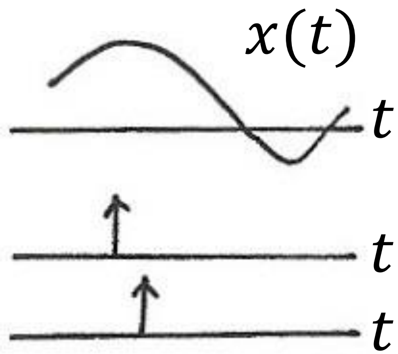
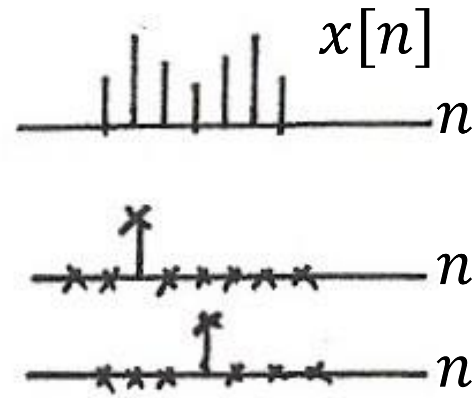
3.0 Fourier Series Representation of Periodic Signals

3.1 Exponential/Sinusoidal Signals as Building Blocks for Many Signals

Time/Frequency Domain Basis Sets

- Time Domain

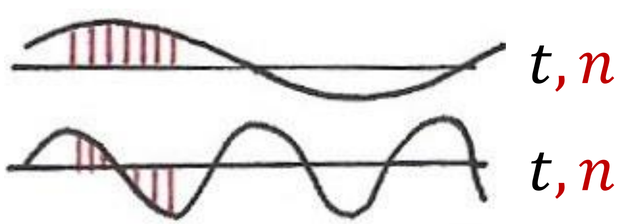




$\{\delta(t-\tau),\ -\infty < \tau < \infty\}$ $\{\delta[n-k],\ k = 0, \pm 1, \pm 2, \cdots\}$

- Frequency Domain

$\{e^{j\omega t},\ -\infty < \omega < \infty\}$ $\{e^{j\omega n}, \omega \in [0, 2\pi]\}$



$$\vec{A} = \sum_k a_k\ \hat{v}_k \text{ (合成)}$$

$$a_j = \vec{A} \cdot \hat{v}_j \text{ (分析)}$$

Signal Analysis (P.32 of 1.0)

$$x(t) = \sum_k a_k x_k(t)$$

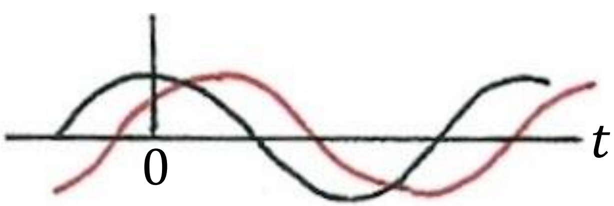
$t = n$ $\quad$ $t = n$

$$x_k(t) = Re\{e^{j\omega_k t}\} = \cos \omega_k t$$

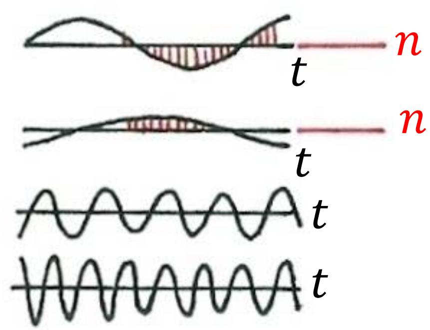
$$a_k = A_k e^{j\phi_k}$$

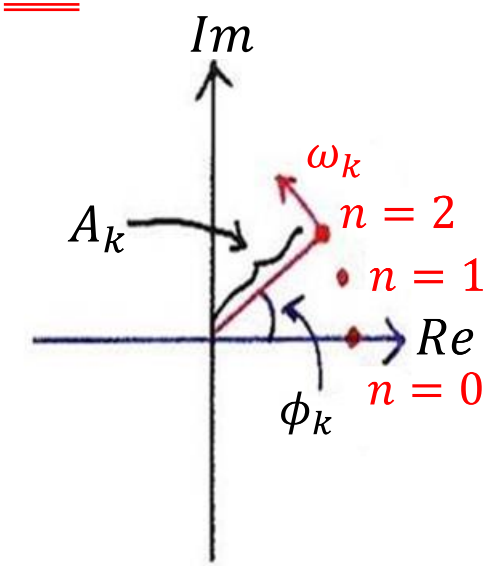
$t = n$

$$Re\{(A_k e^{j\phi_k})(e^{j\omega_k t})\} = A_k \cos(\omega_k t + \phi_k)$$



$$\omega_k = k\omega_0$$





Response of A Linear Time-invariant System to An Exponential Signal

- Initial Observation

$$e^{j\omega_0 t} = x(t) \quad \rightarrow \quad y(t)$$

$$e^{j\omega_0 (t+\tau)} = x(t+\tau) \quad \rightarrow \quad y(t+\tau) \qquad \text{time-invariant}$$

$$e^{j\omega_0 \tau} \cdot x(t) \quad \rightarrow \quad e^{j\omega_0 \tau} \cdot y(t) \qquad \text{scaling property}$$

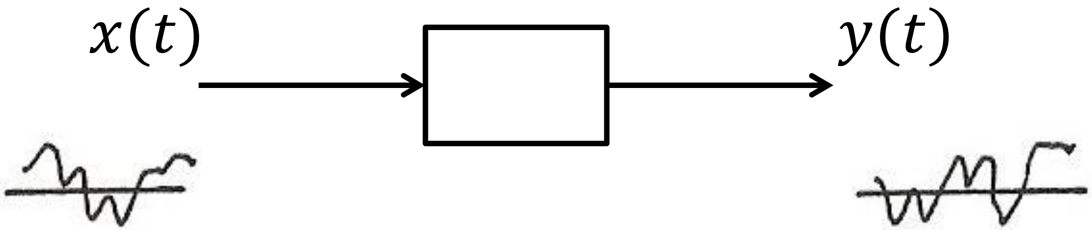
$$\therefore\ y(t+\tau) = e^{j\omega_0 \tau} \cdot y(t)$$

$$\underset{0}{\uparrow} \qquad \underset{t}{\uparrow}$$

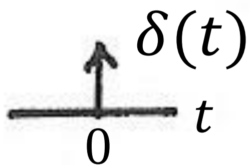
$$y(t) = y(0) e^{j\omega_0 t}$$

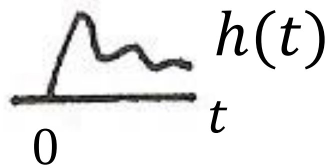
  – if the input has a single frequency component, the output will be exactly the same single frequency component, except scaled by a constant

Input/Output Relationship

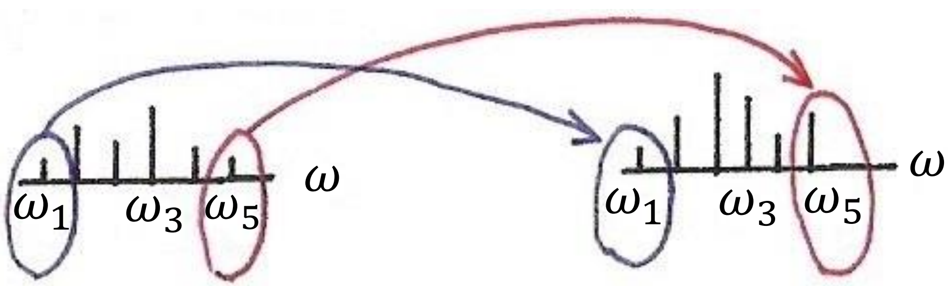


- Time Domain





- Frequency Domain



Response of A Linear Time-invariant System to An Exponential Signal

- More Complete Analysis
  - continuous-time

matrix

vectors

$$A\,x = y$$

$$A\,v = \lambda v$$

eigenvalue

eigenvector

$$x(t) = e^{st},\ s = r + j\omega_0$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\,e^{s(t-\tau)}\,d\tau = \int_{-\infty}^{\infty} e^{st} h(\tau)\,e^{-s\tau}\,d\tau$$

$$= e^{st}\int_{-\infty}^{\infty} h(\tau)\,e^{-s\tau}\,d\tau = H(s)\,e^{st}$$

Response of A Linear Time-invariant System to An Exponential Signal

- More Complete Analysis
  - continuous-time

$$H(s)=\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

Transfer Function
Frequency Response

$x(t)=e^{st}$: eigenfunction of any linear time-invariant system

$H(s)$: eigenvalue associated with the eigenfunction e^{st}

Response of A Linear Time-invariant System to An Exponential Signal

- More Complete Analysis
  - discrete-time

$$x[n] = z^n, \quad z = ce^{j\omega_0}$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] = \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k}$$

$$= z^n \sum_{k=-\infty}^{\infty} h[k]\, z^{-k} = H(z)\, z^n$$

$$H[z] = \sum_{k=-\infty}^{\infty} h[k]\, z^{-k}$$

Transfer Function
Frequency Response

eigenfunction, eigenvalue

System Characterization

- Superposition Property
  - continuous-time

$$x(t) = \sum_k a_k e^{s_k t} \rightarrow y(t) = \sum_k a_k H(s_k) e^{s_k t}$$

  - discrete-time

$$x[n] = \sum_k a_k (z_k)^n \rightarrow y[n] = \sum_k a_k H(z_k)(z_k)^n$$

  - each frequency component never split to other frequency components, no convolution involved
  - desirable to decompose signals in terms of such eigenfunctions

