4.0 Continuous-time Fourier Transform

4.1 From Fourier Series to Fourier Transform

- Fourier Series : for periodic signal

  $x(t) = x(t+T)$, T : fundamental period

  $$\mathrm{x}(\mathrm{t}) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},\ \omega_0 = \frac{2\pi}{T}$$

  as T increases, $\omega_0 = \dfrac{2\pi}{T}$ decreases

  the envelope Ta_k is sampled at closer and closer spacing

  See Fig. 3.6, 3.7, p.193, 195, Fig, 4.2, p.286 of text

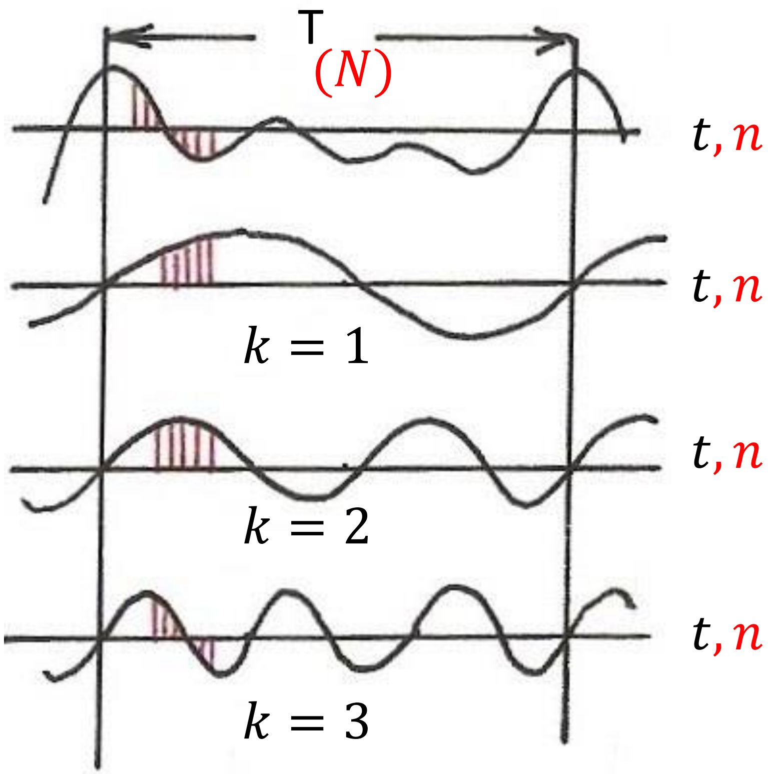
  – aperiodic : $T \to \infty$, $\omega_0 \to 0$

Harmonically Related Exponentials for Periodic Signals (P.11 of 3.0)

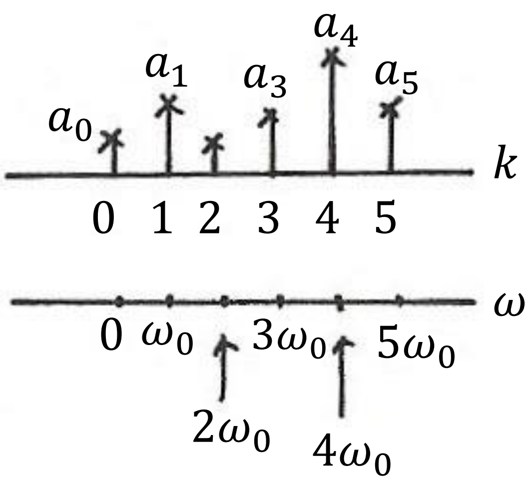
$[n]$

$V = \{x(t)|x(t)$ periodic, fundamental period

$[n]$ $= T(N)\}$

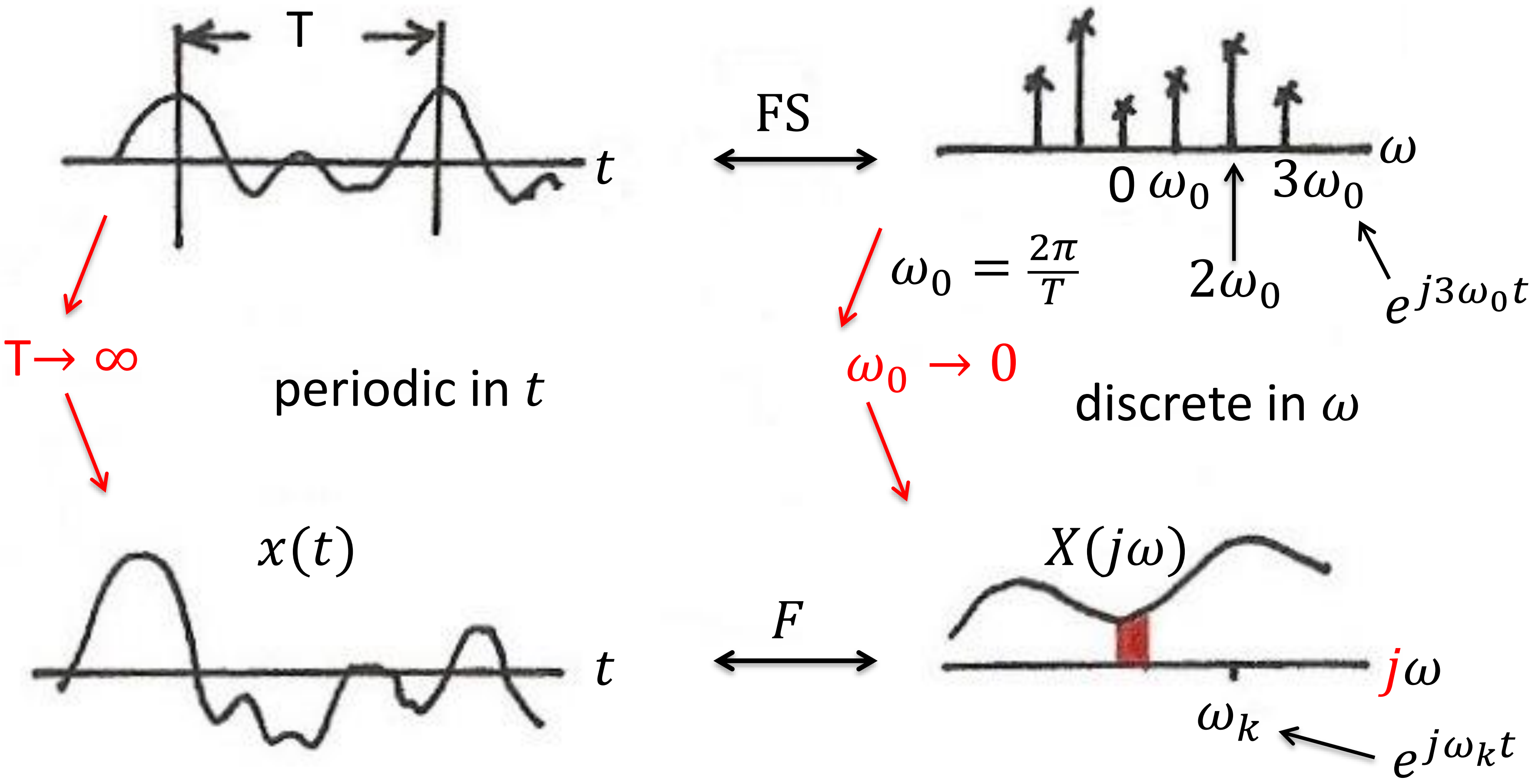


$$\omega_0 = \frac{2\pi}{T(N)}$$



- All with period T: integer multiples of ω_0
  (N)
- Discrete in frequency domain

Fourier Transform



$T \to \infty$

periodic in t

$\omega_0 = \frac{2\pi}{T}$

$\omega_0 \to 0$

discrete in ω

aperiodic in t

continuous in ω

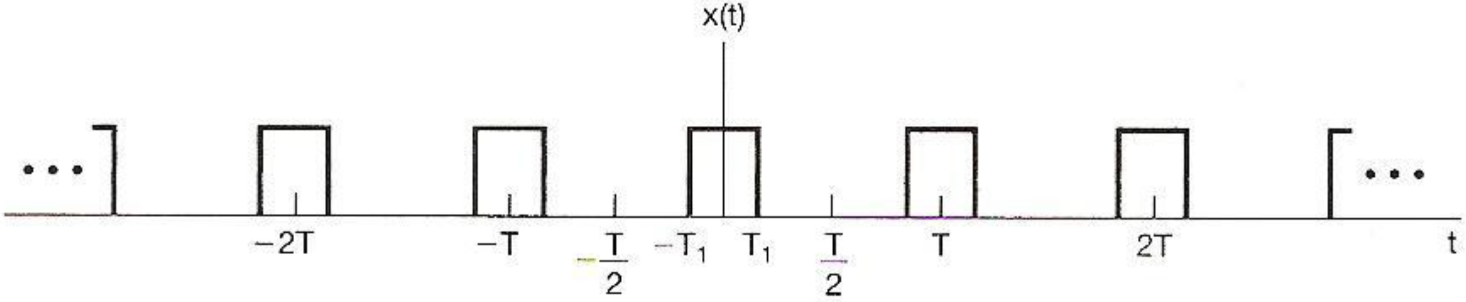


Figure 3.6 Periodic square wave.

