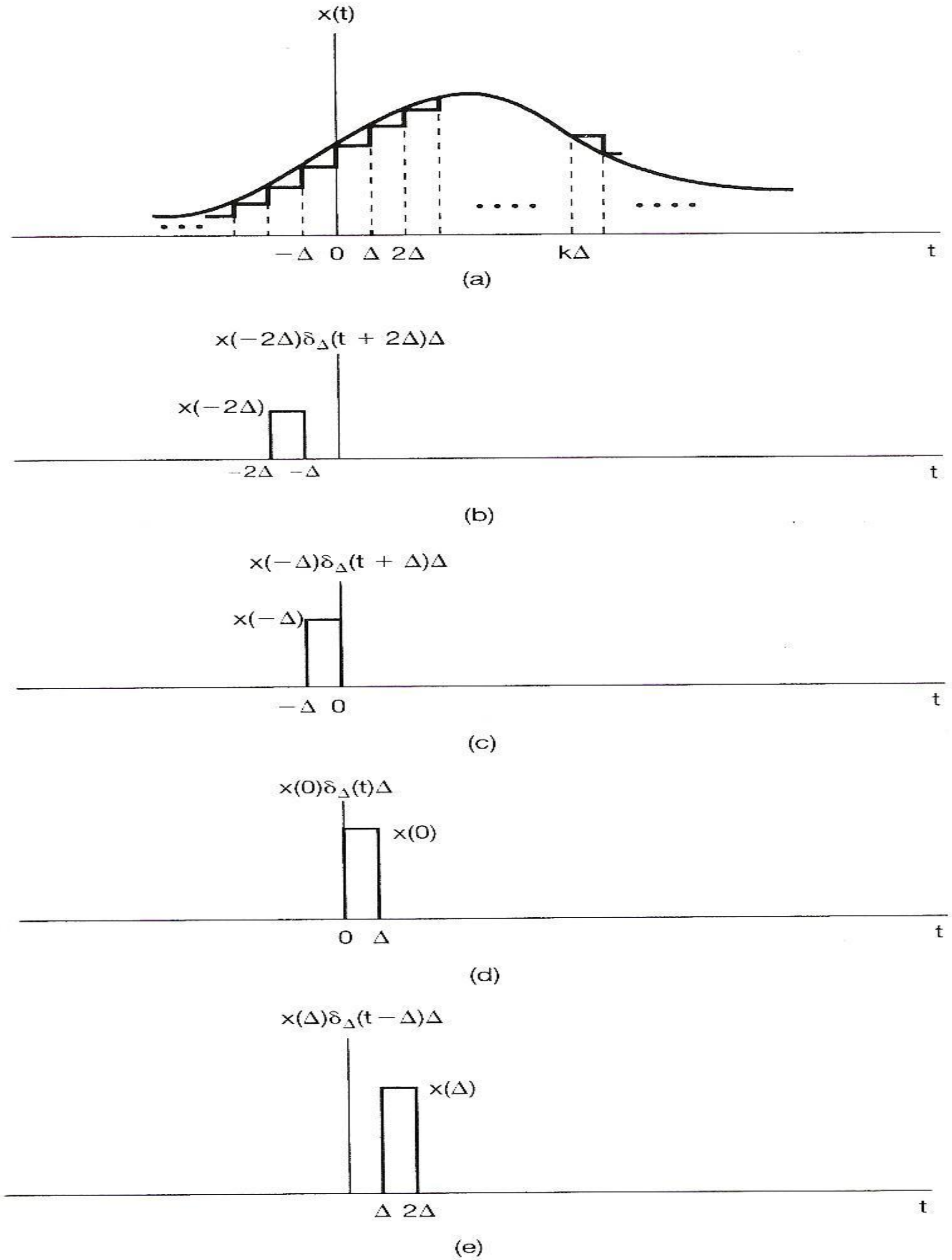


Figure 2.12 Staircase approximation to a continuous-time signal.

Vector Space for Continuous-time Signals

- Vector Space

  $\{x(t), x(t) \text{ defined in } (-\infty, \infty)\} = V$

- Orthonormal Bases(?)