3.2 *Fourier Series Representation of Continuous-time Periodic Signals*

Fourier Series Representation

$x(t) = x(t+T)$, T: fundamental period

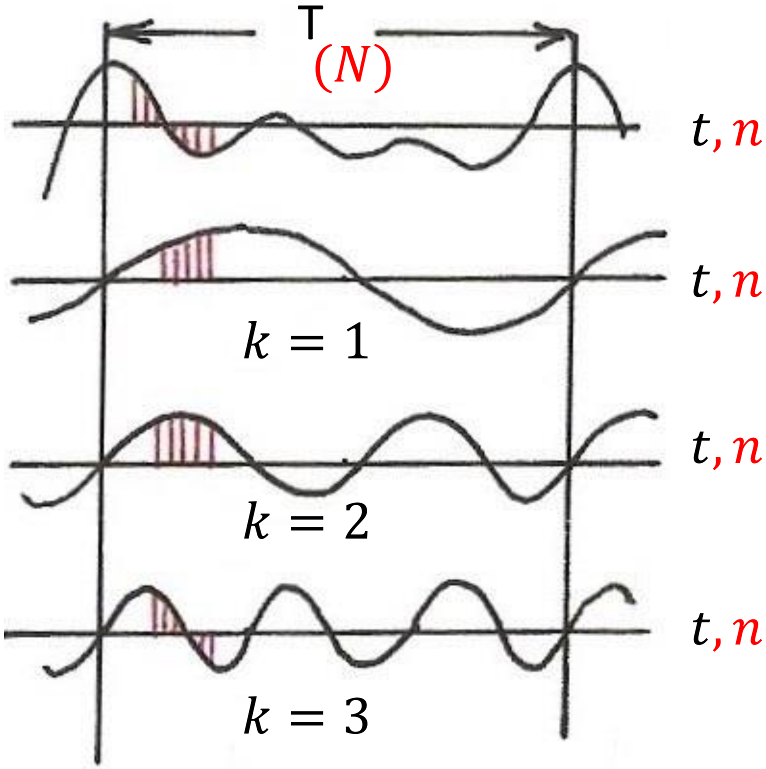
- Harmonically related complex exponentials

$$\left\{\phi_k(t) = e^{jk\omega_0 t}, \mathrm{k} = 0, \pm 1, \pm 2, \ldots.. \right\}, \omega_o = \frac{2\pi}{T}$$

$\phi_k(t)$ with period $\dfrac{T}{|k|}$

all with period T

Harmonically Related Exponentials for Periodic Signals



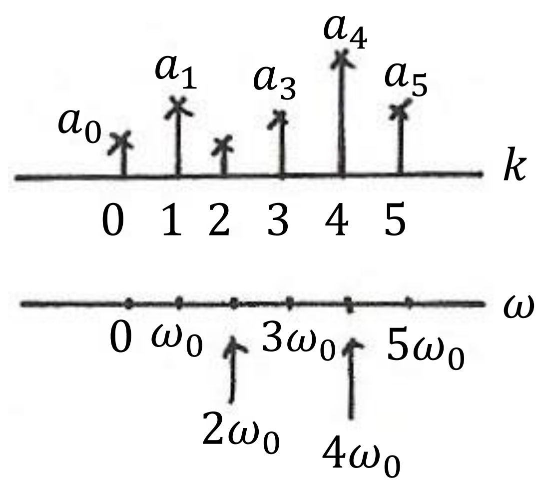
[n]

$V = \{x(t)|x(t)$ periodic, fundamental period

[n]

$= T(N)\}$

$\omega_0 = \frac{2\pi}{T(N)}$



- All with period T: integer multiples of ω_0
  (N)
- Discrete in frequency domain

Fourier Series Representation

- Fourier Series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$$

$a_j \phi_j(t)$: j-th harmonic components

$- \; x(t)$ real

$$a_k^* = a_{-k}$$

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k), \; a_k = A_k e^{j\theta_k}$$

$$= a_0 + 2\sum_{k=1}^{\infty} \left[B_k \cos k\omega_0 t - C_k \sin k\omega_0 t\right], \; a_k = B_k + jC_k$$

Real Signals

$$\left[\cdots a_{-2}\, e^{-j2\omega_0 t} + a_{-1}\, e^{-j\omega_0 t} + a_0 + a_1\, e^{j\omega_0 t} + a_2\, e^{j2\omega_0 t} \cdots\right]^*$$

$$a_{-2} = a_2^*$$

$$a_2^*\, e^{-j2\omega_0 t}$$

For orthogonal basis:

$$\sum_k a_k\, \hat{v}_k = \sum_k b_k\, \hat{v}_k$$

$$\sum_k (a_k - b_k)\, \hat{v}_k = 0 \quad \Rightarrow \quad a_k = b_k$$

(unique representation)

Fourier Series Representation

- Determination of a_k

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \text{(合成)}$$

$$\int_T x(t) e^{-jn\omega_0 t} dt = \int_T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t} dt$$

$$\int_T e^{j(k-n)\omega_0 t} dt = T,\ k = n$$

$$= 0,\ k \neq n$$

$$a_n = \frac{1}{T}\int_T x(t) e^{-jn\omega_0 t} dt, \text{ Fourier series coefficients (分析)}$$

$$a_0 = \frac{1}{T}\int_T x(t) dt, \text{ dc component}$$

Determination of a_k

$$\vec{A} \cdot \hat{v}_n = \left(\sum_k a_k \hat{v}_k\right) \cdot \hat{v}_n$$

$$\hat{v}_k \cdot \hat{v}_n = \begin{cases} T, k = n \\ 0, k \neq n \end{cases}$$

Not unit vector
orthogonal

$$\vec{A} \cdot \hat{v}_n = T a_n$$

$$a_n = \frac{1}{T}(\vec{A} \cdot \hat{v}_n) \quad (分析)$$

Fourier Series Representation

- Vector Space Interpretation
  - vector space

$$\{x(t) : x(t) \text{ is periodic with period } T\}$$

could be a vector space

some special signals (not concerned here) may need to be excluded

$$[x_1(t)] \cdot [x_2(t)] = \int_T x_1(t) x_2^*(t)\, dt$$

Fourier Series Representation

- Vector Space Interpretation
  - orthonormal basis

$$\left[\phi_i(t)\right]\cdot\left[\phi_j(t)\right] = 0, \quad i \neq j$$

$$= T, \; i = j$$

$$\left\{\left(\frac{1}{T}\right)^{1/2}\phi_k(t) = \phi_k'(t), k = 0, \pm 1, \pm 2, \ldots\ldots\right\}$$

is a set of orthonormal basis expanding a vector space of periodic signals with period T

Fourier Series Representation

- Vector Space Interpretation
  - Fourier Series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$$

$$\left(\frac{1}{T}\right)^{1/2} x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k^{'}(t)$$

$$\therefore a_n = \left[\left(\frac{1}{T}\right)^{1/2} x(t)\right] \cdot \left[\phi_n^{'}(t)\right]$$

$$= \left[\left(\frac{1}{T}\right)^{1/2} x(t)\right] \cdot \left[\left(\frac{1}{T}\right)^{1/2} \phi_n(t)\right]$$

$$= \frac{1}{T} \int_T x(t) e^{-jn\omega_0 t} dt$$

Fourier Series Representation

- Completeness Issue
  - Question: Can all signals with period T be represented this way?

    Almost all signals concerned here can, with exceptions very often not important

Fourier Series Representation

- Convergence Issue
  - consider a finite series

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}, \; e_N(t) = x(t) - x_N(t)$$

$$E_N = \int_T \left| e_N(t) \right|^2 dt = \left\| e_N(t) \right\|^2$$

It can be shown

$$E_N = \min \text{ if } a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt, \, k = 0, \pm 1, \ldots \pm N$$

a_k obtained above is exactly the value needed even for a finite series

Truncated Dimensions

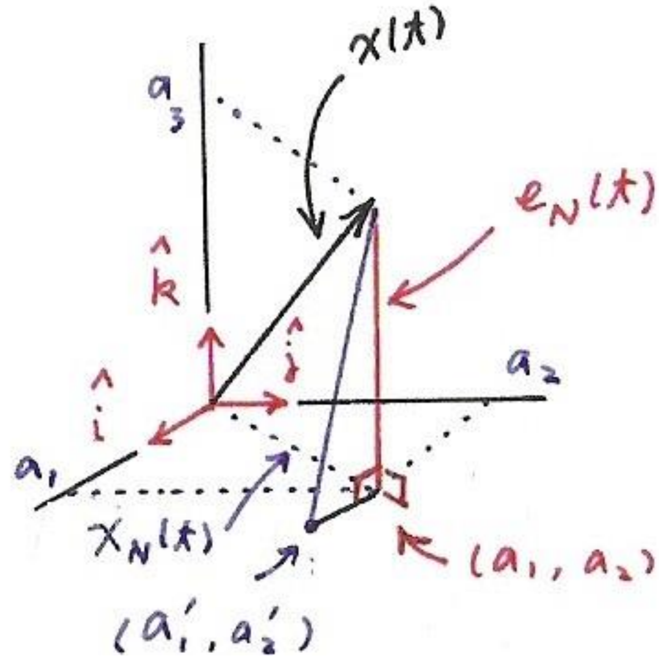
$x_N(t) \Leftarrow \quad \Rightarrow e_N(t)$

$$x(t) = a_0 + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t} + \cdots + a_N e^{jN\omega_0 t} \;\Big|\; \cdots$$
$$+ a_{-1} e^{-j\omega_0 t} + a_{-2} e^{-j2\omega_0 t} + \cdots + a_{-N} e^{-jN\omega_0 t} \;\Big|\; \cdots$$

N

$$x(t) = \overbrace{a_1 \hat{i} + a_2 \hat{j}}^{x_N(t)} + \overbrace{a_3 \hat{k}}^{e_N(t)}$$

$$x_N(t) = a_1' \hat{i} + a_2' \hat{j} \;\Big|_{N=2}$$



$\Rightarrow$ $a_1' = a_1, a_2' = a_2,$

for orthogonal $\hat{i}, \hat{j}, \hat{k}$

- All truncated dimensions are orthogonal to the subspace of dimensions kept.