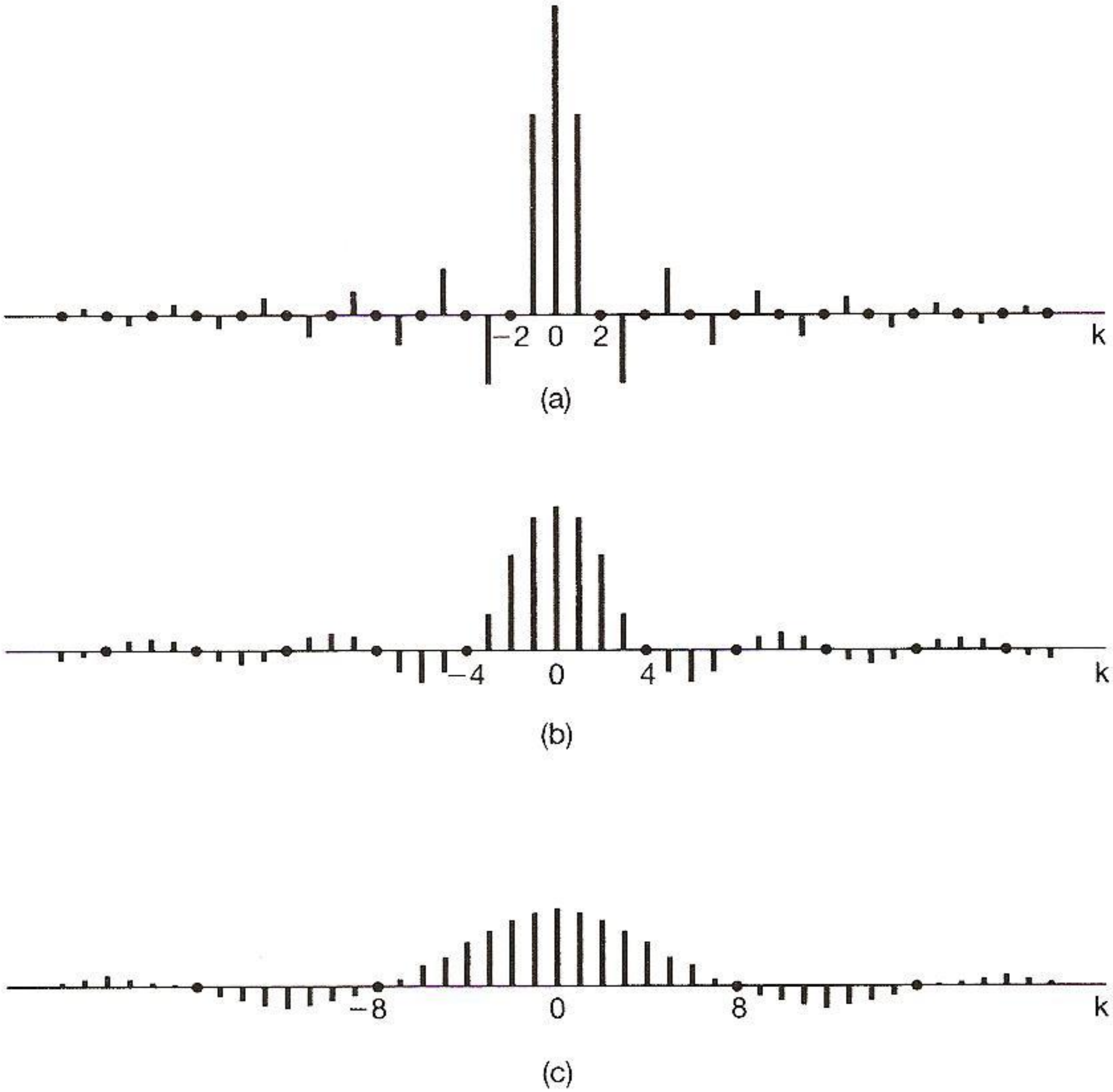


Figure 3.7 Plots of the scaled Fourier series coefficients Ta_k for the periodic square wave with T_1 fixed and for several values of T: (*a*) $T = 4T_1$; (*b*) $T = 8T_1$; (*c*) $T = 16T_1$. The coefficients are regularly spaced samples of the envelope $(2 \sin \omega T_1)/\omega$, where the spacing between samples, $2\pi/T$, decreases as T increases.

the envelope Ta_k is sampled at closer and closer spacing

See Fig. 3.6, 3.7, p.193, 195, Fig, 4.2, p.286 of text

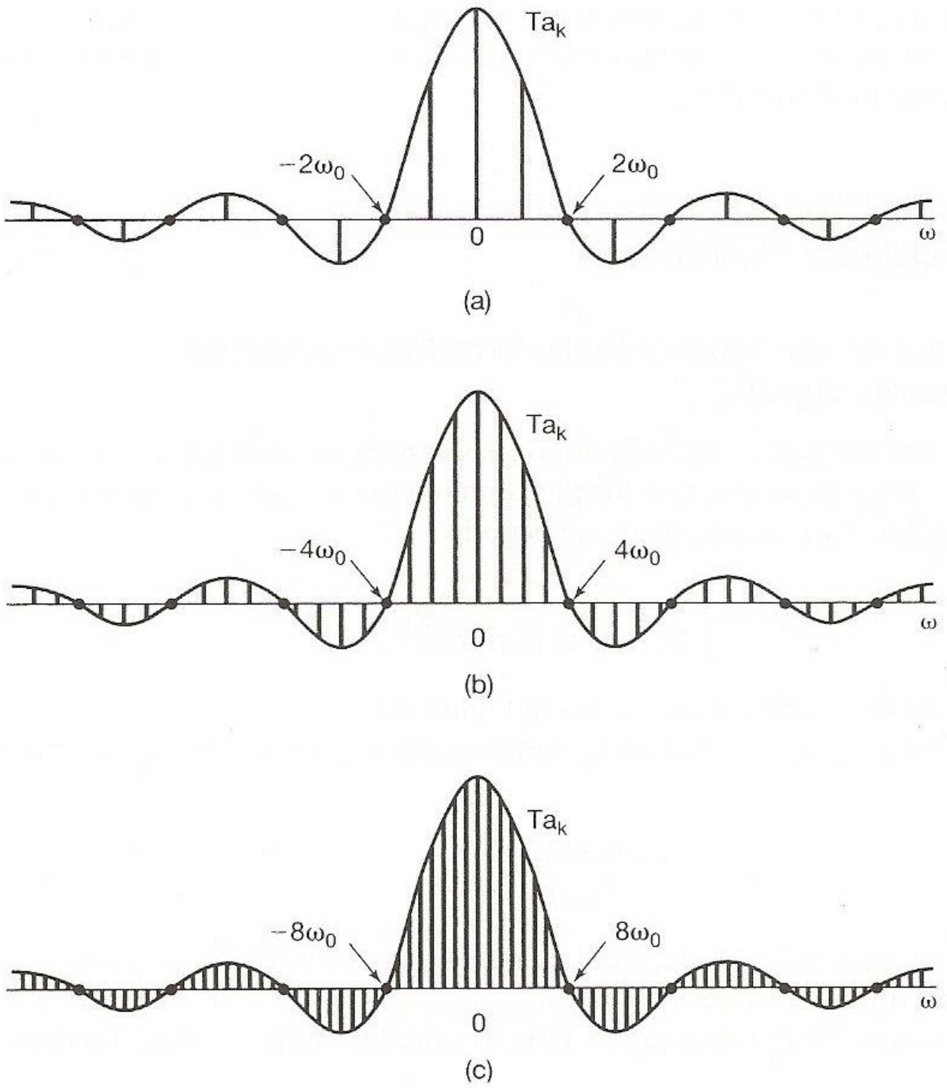


Figure 4.2 The Fourier series coefficients and their envelope for the periodic square wave in Figure 4.1 several values of T (with T_1 fixed): (a) $T = 4T_1$; (b) $T = 8T_1$; (c) $T = 16T_1$.

- Considering $x(t)$, $x(t)=0$ for $| t | > T_1$
  - construct $\tilde{x}(t)$ periodic with period $T > 2T_1$

$$\tilde{x}(t) = x(t) \text{ if } |t| < \frac{T}{2}$$

$$\tilde{x}(t) \to x(t) \text{ if } T \to \infty$$

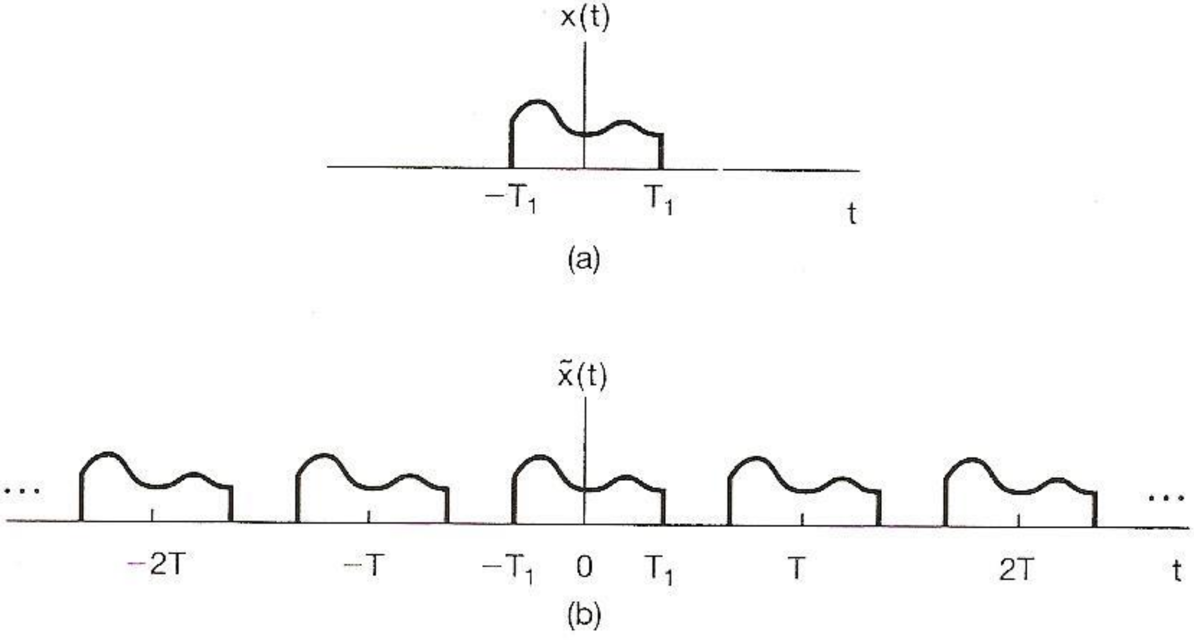


Figure 4.3 (a) Aperiodic signal $x(t)$; (b) periodic signal $\tilde{x}(t)$, constructed to be equal to $x(t)$ over one period.