  $\{\phi_\tau(t) = \delta(t-\tau), \quad \tau \in (-\infty, \infty)\}$

  time-shifted unit impulses, dim=∞

  – Inner-product

$$\delta(t) * \delta(t) = \delta(t) = \int_{-\infty}^{\infty} \delta(\tau)\delta(t-\tau)\,d\tau = \int_{-\infty}^{\infty} \delta(\underline{\tau})\delta(\underline{\tau - t})\,d\tau$$

the distance between two $\delta(t)$'s

Vector Space for Continuous-time Signals

- Orthonormal Bases(?)

  $$\left\{\phi_\tau(t) = \delta(t-\tau),\quad \tau \in (-\infty, \infty)\right\}$$

  - Inner-product

    $$[\phi_{\tau_1}(\mathrm{t})]\cdot[\phi_{\tau_2}(\mathrm{t})] = \left[\delta\left(\mathrm{t} - \tau_1\right)\right]\cdot\left[\delta\left(t-\tau_2\right)\right]$$

    $$= \int_{-\infty}^{\infty} \delta\left(t-\tau_1\right)\delta\left(t-\tau_2\right)dt = \delta\left(\tau_1 - \tau_2\right)$$

    $$= 0,\ \ \tau_1 \neq \tau_2$$

    $$= \delta(0) \neq 1,\ \ \tau_1 = \tau_2$$

  - Not really orthonormal (but are orthogonal), but makes sense under the operational definition

System Characterization

- A Signal expanded by unit impulse bases

$$x(t) = \lim_{\Delta \to 0} \sum_{k=\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)\, d\tau$$

$$\sum_{\tau} \qquad x_{\tau} \qquad \phi_{\tau}(t)$$

(合成)

  - sifting property of $\delta(t)$
    extracting the component of $x(t)$ at $\tau = t$

$$x(t) = [x(\tau)] \cdot [\phi_t(\tau)]$$

(分析)

System Characterization

- A System is a Transformation

$$x(t) \longrightarrow \boxed{\mathbf{H}} \longrightarrow y(t)$$

$$\delta(t) \qquad\qquad\qquad\qquad h(t)$$

  - unit impulse response is the transformation of $\delta(t)$
  - a basis vector is transformed to a set of other bases
  - superposition property for convolution integral

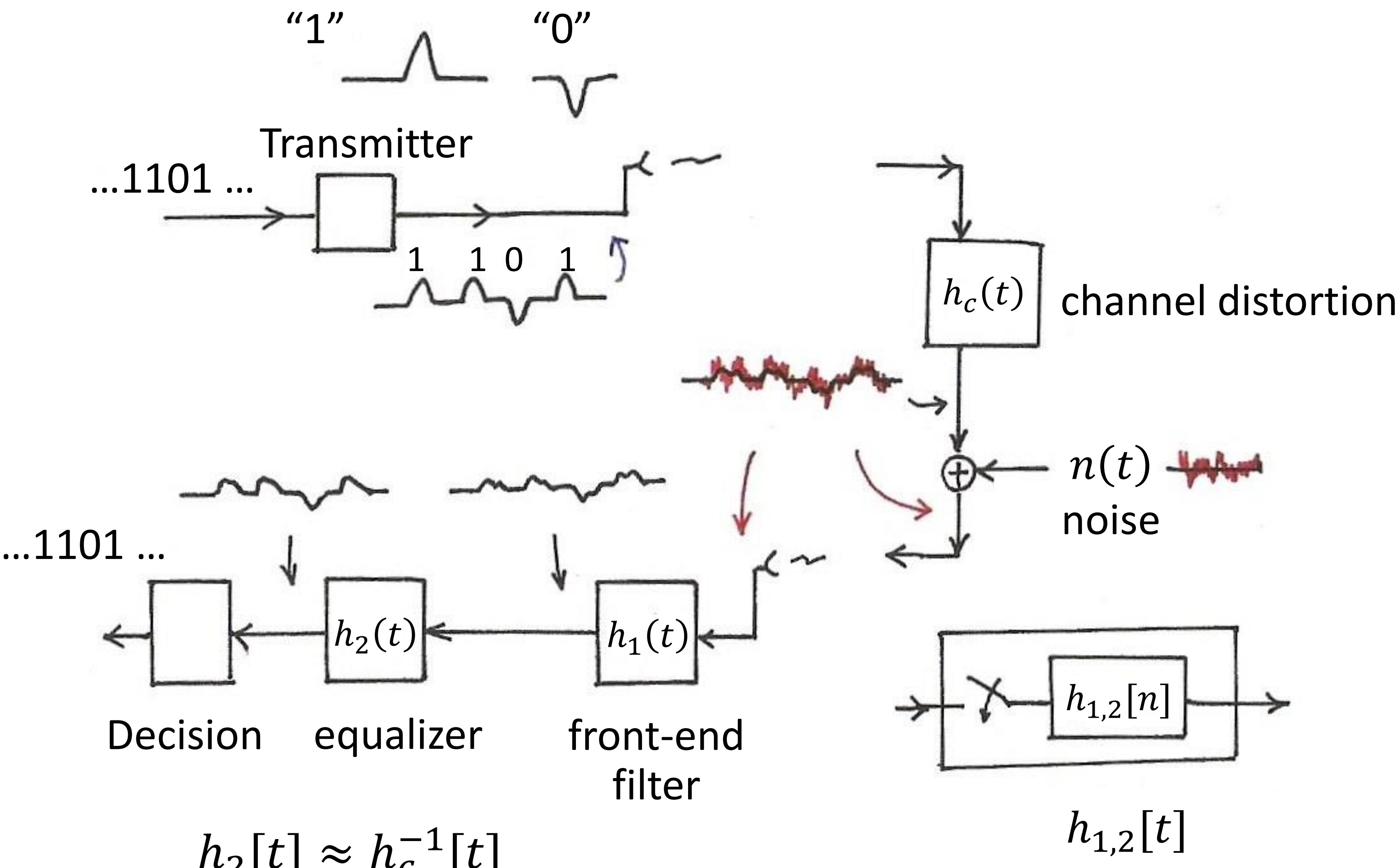
$$\sum_k a_k x_k(t) \rightarrow \sum_k a_k y_k(t)$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