Fourier Series Representation

- Convergence Issue
  - It can be shown

    if $\int_T |x(t)|^2 dt < \infty$

    then all a_k defined above are obtainable (finite), and as $N \rightarrow \infty$, $E_N \rightarrow 0$, or no energy for $e_N(t)$, but $e_N(t)$ may be nonzero for some values

Fourier Series Representation

- Gibbs Phenomenon
  - the partial sum in the vicinity of the discontinuity exhibit ripples whose amplitude does not seem to decrease with increasing N

    See Fig. 3.9, p.201 of text

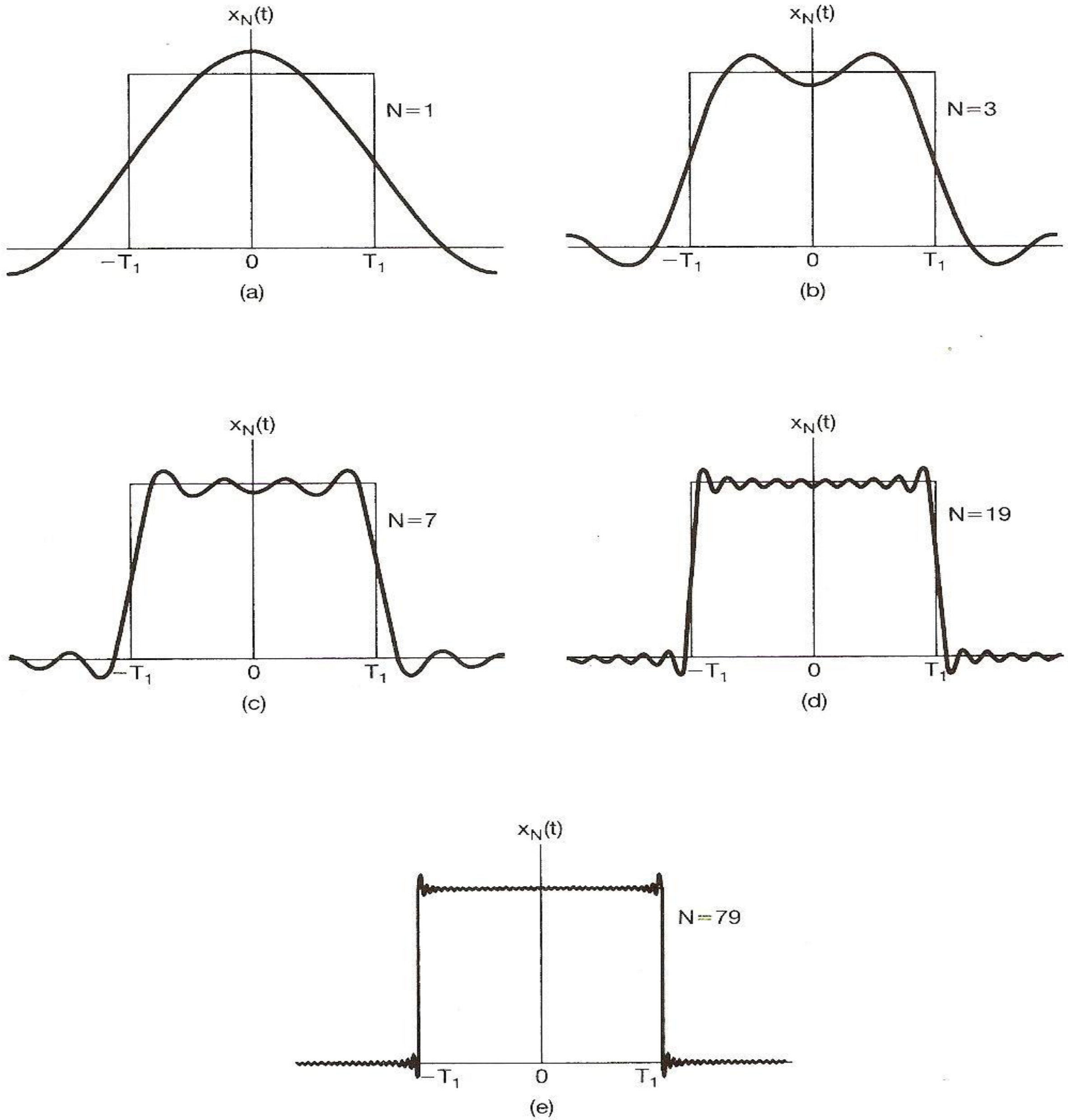


Figure 3.9 Convergence of the Fourier series representation of a square wave: an illustration of the Gibbs phenomenon. Here, we have depicted the finite series approximation $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$ for several values of N.

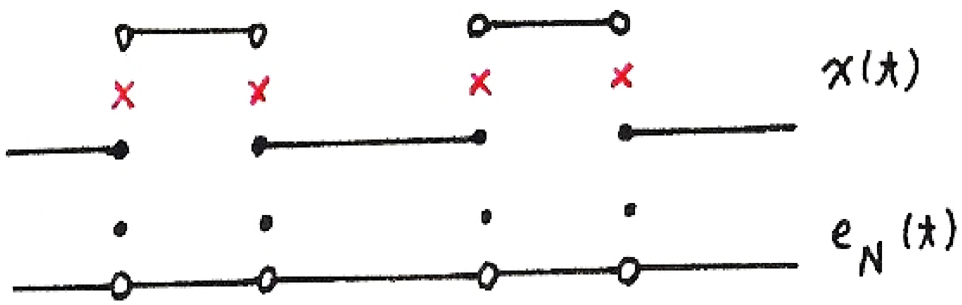
Fourier Series Representation

- Convergence Issue
  - $x(t)$ has no discontinuities

    Fourier series converges to $x(t)$ at every t

    $x(t)$ has finite number of discontinuities in each period

    Fourier series converges to $x(t)$ at every t except at the discontinuity points, at which the series converges to the average value for both sides



All basis signals are continuous, so converge to average values

Fourier Series Representation

- Convergence Issue
  - Dirichlet's condition for Fourier series expansion

    (1) absolutely integrable, $\int_T |x(t)|\, dt < \infty$

    (2) finite number of maxima & minima in a period

    (3) finite number of discontinuities in a period

3.3 Properties of Fourier Series

$$x(t) \xleftrightarrow{FS} a_k$$

- Linearity

$$x(t) \xleftrightarrow{FS} a_k, \;\; y(t) \xleftrightarrow{FS} b_k$$

$$Ax(t) + By(t) \xleftrightarrow{FS} Aa_k + Bb_k$$

$$\vec{x} = (a_1, a_2, a_3, \cdots)$$

$$\vec{y} = (b_1, b_2, b_3, \cdots)$$

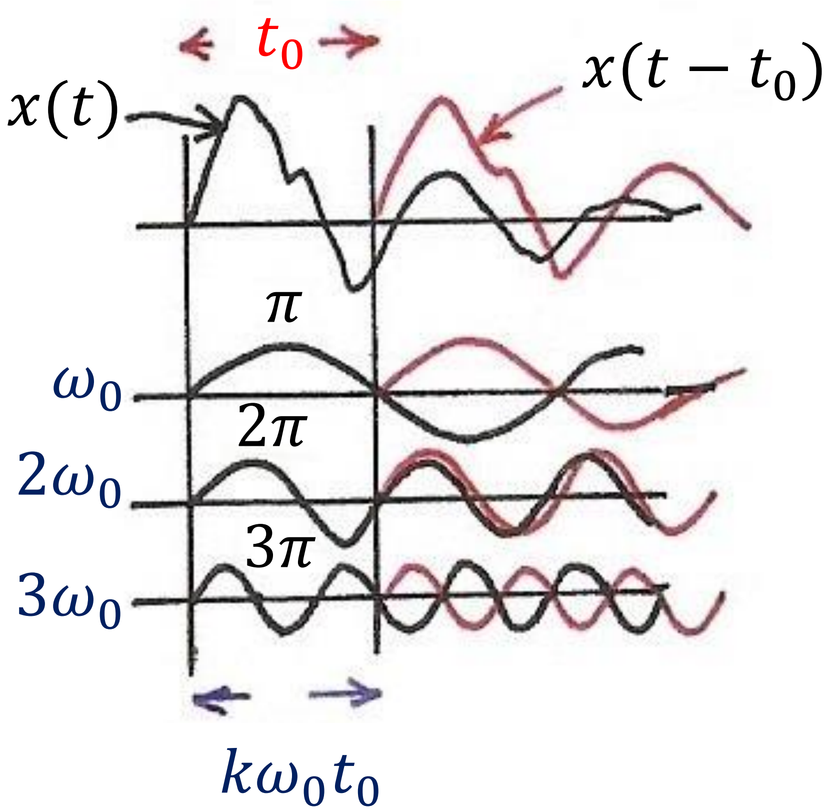
$$A\vec{x} + B\vec{y} = (Aa_1 + Bb_1, Aa_2 + Bb_2, \cdots)$$

- Time Shift

$$x(t-t_0) \overset{FS}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k$$

phase shift linear in frequency with amplitude unchanged

$$a_k\, e^{jk\omega_0 (t-t_0)} = \boxed{e^{-jk\omega_0 t_0}\, a_k}\, e^{jk\omega_0 t}$$



- Time Reversal

$$x(-t) \xleftrightarrow{FS} a_{-k}$$

the effect of sign change for $x(t)$ and a_k are identical

$$\cdots a_{-1} e^{-j\omega_0 t} + a_0 + a_1 e^{j\omega_0 t} + \cdots = x(t)$$

$$\cdots a_{-1} e^{j\omega_0 t} \cdots = x(-t)$$

unique representation for orthogonal basis

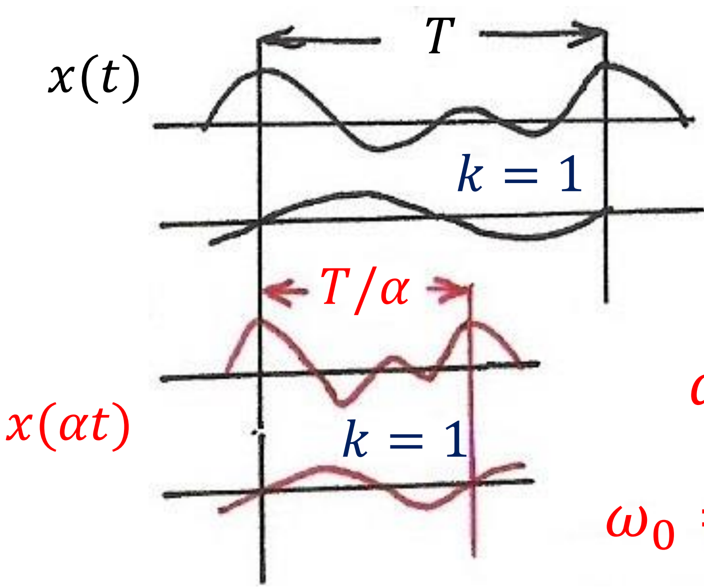
- Time Scaling

  α : positive real number

  $x(\alpha t)$: periodic with period T/α and fundamental frequency $\alpha\omega_0$