- Considering $x(t)$, $x(t)=0$ for $|\,t\,| > T_1$
  - Fourier series for $\tilde{x}(t)$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \boxed{a_k} e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_T \tilde{x}(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\boxed{\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}dt}$$

  - Defining envelope of Ta_k as $X(j\omega)$

$$\boxed{a_k = \frac{1}{T}X(jk\omega_0)} = \frac{1}{T}\boxed{X(j\omega)\Big|_{\omega=k\omega_0}}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

- Considering $x(t)$, $x(t)=0$ for $|\, t \,| > T_1$

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X\left(jk\omega_0 \right) e^{jk\omega_0 t} \omega_0$$

$$\frac{\omega_0}{2\pi} = \frac{1}{T}$$

  - as $T \to \infty, \omega_0 \to 0, \ \tilde{x}(t) \to x(t)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X\left(j\omega \right) e^{j\omega t} d\omega$$

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt:$$ spectrum, frequency domain
Fourier Transform

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega:$$ signal, time domain
Inverse Fourier Transform

Fourier Transform pair, different expressions

$$x(t)\xleftrightarrow{F}X(j\omega)$$

very similar format to Fourier Series for periodic signals

- Convergence Issues
  - Given $x(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$e(t) = \hat{x}(t) - x(t)$$

$$E_e = \int_{-\infty}^{\infty} |\, e(t)\,|^2 dt$$

- Convergence Issues
  - It can be shown

if $\int_{-\infty}^{\infty} \left| x(t) \right|^2 dt < \infty$

$\rightarrow$ (i) $X\left(j\omega\right)$ is obtainable (finite) for every ω

(ii) $E_e = 0$

zero energy for the difference signal
differences at isolated points are possible

$\hat{x}(t)$ converges to $x(t)$ at continuous points,
but to averages at discontinuities

- Convergence Issues
  - Dirichlet’s conditions

    (1) absolutely integrable $\int_{-\infty}^{\infty} |\ x(t)\ |\ dt < \infty$

    (2) finite number of maxima and minima within any finite interval

    (3) finite number of discontinuities with finite values within any finite interval

Examples

- Example 4.4, p.293 of text

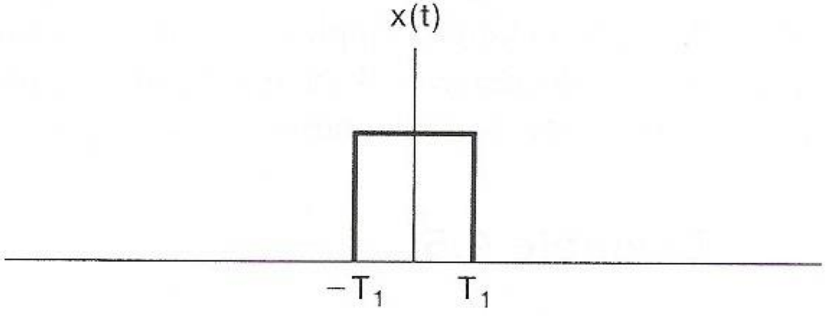
$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$

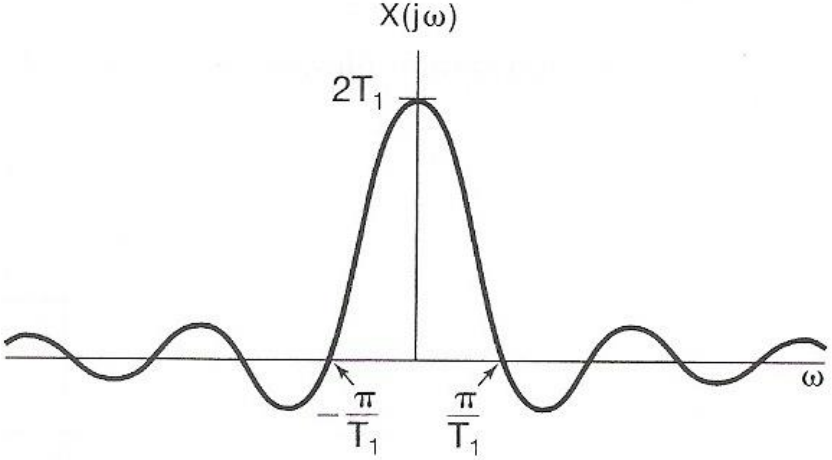
$$\text{then } X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\, dt$$

$$= \frac{2\sin \omega T_1}{\omega}$$

$$= 2T_1 \operatorname{sin}c(\frac{\omega T_1}{\pi})$$

$$\text{where } \operatorname{sin}c(\theta) = (\frac{\sin \pi\theta}{\pi\theta})$$





Examples

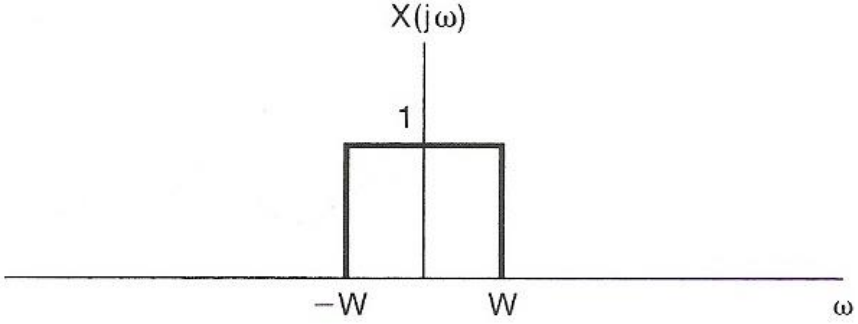
- Example 4.5, p.294 of text

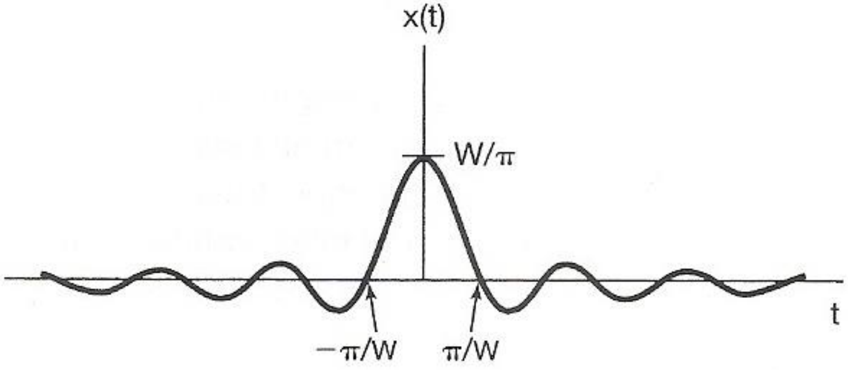
$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$

then $x(t) = \frac{1}{2\pi} \int_{-W}^{W} e^{j\omega t}\, d\omega$

$$= \frac{\sin Wt}{\pi t}$$

$$= \frac{W}{\pi} \text{sinc}\left(\frac{Wt}{\pi}\right)$$





- Fourier Transform for Periodic Signals – Unified Framework
  - Given $x(t)$

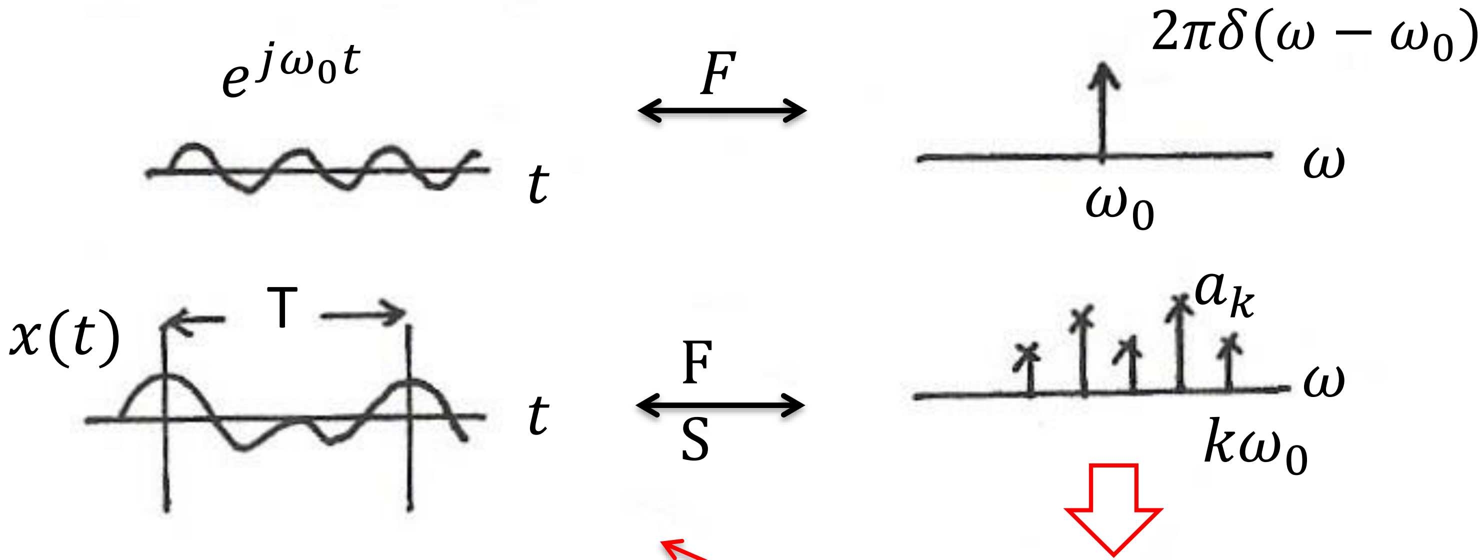
assume $X(j\omega) = 2\pi\delta(\omega - \omega_0)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}$$

$$e^{j\omega_0 t} \overset{F}{\longleftrightarrow} 2\pi\delta(\omega - \omega_0)$$

(easy in one way)

Unified Framework: Fourier Transform for Periodic Signals



- If $x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$$X(j\omega)=\sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega-k\omega_0)$$

$$\begin{tikzcd} x(t) \arrow[r, leftrightarrow, "FS"] \arrow[rr, leftrightarrow, bend right, "F"'] & a_k \arrow[r] & X(j\omega) \end{tikzcd}$$

Examples

- Example 4.7, p.298 of text

$$x(t) = \cos\omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

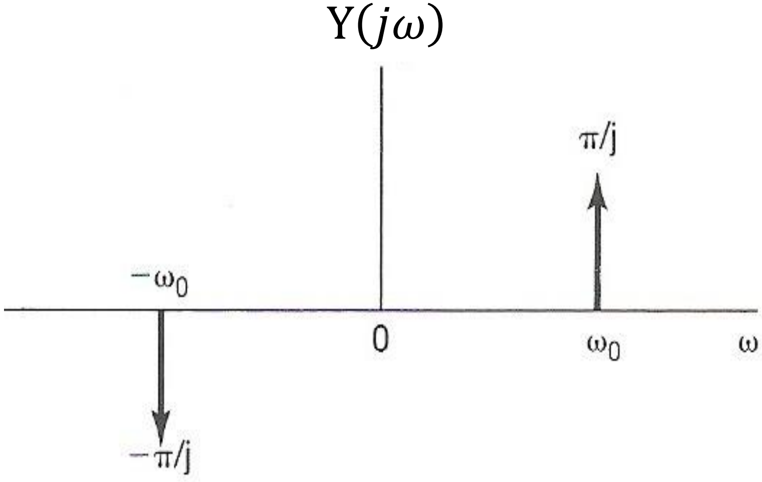
$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