- Can’t be represented by a matrix due to the discrete property of the matrices, but the concept is the same

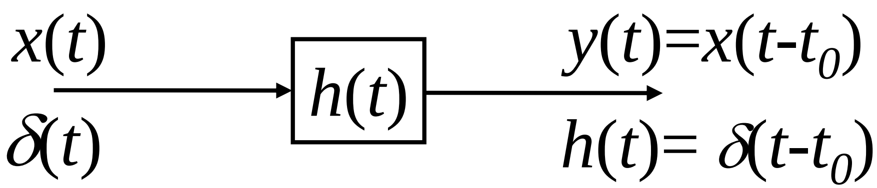
Data Transmission



$$h_2[t] \approx h_c^{-1}[t]$$

Examples

- Example 2.11, p.110 of text
  - time shift of signals

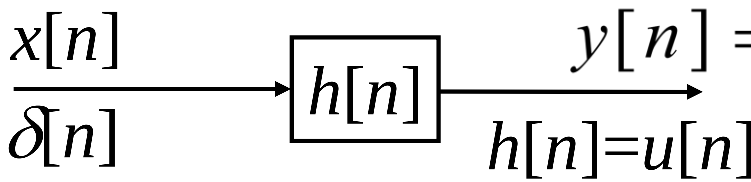


$$x(t-t_0)=x(t)*\delta\left(t-t_0\right)$$

  - convolution of a signal with a shifted impulse is the signal itself but shifted

Examples

- Example 2.12, p.111 of text
  - running sum or accumulation



$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]u[n-k]$$

  - first difference is the inverse

$$y[n]=x[n]-x[n-1]$$

$$h_1[n]=\delta[n]-\delta[n-1]$$

  - $u[n]*\{\delta[n]-\delta[n-1]\}=u[n]-u[n-1]=\delta[n]$

Examples

- Example 2.15, p.123 of text

$$y[n] - \frac{1}{2} y[n-1] = x[n]$$

$$y[n] = x[n] + \frac{1}{2} y[n-1]$$

  - initial rest condition

$$x[n] = 0,\ n \le -1 \quad \text{imples} \quad y[n] = 0,\ n \le -1$$

$$x[n] = \delta[n]$$

$$y[0] = x[0] + \tfrac{1}{2} y[-1] = 1$$

$$y[1] = x[1] + \tfrac{1}{2} y[0] = \tfrac{1}{2}$$

$$y[2] = x[2] + \tfrac{1}{2} y[1] = (\tfrac{1}{2})^2$$

$$\vdots$$

$$y[\mathrm{n}] = \left(\frac{1}{2}\right)^n u[n] = h[n]$$

  - infinite impulse response (IIR)

Problem 2.51(a), p.154 of text

- System A is linear, time-invariant with

$$h[n] = (\frac{1}{2})^n\, u[n]$$

- System B is linear but time-varying with

$$y[n] = nx[n]$$

Show that the commutativity property does not hold for $[A - B]$ and $[B - A]$ cascade structure