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t}$$

  a_k unchanged, but $x(\alpha t)$ and each harmonic component are different



$$\omega_0 = \frac{2\pi}{T}$$

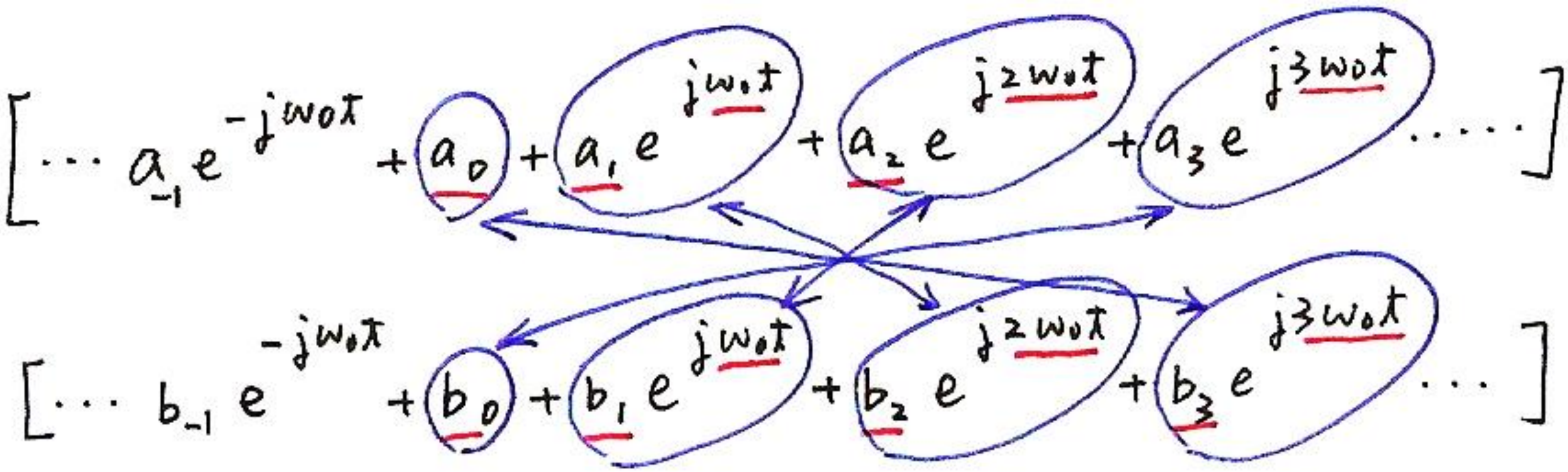
$\alpha > 1$

$$\omega_0 = \frac{2\pi}{T/\alpha} = \alpha\left(\frac{2\pi}{T}\right)$$

- Multiplication

$$x(t) \xleftrightarrow{FS} a_k, \quad y(t) \xleftrightarrow{FS} b_k$$

$$x(t)y(t) \xleftrightarrow{FS} d_k = \sum_{j=-\infty}^{\infty} a_j b_{k-j} = a_k * b_k$$



$$\underbrace{(\cdots a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 \cdots)}_{d_3 = \sum_j a_j b_{3-j}} e^{j3\omega_0 t}$$

- Conjugation

$$x^*(t) \overset{FS}{\longleftrightarrow} a^*_{-k}$$

$$a_{-k} = a_k^*, \text{ if } x(t) \text{ real}$$

$$\left[\cdots a_{-1} e^{-j\omega_0 t} + a_0 + a_1 e^{j\omega_0 t} + \cdots\right]^*$$

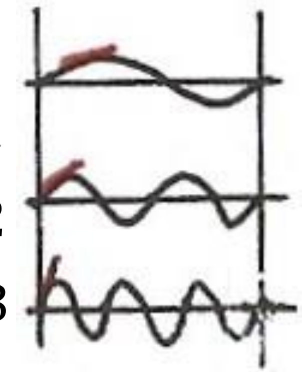
$$a^*_{-1} e^{j\omega_0 t}$$

unique representation

- Differentiation

$$\frac{dx(t)}{dt} \overset{FS}{\longleftrightarrow} jk\omega_0 a_k$$

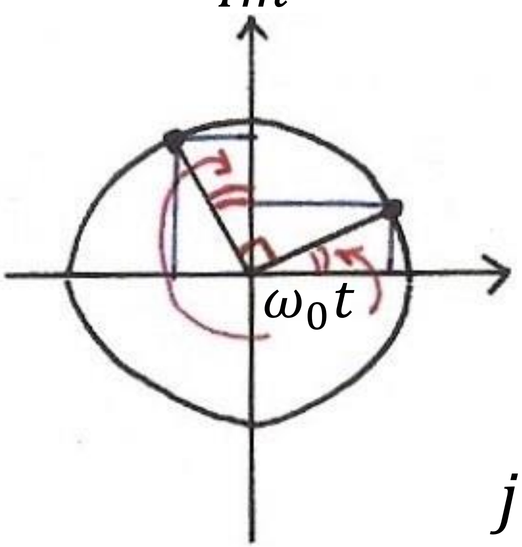
$$\frac{d}{dt}\left(a_k e^{jk\omega_0 t}\right) = \boxed{j\,k\omega_0\,a_k}\, e^{jk\omega_0 t}$$



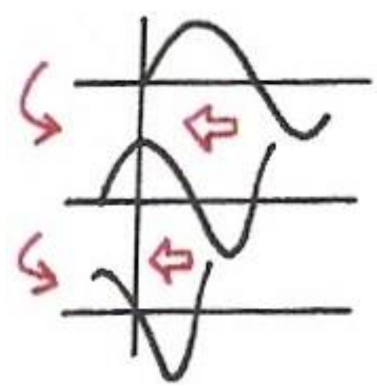
$$j \cdot [\cos\omega_0 t + j\sin\omega_0 t]$$

$$\frac{d}{dt} \qquad \frac{d}{dt}$$

$$= -\sin\omega_0 t + j\cos\omega_0 t$$



$$j = e^{j90^\circ}$$



- Parseval's Relation

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

$$\| \vec{A} \|^2 = \sum_k |a_k|^2$$

but $\hat{v}_i \cdot \hat{v}_j = T\delta_{ij}$

total average power in a period *T*

$$\frac{1}{T}\int_T \left|a_k e^{jk\omega_0 t}\right|^2 dt = |a_k|^2$$

average power in the *k*-th harmonic component in a period *T*

3.4 Fourier Series Representation of Discrete-time Periodic Signals

Fourier Series Representation

$x[n] = x[n+N]$, periodic with fundamental period N

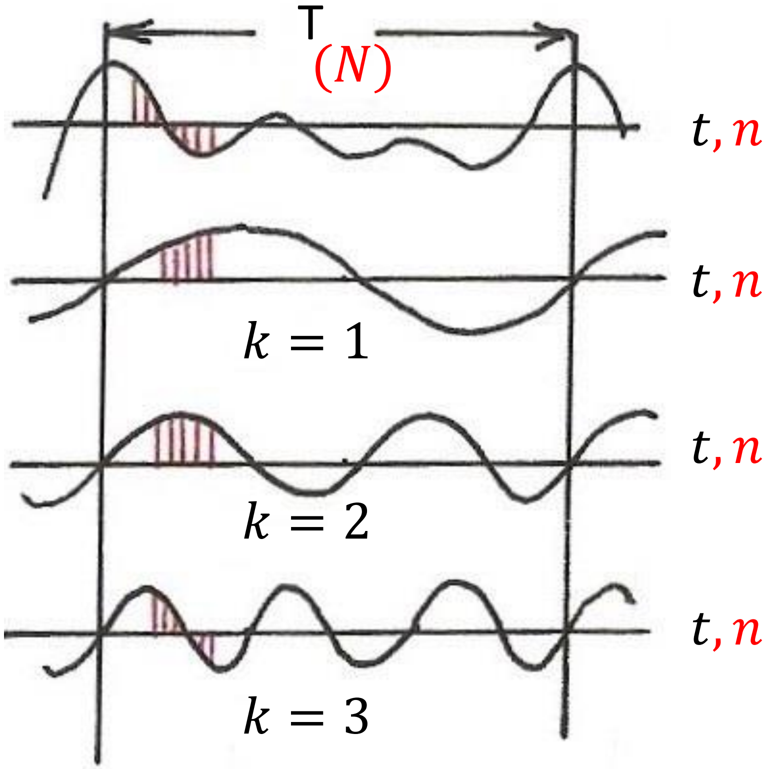
- Harmonically related signal sets

$$\left\{\phi_k[n] = e^{jk\left(\frac{2\pi}{N}\right)n}, \text{k} = 0, \pm 1, \pm 2, \ldots\ldots \right\}$$

all with period N, $\omega_0 = \dfrac{2\pi}{N}$

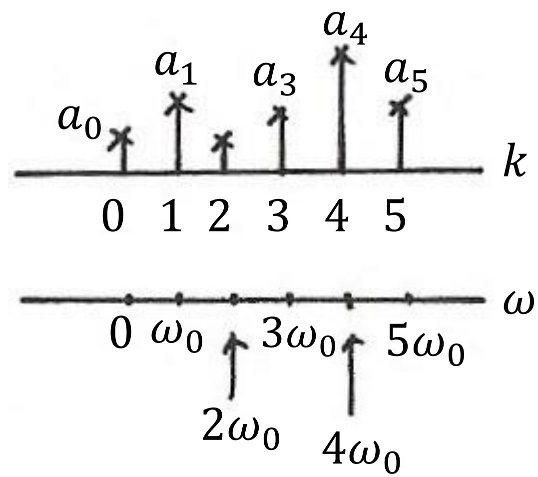
$\phi_{k+rN}[n] = \phi_k[n]$, only N distinct signals in the set

Harmonically Related Exponentials for Periodic Signals (P. 11 of 3.0)