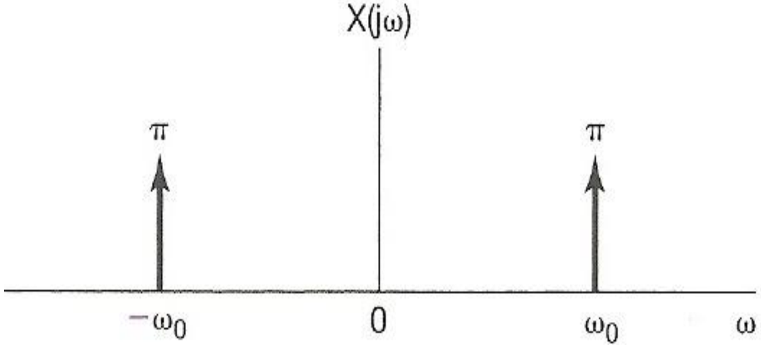
$$FS\colon\ a_1 = a_{-1} = \frac{1}{2}, \qquad a_k = 0 \quad \text{else}$$

$$y(t) = \sin\omega_0 t = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})$$

$$\mathrm{Y}(j\omega) = \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$$

$$FS\colon\ a_1 = -a_{-1} = \frac{1}{2j},\ \ a_k = 0 \ \ \text{else}$$





4.2 Properties of Continuous-time Fourier Transform

$$x(t) \xleftrightarrow{F} X(j\omega)$$

- Linearity

$$x(t) \xleftrightarrow{F} X(j\omega), \; y(t) \xleftrightarrow{F} Y(j\omega)$$

$$ax(t) + by(t) \xleftrightarrow{F} aX(j\omega) + bY(j\omega)$$

Linearity (P.27 of 3.0)

$$x(t) \xleftrightarrow{FS} a_k$$

- Linearity

$$x(t) \xleftrightarrow{FS} a_k, \quad y(t) \xleftrightarrow{FS} b_k$$

$$Ax(t) + By(t) \xleftrightarrow{FS} Aa_k + Bb_k$$

$$\vec{x} = (a_1, a_2, a_3, \cdots)$$

$$\vec{y} = (b_1, b_2, b_3, \cdots)$$

$$A\vec{x} + B\vec{y} = (Aa_1 + Bb_1, Aa_2 + Bb_2, \cdots)$$

- Time Shift

$$x(t - t_0) \xleftrightarrow{F} e^{-j\omega t_0} X(j\omega)$$

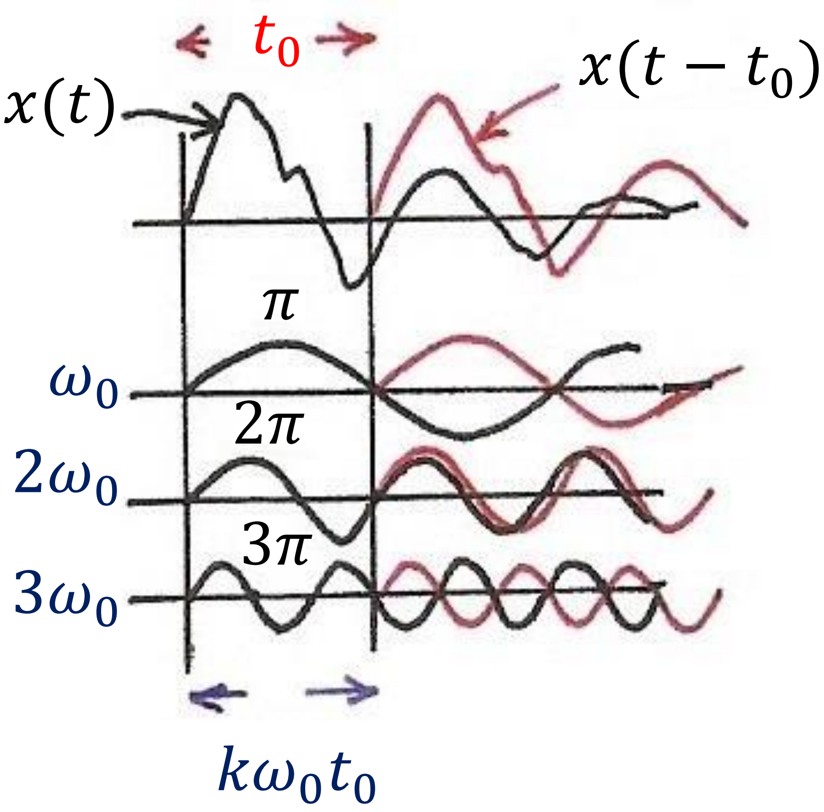
linear phase shift (linear in frequency) with amplitude unchanged

Time Shift (P.28 of 3.0)

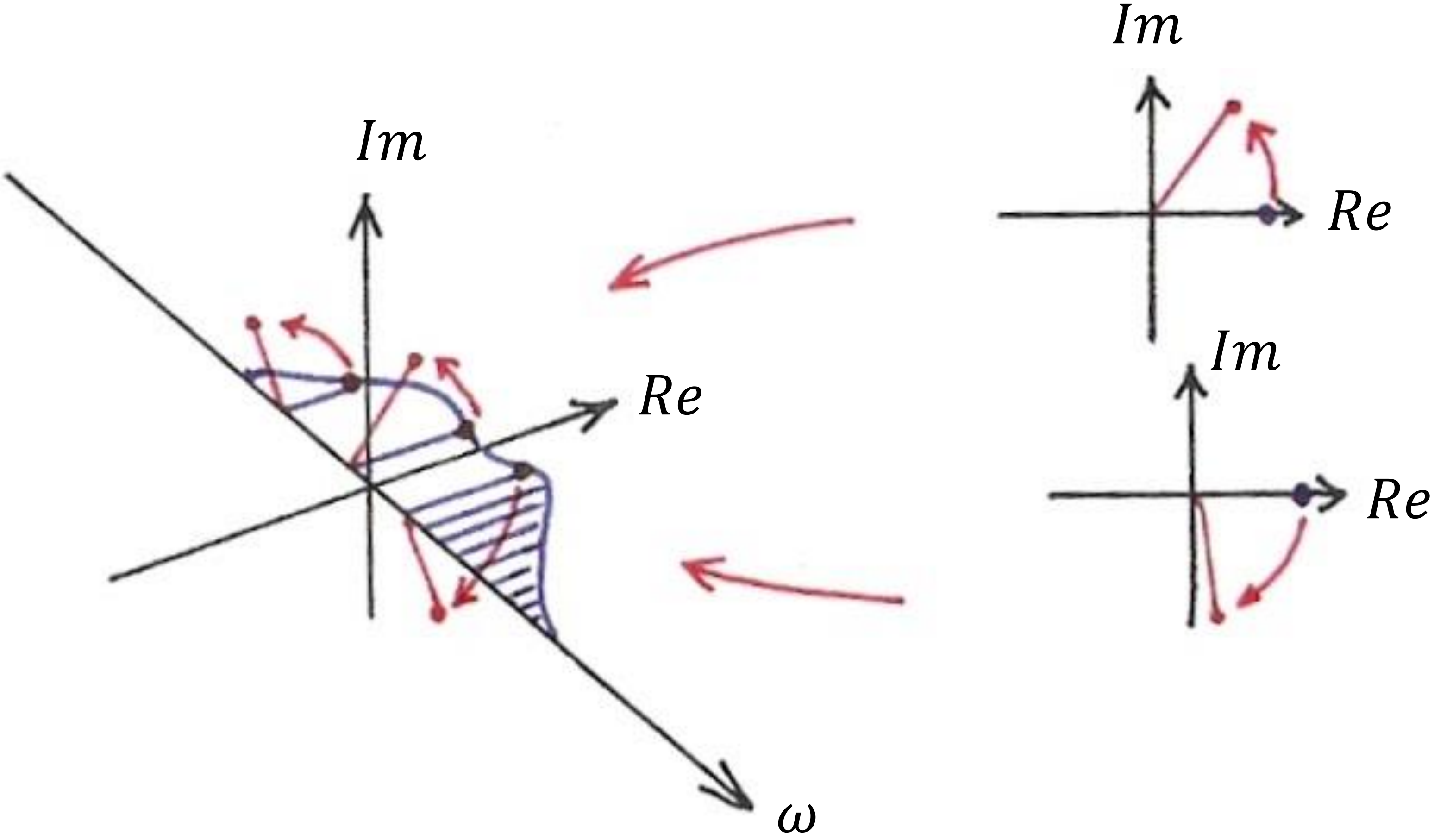
$$x(t - t_0) \xleftrightarrow{FS} e^{-jk\omega_0 t_0} a_k$$

phase shift linear in frequency with amplitude unchanged

$$a_k\, e^{jk\omega_0 (t - t_0)} = \boxed{e^{-jk\omega_0 t_0}\, a_k}\, e^{jk\omega_0 t}$$



Time Shift



Examples (P.18 of 4.0)

- Example 4.7, p.298 of text

$$x(t) = \cos \omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

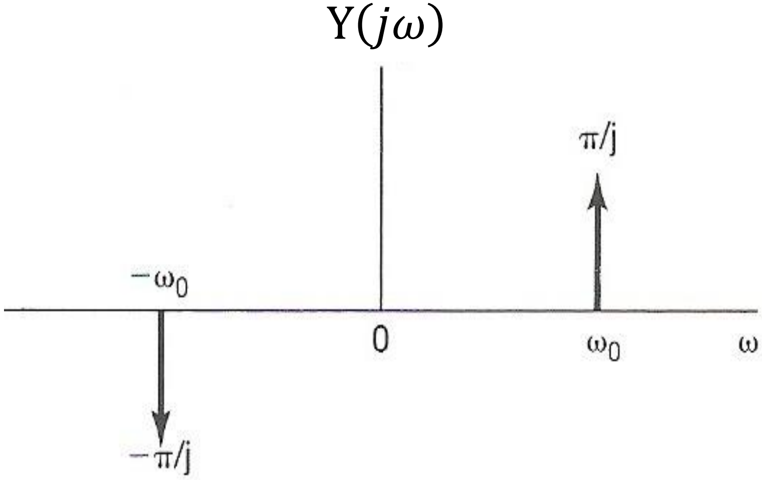
$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