- For $[A - B]$

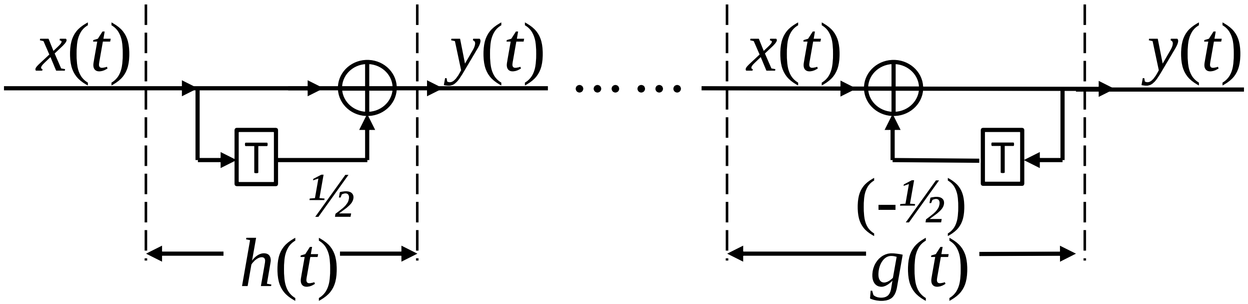
$$x_1[n] = \delta[n],\, y_1[n] = (\frac{1}{2})^n\, u[n],\, z_1[n] = n(\frac{1}{2})^n\, u[n]$$

- For $[B - A]$

$$x_1[n] = \delta[n],\, y_1[n] = 0,\, z_1[n] = 0$$

Problem 2.64, p.166 of text

- A simplified echo system and its inverse



$$h(t) = \delta(t) + \tfrac{1}{2}\delta(t-T)$$

$$g(t) = \delta(t) + \sum_{k=1}^{\infty}\left(-\tfrac{1}{2}\right)^k \delta(t-kT) = \sum_{k=0}^{\infty}\left(-\tfrac{1}{2}\right)^k \delta(t-kT)$$

$$h(t) * g(t) = [\delta(t) + \tfrac{1}{2}\delta(t-T)] * [\sum_{k=0}^{\infty}\left(-\tfrac{1}{2}\right)^k \delta(t-kT)]$$

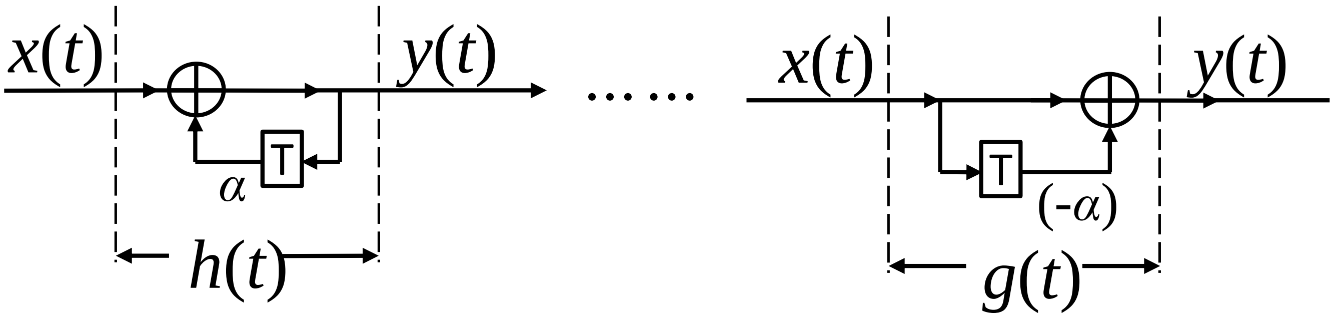
$$\delta(t-a) * \delta(t-b) = \delta(t-(a+b))$$

$$h(t) * g(t) = \sum_{k=0}^{\infty}\left(-\tfrac{1}{2}\right)^k \delta(t-kT) + \left(\tfrac{1}{2}\right)\sum_{k=0}^{\infty}\left(-\tfrac{1}{2}\right)^k \delta(t-(k+1)T) = \delta(t)$$

$$-\sum_{k=0}^{\infty}\left(-\tfrac{1}{2}\right)^{k+1} \delta(t-(k+1)T) = -\sum_{k=1}^{\infty}\left(-\tfrac{1}{2}\right)^k \delta(t-kT)$$

Problem 2.64, p.166 of text

- A more realistic model



$$h(t) = \sum_{k=0}^{\infty} \alpha^k \delta(t-kT)$$

$$g(t) = \delta(t) - \alpha\delta(t-T)$$

$$h(t) * g(t) = [\sum_{k=0}^{\infty} \alpha^k \delta(t-kT)] * [\delta(t) - \alpha\delta(t-T)] = \delta(t)$$

– stability analysis

$0<\alpha<1$, $\int_{-\infty}^{\infty} |h(t)| dt = \sum_{k=0}^{\infty} \alpha^k$, stable

$\alpha>1$, $\int_{-\infty}^{\infty} |h(t)| dt$ not integrable, "NOT" stable