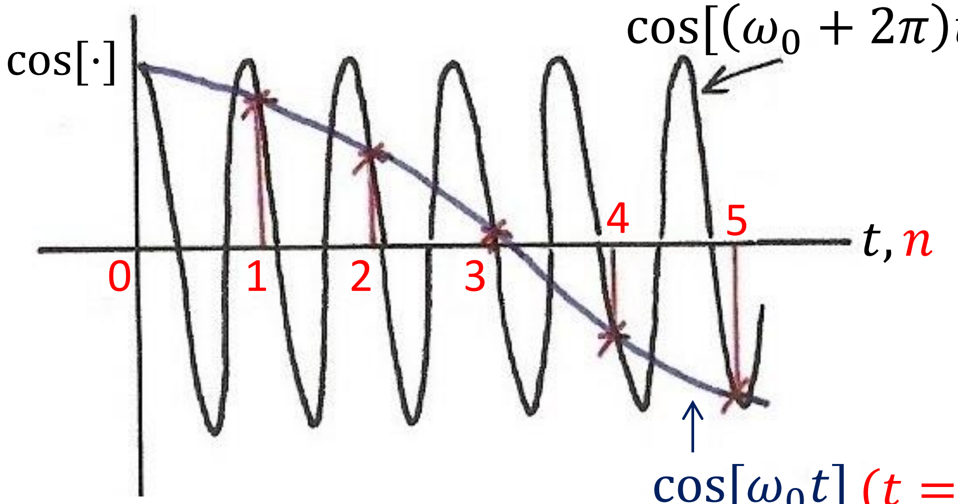
$V = \{x(t)\,[n] \mid x(t)\,[n]$ periodic, fundamental period $= T\,(N)\}$

$\omega_0 = \frac{2\pi}{T\,(N)}$



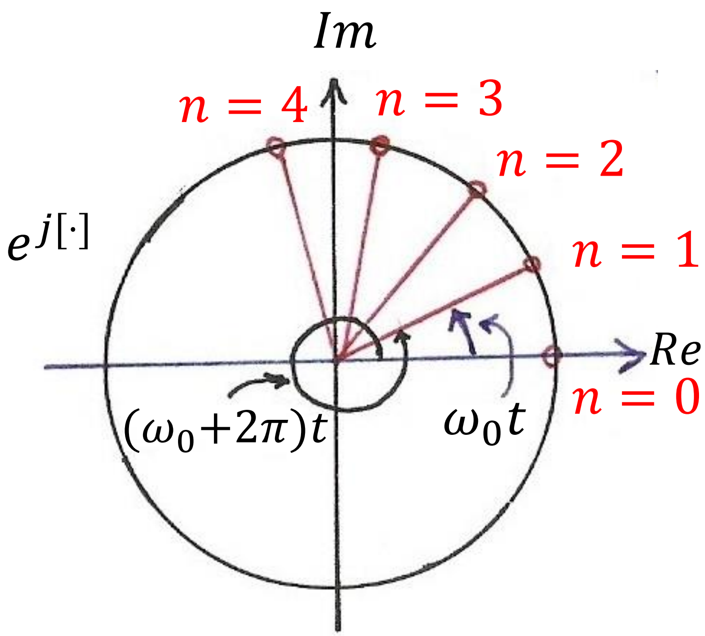
- All with period T (N): integer multiples of ω_0
- Discrete in frequency domain

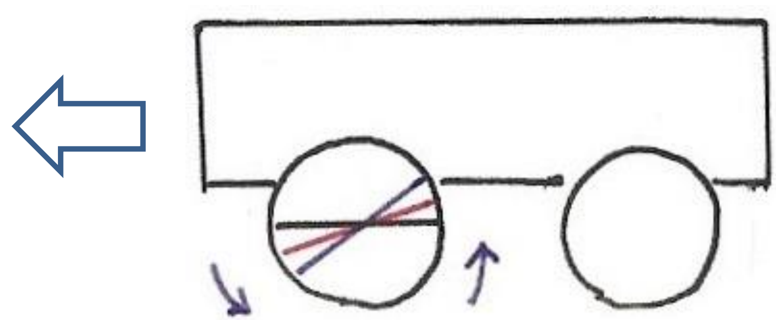
Continuous/Discrete Sinusoidals (P.36 of 1.0)



$$\begin{cases} \cos \omega_0 t \neq \cos(\omega_0 + 2\pi)t \\ \cos \omega_0 n = \cos(\omega_0 + 2\pi)n \end{cases}$$

$$\begin{cases} e^{j\omega_0 t} \neq e^{j(\omega_0+2\pi)t} \\ e^{j\omega_0 n} = e^{j(\omega_0+2\pi)n} \end{cases}$$





Exponential/Sinusoidal Signals (P.42 of 1.0)

- Harmonically related discrete-time signal sets

$$\{\phi_k[n] = e^{jk(\frac{2\pi}{N})n} \quad , \qquad \text{k} = 0, \pm 1, \pm 2, ...\}$$

all with common period N

$$\phi_{k+N}[n] = \phi_k[n]$$

This is different from continuous case. Only *N* distinct signals in this set.

Fourier Series Representation (P.14 of 3.0)

- Determination of a_k

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\int_T x(t) e^{-jn\omega_0 t} dt = \int_T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t} dt$$

$$\int_T e^{j(k-n)\omega_0 t} dt = T, \ k = n$$

$$= 0, \ k \neq n$$

$$a_n = \frac{1}{T} \int_T x(t) e^{-jn\omega_0 t} dt,$$ Fourier series coefficients

$$a_0 = \frac{1}{T} \int_T x(t) dt,$$ dc component

Determination of a_k (P.15 of 3.0)

$$\vec{A} \cdot \hat{v}_n = \left(\sum_k a_k \hat{v}_k\right) \cdot \hat{v}_n$$

$$\hat{v}_k \cdot \hat{v}_n = \begin{cases} T, k = n \\ 0, k \neq n \end{cases}$$

Not unit vector
orthogonal

$$\vec{A} \cdot \hat{v}_n = T a_n$$

$$a_n = \frac{1}{T}(\vec{A} \cdot \hat{v}_n)$$ (分析)

Fourier Series Representation

- Fourier Series

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \qquad (合成)$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n} \qquad (分析)$$

$a_{k+rN} = a_k$, repeat with period N

Note: both $x[n]$ and a_k are discrete, and periodic with period N, therefore summed over a period of N

– $\vec{A} = \sum_k a_k \hat{v}_k$ (合成)

$a_k = \vec{A} \cdot \hat{v}_k$ (分析)

– N different values in $x[n]$

N-dimensional vector space

Orthogonal Basis

$$\sum_{n=<N>} e^{jk\left(\frac{2\pi}{N}\right)n} = N, k = 0, \pm N, \ \pm 2N, \ldots\ldots$$

$$= 0, \text{else}$$

$$\sum_{n=<N>} e^{j(k-l)\left(\frac{2\pi}{N}\right)n} \ \ = N, \qquad k - l = 0, \pm N, \pm 2N, \cdots$$

$$= 0\ , \qquad \text{else}$$

$$\|$$

$$\left[e^{jk(2\pi/N)n}\right] \cdot \left[e^{jl(2\pi/N)n}\right] = \hat{v}_k \cdot \hat{v}_l$$

Fourier Series Representation

- Vector Space Interpretation

$$\{x[n], x[n] \text{ is periodic with period } N\}$$

is a vector space

$$(x_1[n])\cdot(x_2[n]) = \sum_{k=<N>} x_1[k]x_2^*[k]$$

$$(\phi_i[n])\cdot(\phi_j[n]) = N,\ i = j + rN$$

$$= 0,\ \text{else}$$

Fourier Series Representation

- Vector Space Interpretation

$$\left\{ \left(\frac{1}{N}\right)^{1/2} \phi_k[n] = \phi'_k[n],\ k = < N > \right\}$$

a set of orthonormal bases

$$x[n] = \sum_{k=<N>} a_k \phi_k[n]$$

$$a_k = \left(\left(\frac{1}{N}\right)^{1/2} x[n] \right) \cdot \left(\left(\frac{1}{N}\right)^{1/2} \phi_k[n] \right)$$

$$= \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n}$$

Fourier Series Representation

- No Convergence Issue, No Gibbs Phenomenon, No Discontinuity Issue
  - $x[n]$ has only N parameters, represented by N coefficients

    sum of N terms gives the exact value
  - N odd

    $$x[n]_M = \sum_{k=-M}^{M} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

    $$x[n]_M = x[n], \text{ if } M = \frac{(N-1)}{2}$$
  - N even

    $$x[n]_M = \sum_{k=-M+1}^{M} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