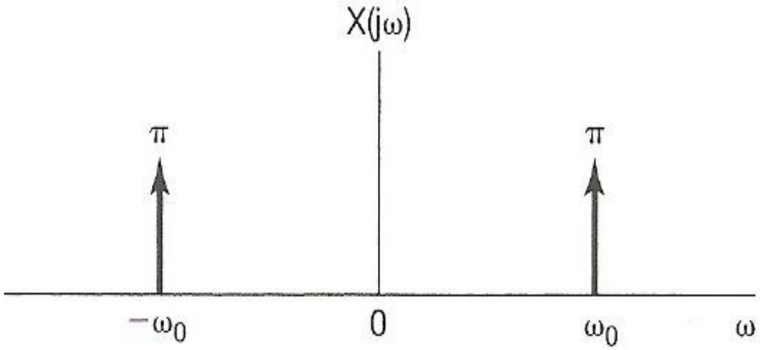
$$FS:\ a_1 = a_{-1} = \frac{1}{2}, \qquad a_k = 0 \quad \text{else}$$

$$y(t) = \sin \omega_0 t = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})$$

$$\mathrm{Y}(j\omega) = \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$$

$$FS:\ a_1 = -a_{-1} = \frac{1}{2j},\ \ a_k = 0 \ \ \text{else}$$

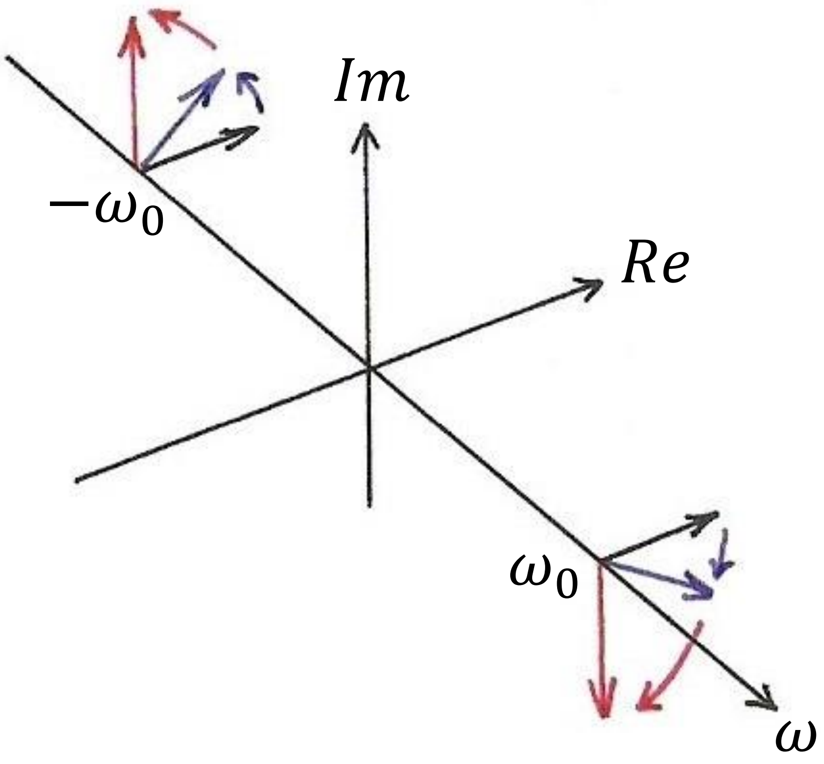


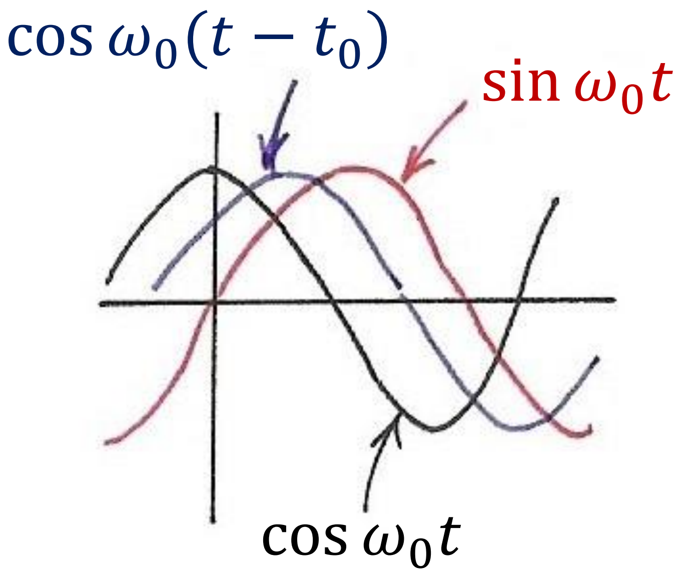


Sinusoidals

$$\cos\omega_0 t \overset{F}{\leftrightarrow} \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)],\ \ \tfrac{1}{2}[e^{j\omega_0 t}+e^{-j\omega_0 t}]$$

$$\sin\omega_0 t \overset{F}{\leftrightarrow} \frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)],\ \tfrac{1}{2j}[e^{j\omega_0 t}-e^{-j\omega_0 t}]$$





$$x(t-t_0)\leftrightarrow e^{-j\omega t_0}\cdot X(j\omega)$$

- Conjugation

$$x^*(t) \xleftrightarrow{F} X^*(-j\omega)$$

if $x(t)$ real, $X(j\omega)$ conjugate symmetric

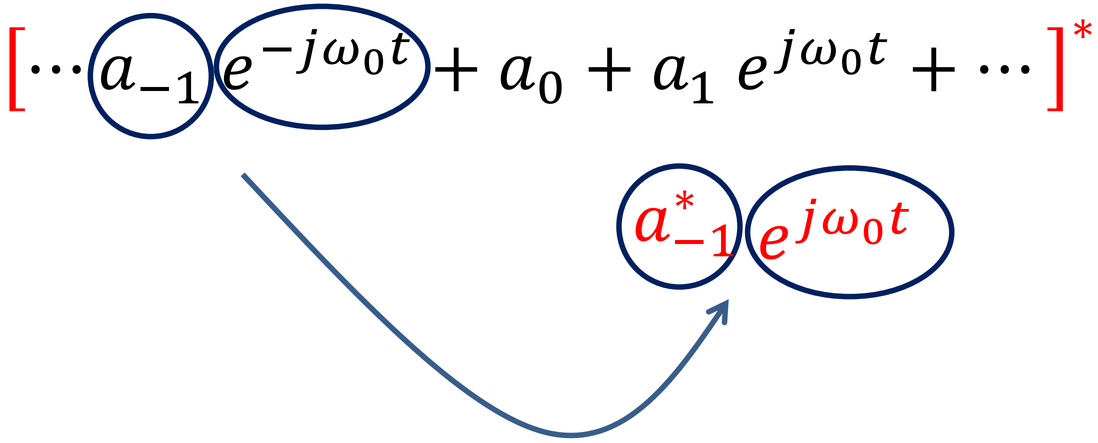
$$X(-j\omega) = X^*(j\omega),\ x(t)\ \text{real}$$

even/odd properties

- Conjugation (P.32 of 3.0)

$$x^*(t) \xleftrightarrow{FS} a^*_{-k}$$

$$a_{-k} = a_k^*, \text{ if } \; x(t) \, \text{real}$$



$$\left[\cdots a_{-1} e^{-j\omega_0 t} + a_0 + a_1\, e^{j\omega_0 t} + \cdots\right]^*$$

$$a^*_{-1} e^{j\omega_0 t}$$

unique representation

Conjugation

$$\left[\int_{-\infty}^{\infty} \cdots \boxed{X(-j\omega_k)}\,\boxed{e^{-j\omega_k t}} + \cdots + X(j\omega_k)e^{j\omega_k t} + \cdots d\omega\right]^*$$

$$\Downarrow$$

$$\boxed{X^*(-j\omega_k)}\,\boxed{e^{j\omega_k t}}$$

Unique representation for orthogonal bases

- Conjugation

$$x^*(t) \xleftrightarrow{F} X^*(-j\omega)$$

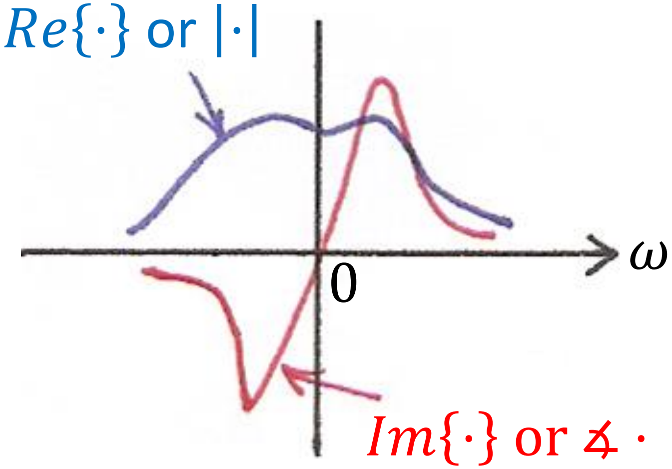
Even/Odd Properties

- Conjugation Property

$$x^*(t) \xleftrightarrow{F} X^*(-j\omega)$$

$$X(-j\omega) = X^*(j\omega) \text{ if } x(t) \text{ is real}$$

$$X(j\omega) = \mathrm{Re}\{X(j\omega)\} + j\mathrm{Im}\{X(j\omega)\}$$



- if $x(t)$ is real

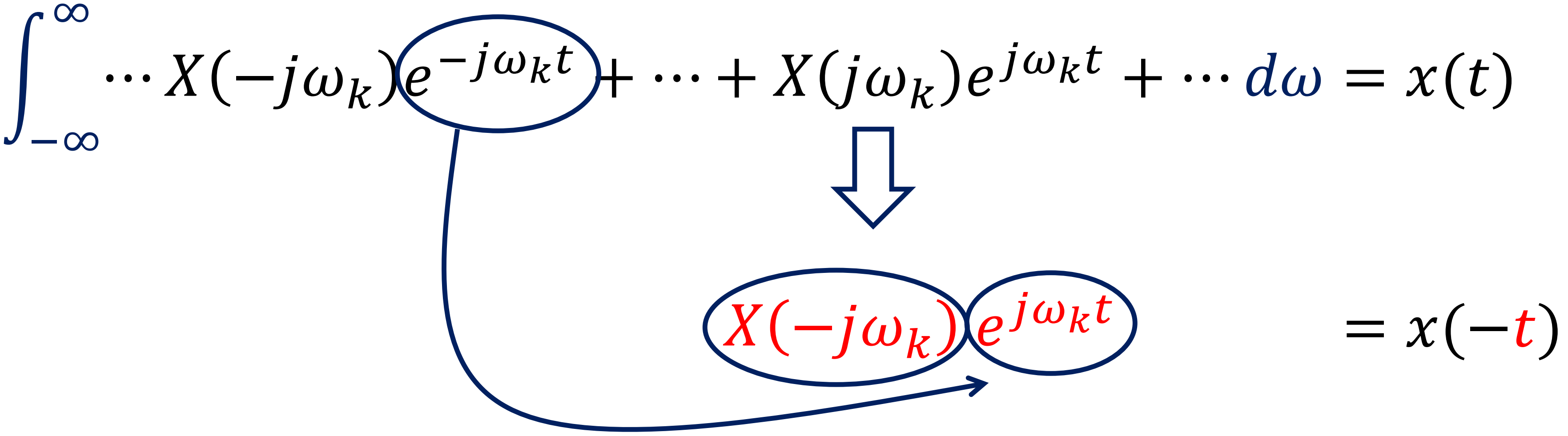
$$\mathrm{Re}\{X(-j\omega)\} = \mathrm{Re}\{X^*(j\omega)\} = \mathrm{Re}\{X(j\omega)\} \quad \text{real part is even}$$

$$\mathrm{Im}\{X(j\omega)\} = -\mathrm{Im}\{X^*(j\omega)\} = -\mathrm{Im}\{X(-j\omega)\} \text{ imaginary part is odd}$$

$$X(j\omega) = |X(j\omega)| e^{j\angle X(j\omega)}$$

$$|X(j\omega)| \text{ is even but } \angle X(j\omega) \text{ is odd}$$

Time Reversal



$$\int_{-\infty}^{\infty} \cdots X(-j\omega_k)e^{-j\omega_k t} + \cdots + X(j\omega_k)e^{j\omega_k t} + \cdots d\omega = x(t)$$

$$\Downarrow$$

$$X(-j\omega_k)e^{j\omega_k t} \qquad = x(-t)$$

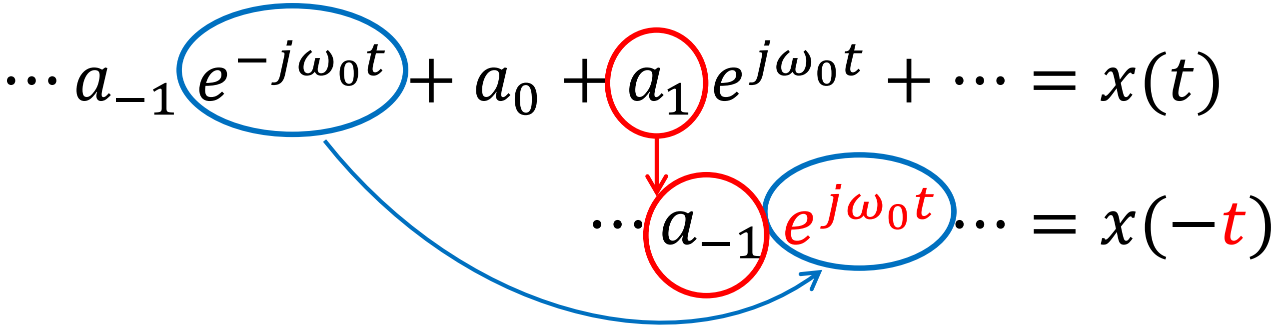
Unique representation for orthogonal bases

$$x(-t) \xleftrightarrow{F} X(-j\omega)$$

- Time Reversal (P.29 of 3.0)

$$x(-t) \xleftrightarrow{FS} a_{-k}$$

the effect of sign change for $x(t)$ and a_k are identical



$$\cdots a_{-1} e^{-j\omega_0 t} + a_0 + a_1 e^{j\omega_0 t} + \cdots = x(t)$$

$$\cdots a_{-1} e^{j\omega_0 t} \cdots = x(-t)$$

unique representation for orthogonal basis

Even/Odd Properties

- $x(t)$ both real and even
  - Time Reversal

$$x(-t) \xleftrightarrow{F} X(-j\omega)$$

$$X^*(j\omega) = X(-j\omega) \xleftrightarrow{F} x(-t) = x(t) \xleftrightarrow{F} X(j\omega),$$

$\therefore\ X(j\omega)$ is real

$\therefore\ X(j\omega)$ both real and even, example : cosine

- $x(t)$ real and odd

$$X^*(j\omega) = X(-j\omega) \xleftrightarrow{F} x(-t) = -x(t) \xleftrightarrow{F} -X(j\omega)$$

$$\mathrm{Re}\{X^*(j\omega)\} = \mathrm{Re}\{X(j\omega)\} = -\mathrm{Re}\{X(j\omega)\} = 0$$

$\therefore\ X(j\omega)$ pure imaginary and odd, example : sine

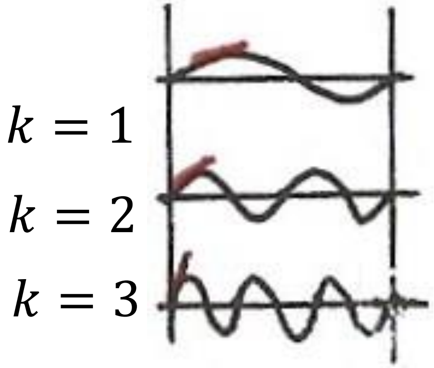
- Differentiation/Integration

$$\frac{dx(t)}{dt} \xleftrightarrow{F} j\omega X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{F} \frac{1}{j\omega} X(j\omega) + \pi \underset{\substack{\uparrow \\ \text{dc term}}}{X(j0)}\delta(\omega)$$

- Differentiation (P.33 of 3.0)

$$\frac{dx(t)}{dt} \xleftrightarrow{FS} jk\omega_0 a_k$$

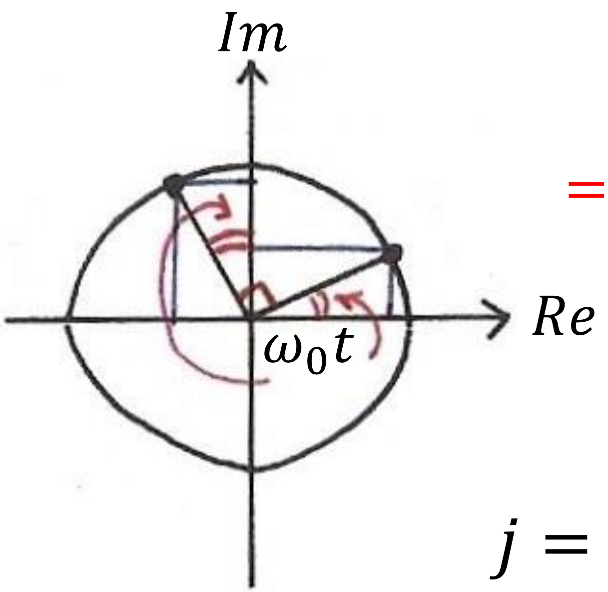


$$\frac{d}{dt}\left(a_k e^{jk\omega_0 t}\right) = \boxed{\underline{\underline{j}}\ \underline{\underline{k}}\omega_0\ a_k}\, e^{jk\omega_0 t}$$

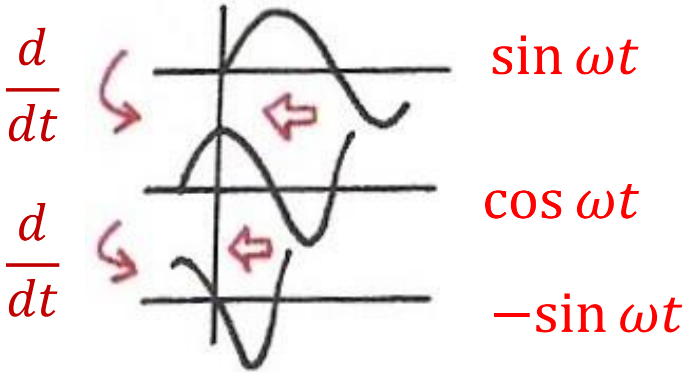
$$j \cdot [\underline{\cos\omega_0 t} + j\,\underline{\sin\omega_0 t}]$$

$$\frac{d}{dt}\Big\downarrow \qquad\qquad \frac{d}{dt}\Big\downarrow$$

$$= \underline{-\sin\omega_0 t} + j\,\underline{\cos\omega_0 t}$$



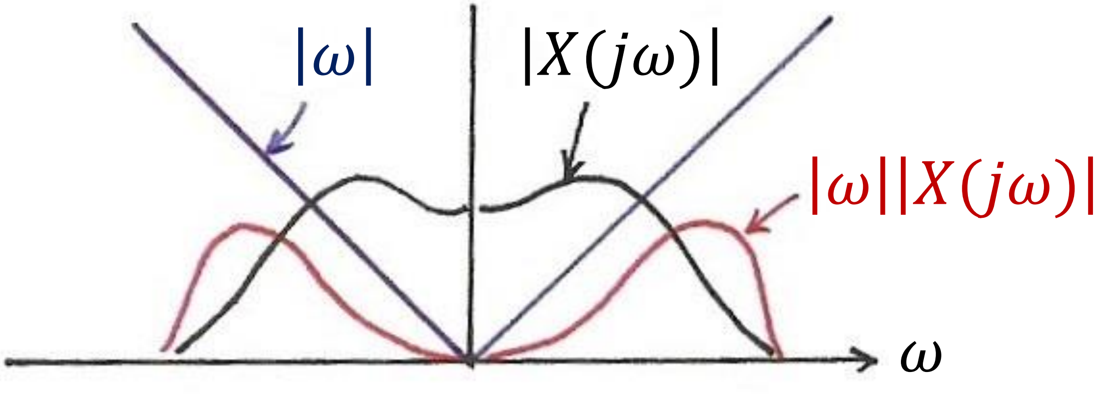
$$j = e^{j90^\circ}$$



Differentiation

$$\frac{dx(t)}{dt} \xleftrightarrow{F} j\omega X(j\omega)$$

$$|j\omega \cdot X(j\omega)| = |\omega| \cdot |X(j\omega)|$$



Enhancing higher frequencies

De-emphasizing lower frequencies

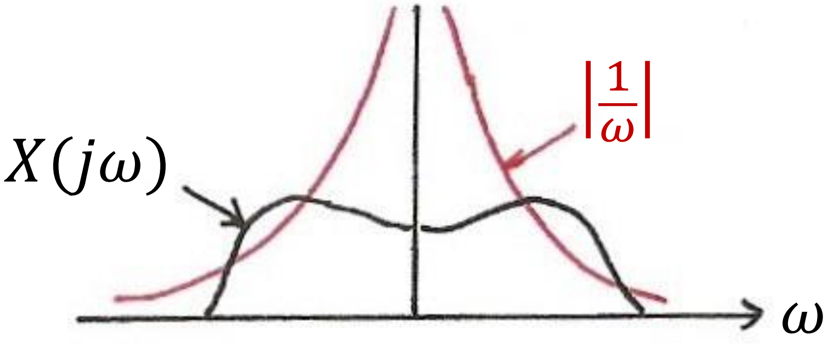
Deleting DC term (=0 for ω=0)