    $$x[n]_M = x[n], \text{ if } M = \frac{N}{2}$$

See Fig. 3.18, P.220 of text

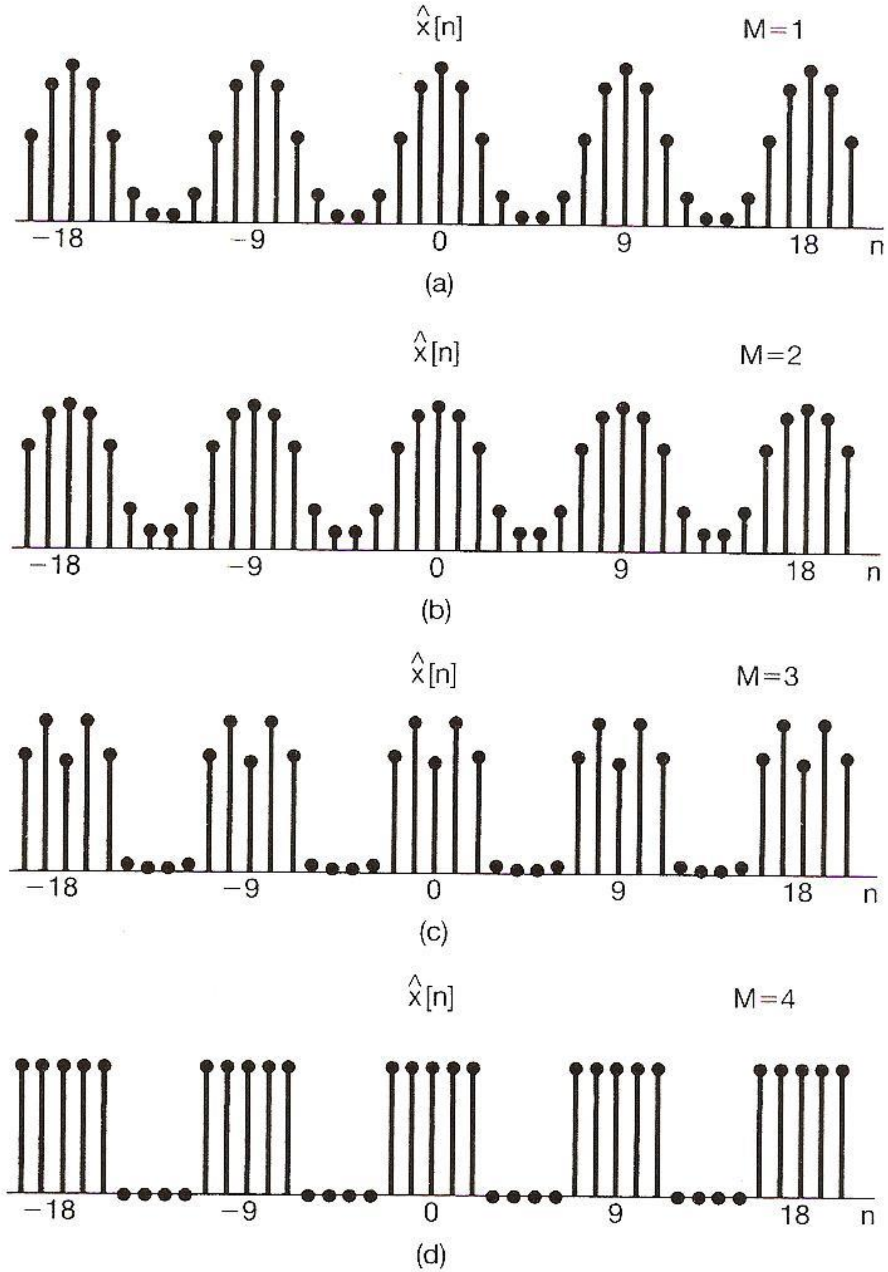


Figure 3.18 Partial sums of eqs. (3.106) and (3.107) for the periodic square wave of Figure 3.16 with $N = 9$ and $2N_1 + 1 = 5$: (a) $M = 1$; (b) $M = 2$; (c) $M = 3$; (d) $M = 4$.

Properties

- Primarily Parallel with those for continuous-time

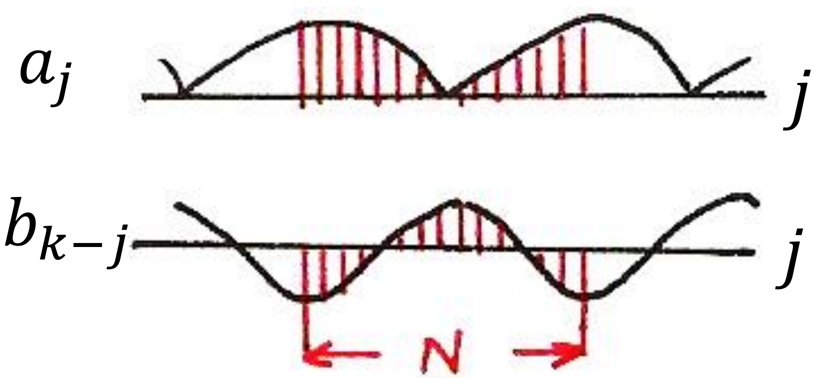
$$x[n] \stackrel{FS}{\longleftrightarrow} a_k$$

- Multiplication

$$x[n] \stackrel{FS}{\longleftrightarrow} a_k, \;\; y[n] \stackrel{FS}{\longleftrightarrow} b_k$$

$$x[n]y[n] \stackrel{FS}{\longleftrightarrow} d_k = \sum_{j=<N>} a_j b_{k-j}$$

periodic convolution



Time Shift

$$x(t - t_0) \leftrightarrow e^{-jk\omega_0 t_0} a_k$$

$$x[n - n_0] \leftrightarrow e^{-jk\omega_0 n_0} a_k$$

$$x[n - 1] \leftrightarrow e^{-jk(\frac{2\pi}{N})} a_k$$

First Difference

$$x[n] - x[n-1] \overset{FS}{\longleftrightarrow} \left(1 - e^{-jk\left(\frac{2\pi}{N}\right)}\right) a_k$$

Properties

- Parseval's Relation

$$\frac{1}{N}\sum_{k=<N>}\left|x[n]\right|^2 = \sum_{k=<N>}\left|a_k\right|^2$$

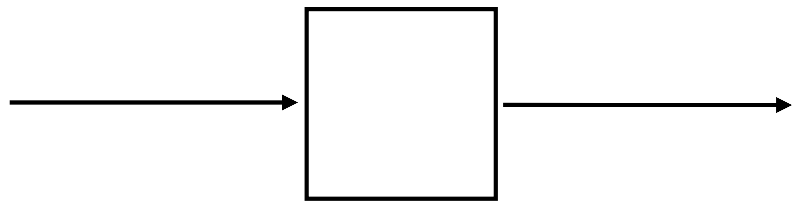
↑ average power in a period

↑ average power in a period for each harmonic component

3.5 Application Example

System Characterization

$x[n], x(t)$
$\delta[n], \delta(t)$



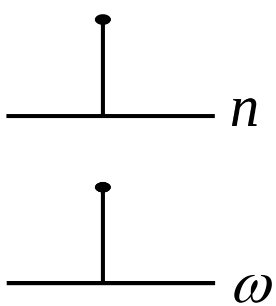
$y[n], y(t)$
$h[n], h(t)$

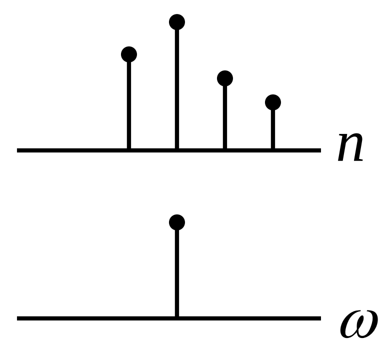
z^n , e^{st}

$e^{j\omega n}$, $e^{j\omega t}$

$H(z)z^n$, $H(s)e^{st}$, $z=e^{j\omega}$, $s=j\omega$

$H(e^{j\omega})e^{j\omega n}$, $H(j\omega)e^{j\omega t}$





Superposition Property

– Continuous-time

$$x(t)=\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_0 t}\rightarrow y(t)=\sum_{k=-\infty}^{\infty}a_k H\left(jk\omega_0\right)e^{jk\omega_0 t}$$

$$a_k \rightarrow a_k H\left(jk\omega_0\right)$$

– Discrete-time

$$x[n]=\sum_{k=<N>}a_k e^{jk\left(\frac{2\pi}{N}\right)n}\rightarrow y[n]=\sum_{k=<N>}a_k H\left(e^{jk\left(\frac{2\pi}{N}\right)}\right)e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$a_k \rightarrow a_k H\left(e^{jk\left(\frac{2\pi}{N}\right)}\right)$$

– $H(j\omega)$, $H(e^{j\omega})$ frequency response, or transfer function

Filtering

modifying the amplitude/ phase of the different frequency components in a signal, including eliminating some frequency components entirely

– frequency shaping, frequency selective

- Example 1

$$y[n] - ay[n-1] = x[n]$$

$$H\left(e^{j\omega}\right) = \frac{1}{1 - ae^{-j\omega}}$$

See Fig. 3.34, P.246 of text

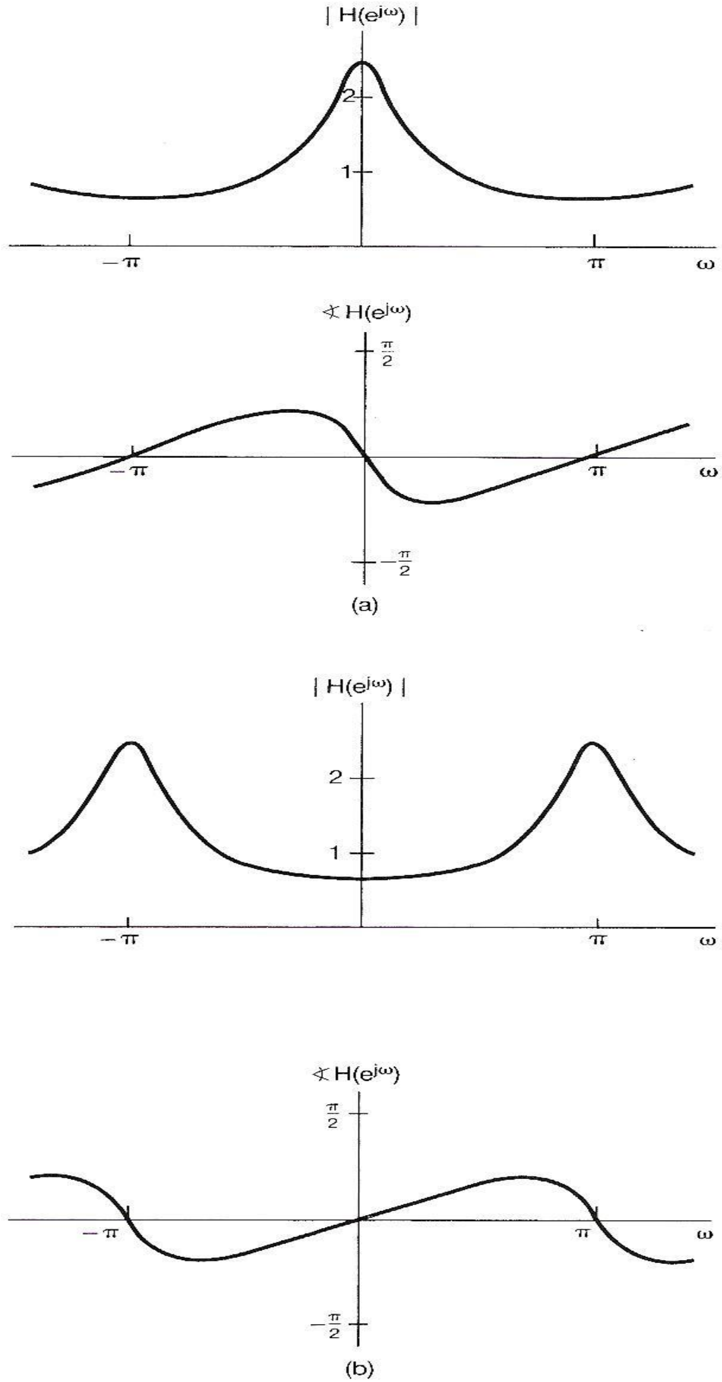


Figure 3.34 Frequency response of the first-order recursive discrete-time filter of eq. (3.151): (a) $a =$ [illegible] (b) $a = -0.6$.