Integration

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{F} \frac{1}{j\omega}X(j\omega)+\pi \underset{\substack{\uparrow \\ \text{dc term}}}{X(j0)}\delta(\omega)$$

$$\frac{1}{j} = e^{-j90^\circ}$$

$$\left|\frac{1}{j\omega}\right| \cdot |X(j\omega)| = \left|\frac{1}{\omega}\right| \cdot |X(j\omega)|$$



Enhancing lower frequencies (accumulation effect)

De-emphasizing higher frequencies

(smoothing effect)

Undefined for ω=0

- Parseval's Relation

$$\int_{-\infty}^{\infty} |\, x(t) \,|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\, X(j\omega) \,|^2 d\omega$$

total energy: energy per unit time integrated over the time

total energy: energy per unit frequency integrated over the frequency

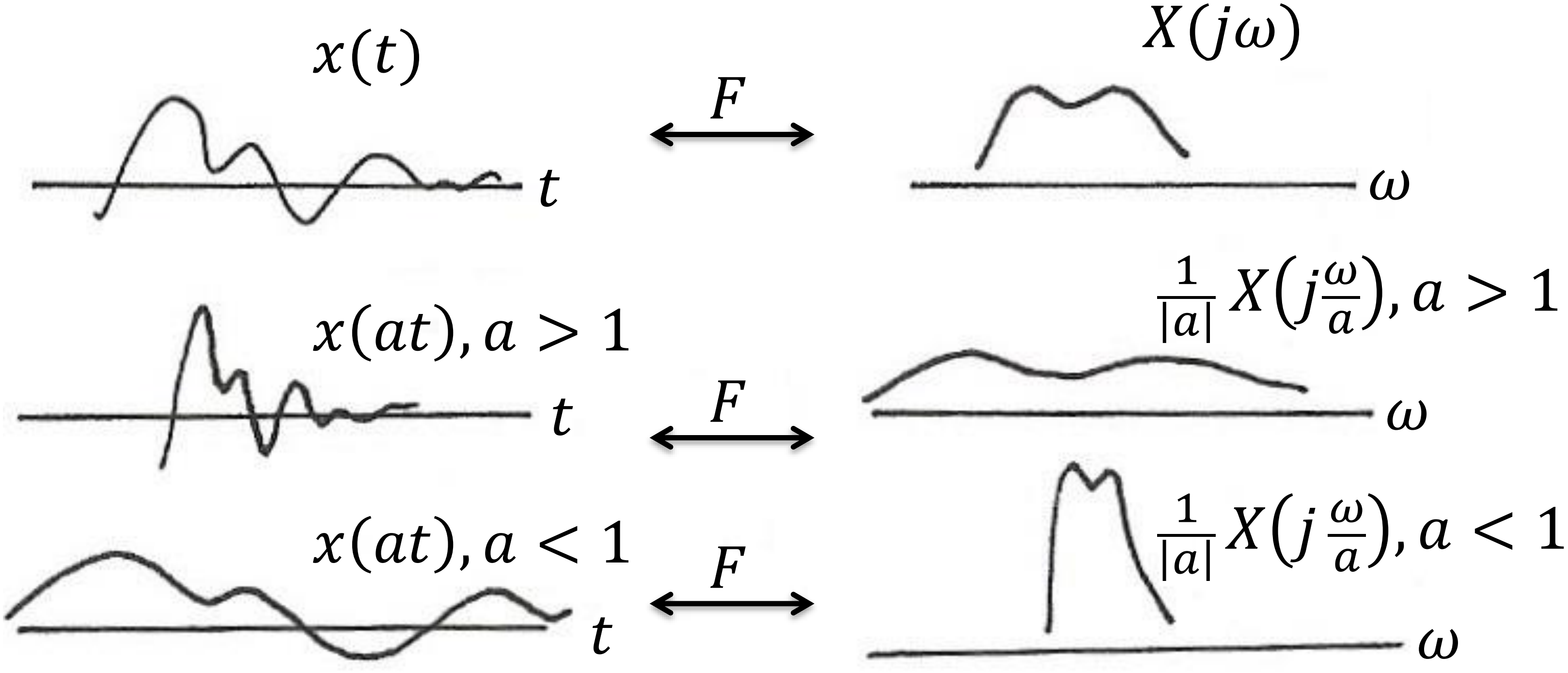
$$\vec{A} = \sum_i a_i \, \hat{v}_i = \sum_k b_k \, \hat{u}_k$$

$$\| \vec{A} \|^2 = \sum_i |a_i|^2 = \sum_k |b_k|^2$$

- Time/Frequency Scaling

$$x(at) \xleftrightarrow{F} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

$$x(-t) \xleftrightarrow{F} X(-j\omega)$$ (time reversal)



See Fig. 4.11, p.296 of text

- inverse relationship between signal “width” in time/frequency domains

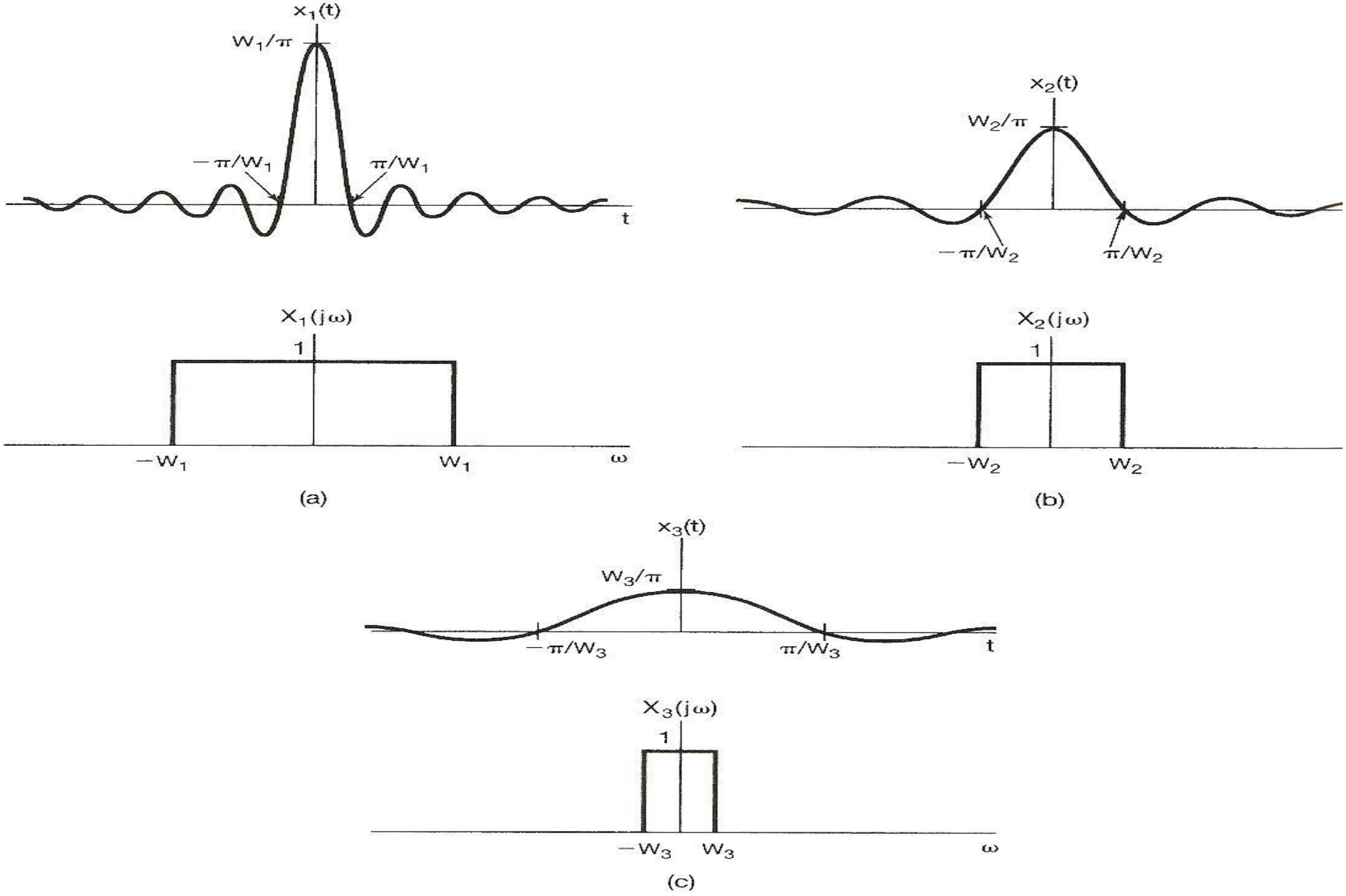
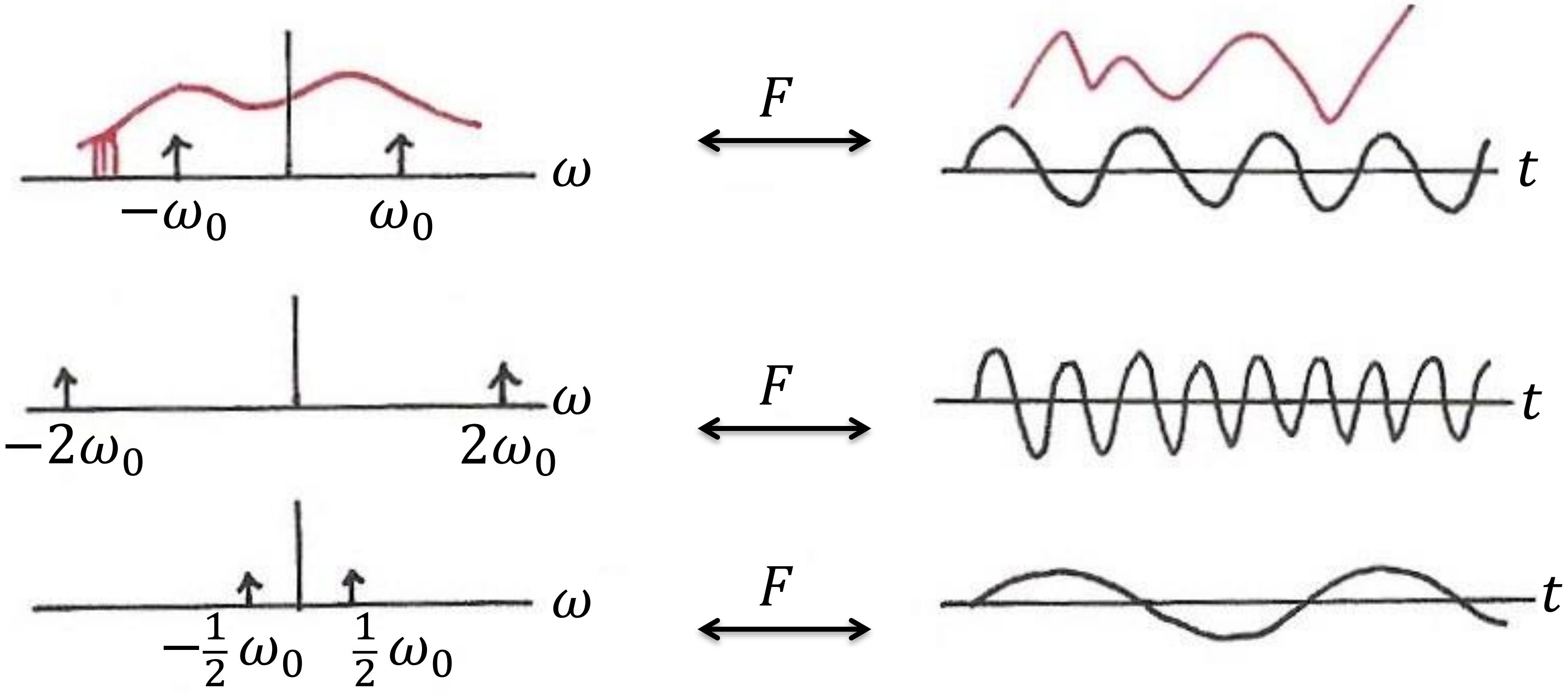