Filtering

- Example 2

$$y[n] = \frac{1}{N+M+1} \sum_{k=-N}^{M} x[n-k]$$

$$h[n] = 1/(N+M+1), \quad -N \le n \le M$$

$$= 0, \qquad\qquad , \text{else}$$

See Fig. 3.36, P.248 of text

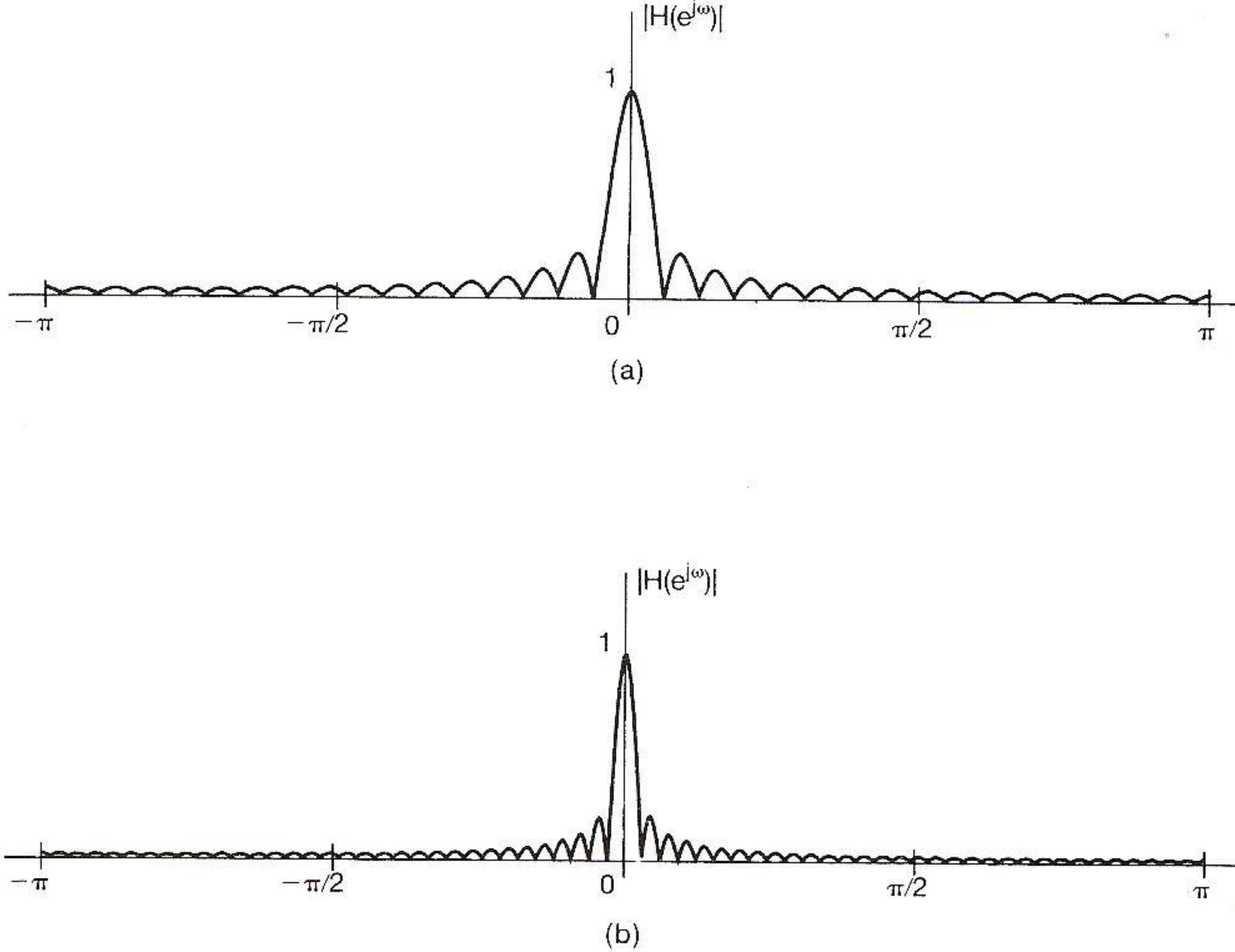


Figure 3.36 Magnitude of the frequency response for the lowpass moving-average filter of eq. (3.162): (a) $M = N = 16$; (b) $M = N = 32$.

Examples

- Example 3.5, p.193 of text

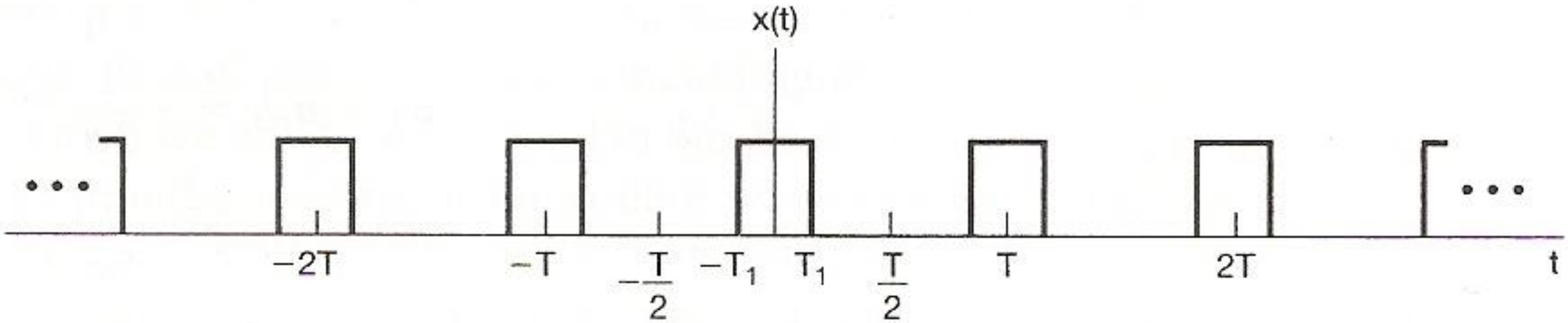


Figure 3.6 Periodic square wave.

Examples

- Example 3.5, p.193 of text

$$a_0 = \frac{1}{T} \int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$$

$$a_k = \frac{1}{T} \int_{-T_1}^{T_1} e^{-jk\omega_0 t} dt$$

$$= \frac{\sin\left(k\omega_0 T_1\right)}{k\pi}, k \neq 0$$

Examples

- Example 3.5, p.193 of text

(a)

(b)

(c)

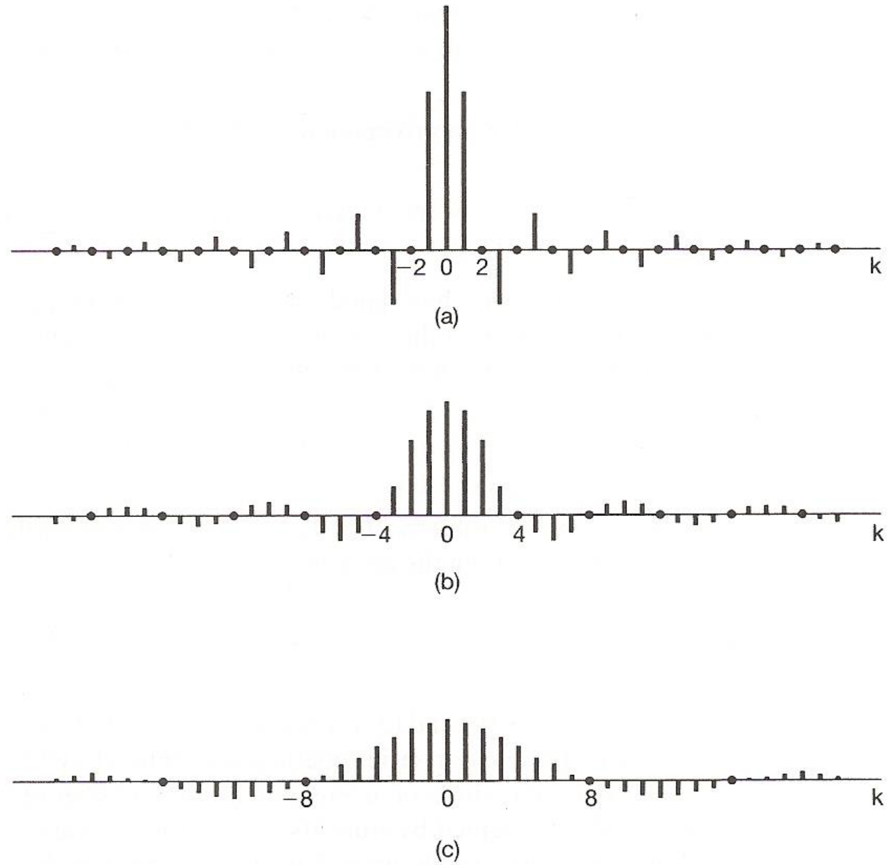


Figure 3.7 Plots of the scaled Fourier series coefficients Ta_k for the periodic square wave with T_1 fixed and for several values of T: (*a*) $T = 4T_1$; (*b*) $T = 8T_1$; (*c*) $T = 16T_1$. The coefficients are regularly spaced samples of the envelope $(2 \sin \omega T_1)/\omega$, where the spacing between samples, $2\pi/T$, decreases as T increases.

Examples

- Example 3.8, p.208 of text

(a) $x(t) \longrightarrow a_k$

(b) $g(t) \longrightarrow c_k$

(c) $q(t) \longrightarrow b_k$

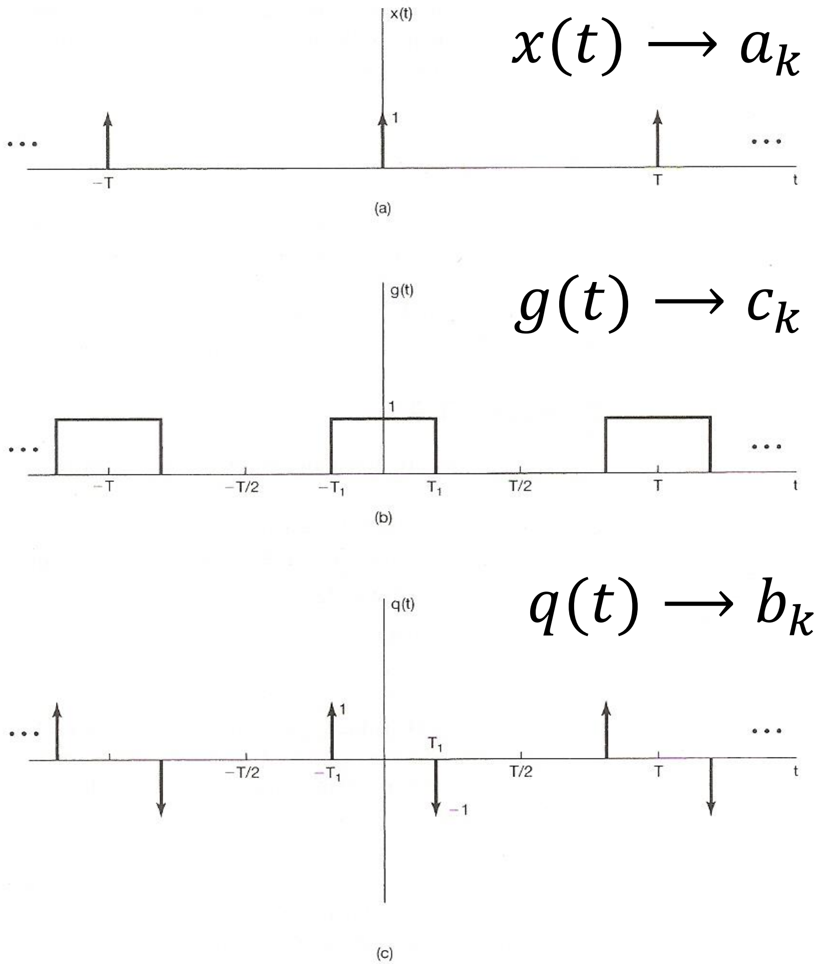


Figure 3.12 (a) Periodic train of impulses; (b) periodic square wave; (c) derivative of the periodic square wave in (b).

Examples

- Example 3.8, p.208 of text

$$x(t) \to a_k, q(t) \to b_k, g(t) \to c_k,$$

$$q(t) = x(t + T_1) - x(t - T_1), \qquad b_k = e^{jk\omega_0 T_1} a_k - e^{-jk\omega_0 T_1} a_k$$

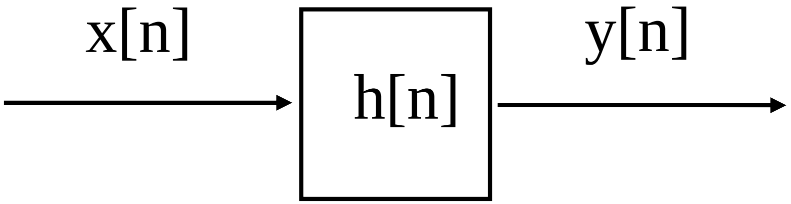
$$q(t) = \frac{d}{dt} g(t), \qquad b_k = jk\omega_0 c_k$$

$$a_k = \frac{1}{T}$$

$$c_0 = \frac{2T_1}{T}, k = 0, \quad c_k = \frac{\sin(k\omega_0 T_1)}{k\pi}, k \neq 0$$

Examples

- Example 3.17, p.230 of text



$$h[n] = \alpha^n u[n]\ , \quad |\alpha| < 1$$

$$x[n] = \cos\left(\frac{2\pi n}{N}\right) = \frac{1}{2} e^{j(\frac{2\pi}{N})n} + \frac{1}{2} e^{-j(\frac{2\pi}{N})n}$$

$$H\left(e^{j\omega}\right) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \frac{1}{1-\alpha e^{-j\omega}}$$

$$y[n] = \frac{1}{2} H\left(e^{j\frac{2\pi}{N}}\right) e^{j(\frac{2\pi}{N})n} + \frac{1}{2} H\left(e^{-j\frac{2\pi}{N}}\right) e^{-j(\frac{2\pi}{N})n}$$

$$= r\cos\left(\frac{2\pi n}{N} + \theta\right)$$

$$\text{where} \quad re^{j\theta} = \frac{1}{1-\alpha e^{-j\frac{2\pi}{N}}}$$

Problem 3.66, p.275 of text

- $\{\phi_i(t), i = 0, \pm 1, \pm 2, \ldots\}$ a set of orthonormal functions over $[a, b]$

  $\int_a^b \phi_i(t)\phi_j^*(t)dt = \delta_{ij}$

  for a signal $x(t)$ over $[a, b]$, $\hat{x}_N(t) = \sum_{i=-N}^{N} a_i \phi_i(t)$, $e_N(t) = x(t) - \hat{x}_N(t)$

  $E_N = \int_a^b |e_N(t)|^2 dt$

- It can be shown $E_N = \min$ when $a_i = \int_a^b x(t)\phi_i^*(t)dt$

  $a_i = b_i + \mathrm{j}c_i$

  $\frac{\partial E_N}{\partial b_i} = 0$, $\frac{\partial E_N}{\partial c_i} = 0$, $i = 0, \pm 1, \pm 2 \ldots$

- For basis functions not normalized

  $\int_a^b \phi_i(t)\phi_j^*(t)dt = A\delta_{ij}$

  $a_i = \frac{1}{A}\int_a^b x(t)\phi_i^*(t)dt$

<u>Problem 3.70</u>, p.281 of text

- 2-dimensional signals

$$x(t_1, t_2) = x(t_1 + T_1, t_2 + T_2), \text{ all } t_1, t_2$$

$$x(t_1, t_2) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{mn} e^{j(m\omega_{10} t_1 + n\omega_{20} t_2)}$$

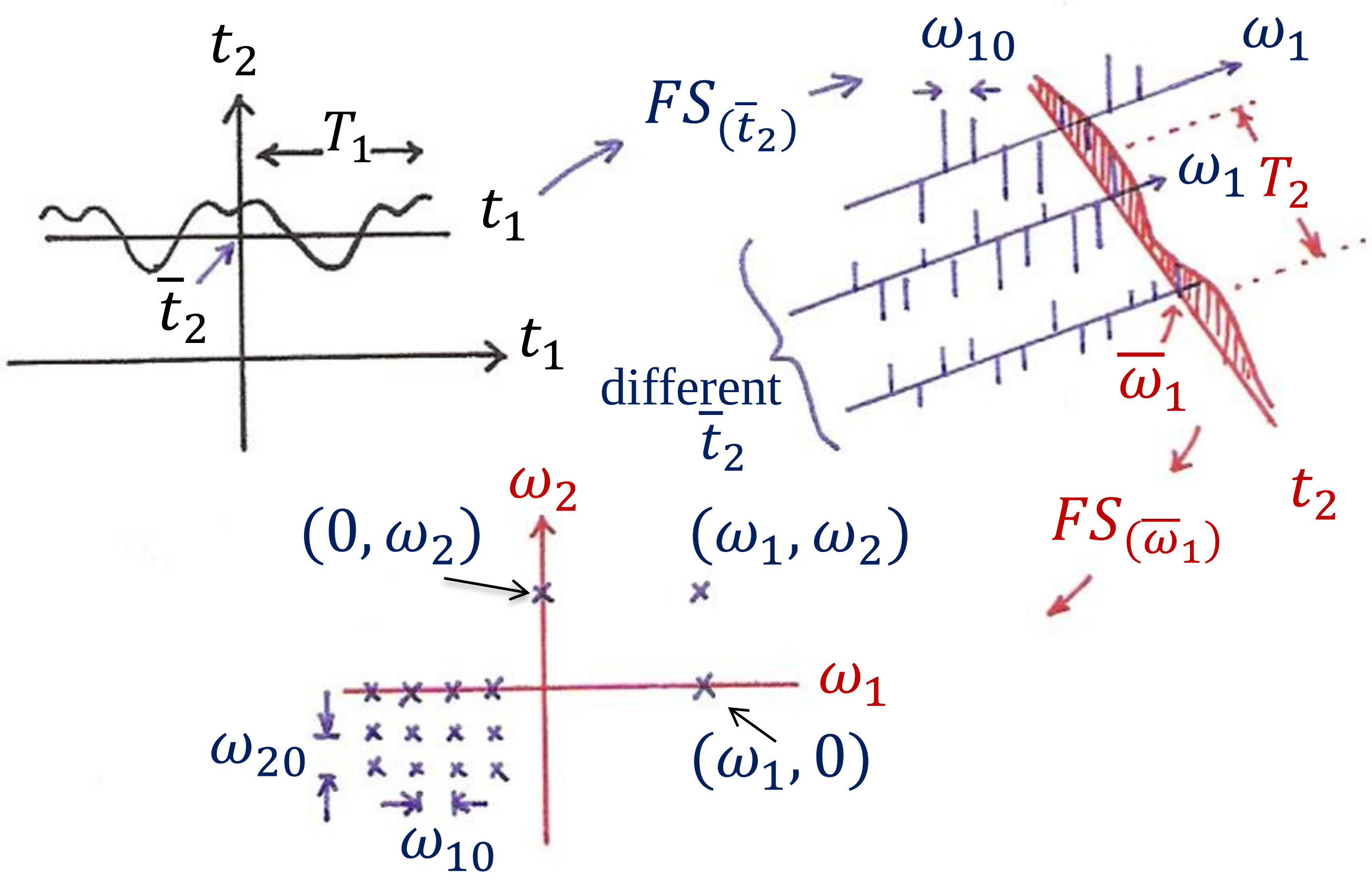
$$\omega_{10} = \frac{2\pi}{T_1}, \omega_{20} = \frac{2\pi}{T_2}$$

$$a_{mn} = \frac{1}{T_1 T_2} \int_{T_1} \int_{T_2} x(t_1, t_2) e^{-jm\omega_{10} t_1} e^{-jn\omega_{20} t_2} dt_1 dt_2$$

$$= \frac{1}{T_2} \int_{T_2} \left[\frac{1}{T_1} \int_{T_1} x(t_1, t_2) e^{-jm\omega_{10} t_1} dt_1 \right] e^{-jn\omega_{20} t_2} dt_2$$

Problem 3.70, p.281 of text

- 2-dimensional signals



<u>Problem 3.70</u>, p.281 of text

- 2-dimensional signals