Figure 4.11 Fourier transform pair of Figure 4.9 for several different values of *W*.

Single Frequency

$$\cos \omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$



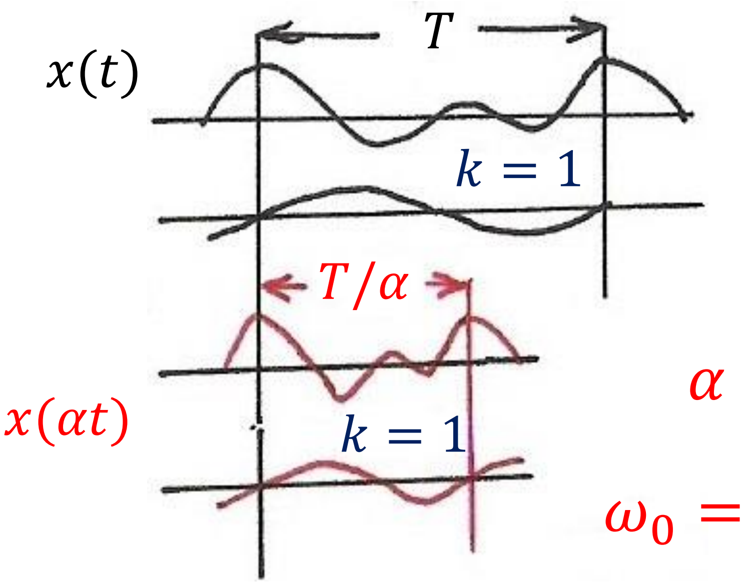
- Time Scaling (P.30 of 3.0)

  α : positive real number

  $x(\alpha t)$: periodic with period T/α and fundamental frequency $\alpha\omega_0$

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t}$$

  a_k unchanged, but $x(\alpha t)$ and each harmonic component are different

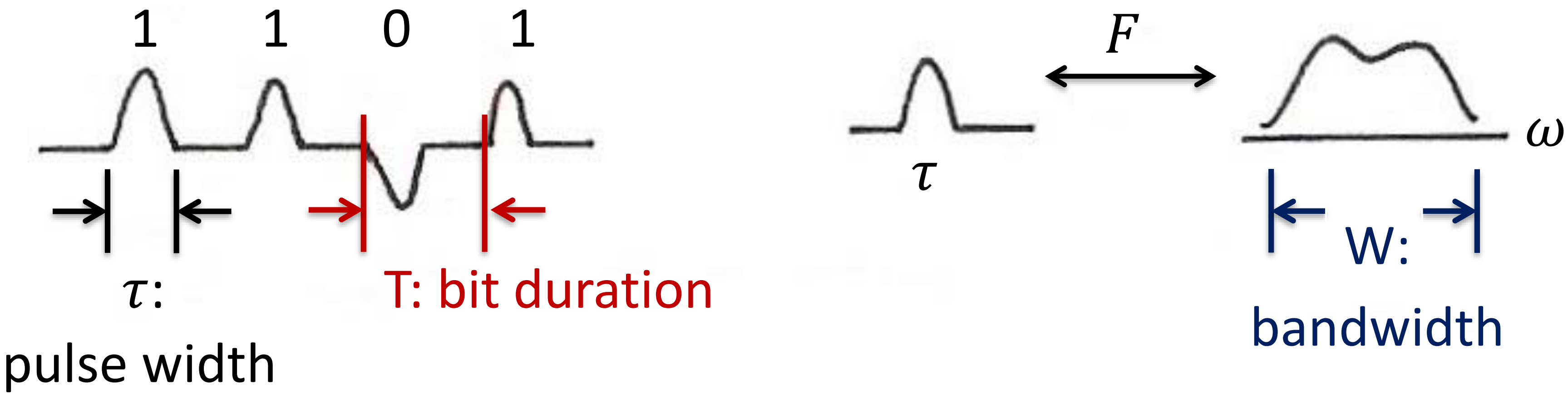


$$\omega_0 = \frac{2\pi}{T}$$

$\alpha > 1$

$$\omega_0 = \frac{2\pi}{T/\alpha} = \alpha(\frac{2\pi}{T})$$

Data Transmission



$$W \propto \frac{1}{\tau} \geq \frac{1}{T} = r \text{ : bit rate}$$

(required bandwidth) ∝ (bit rate)

- Duality

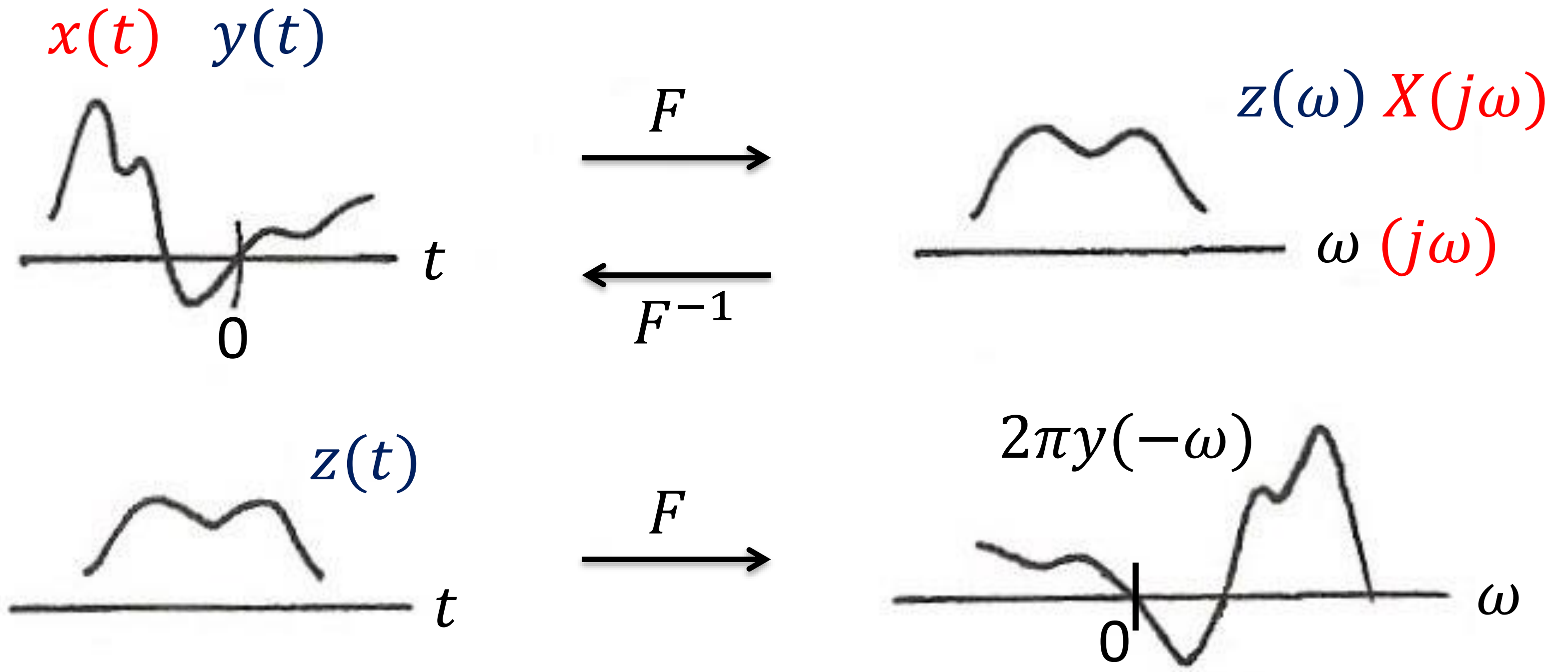
$$x(t)\xleftrightarrow{F}X(j\omega)\Rightarrow y(t)\xleftrightarrow{F}z(\omega)$$

$$z(t)\xleftrightarrow{F}2\pi y(-\omega)$$

  – time/frequency domains are kind of "symmetric" except for a sign change (and a factor of 2π) --- "two domains"

    See Fig. 4.17, p.310 of text

Duality



$$x(t) \xleftrightarrow{F} X(j\omega) \Rightarrow y(t) \xleftrightarrow{F} z(\omega)$$

$$z(t) \xleftrightarrow{F} 2\pi y(-\omega)$$

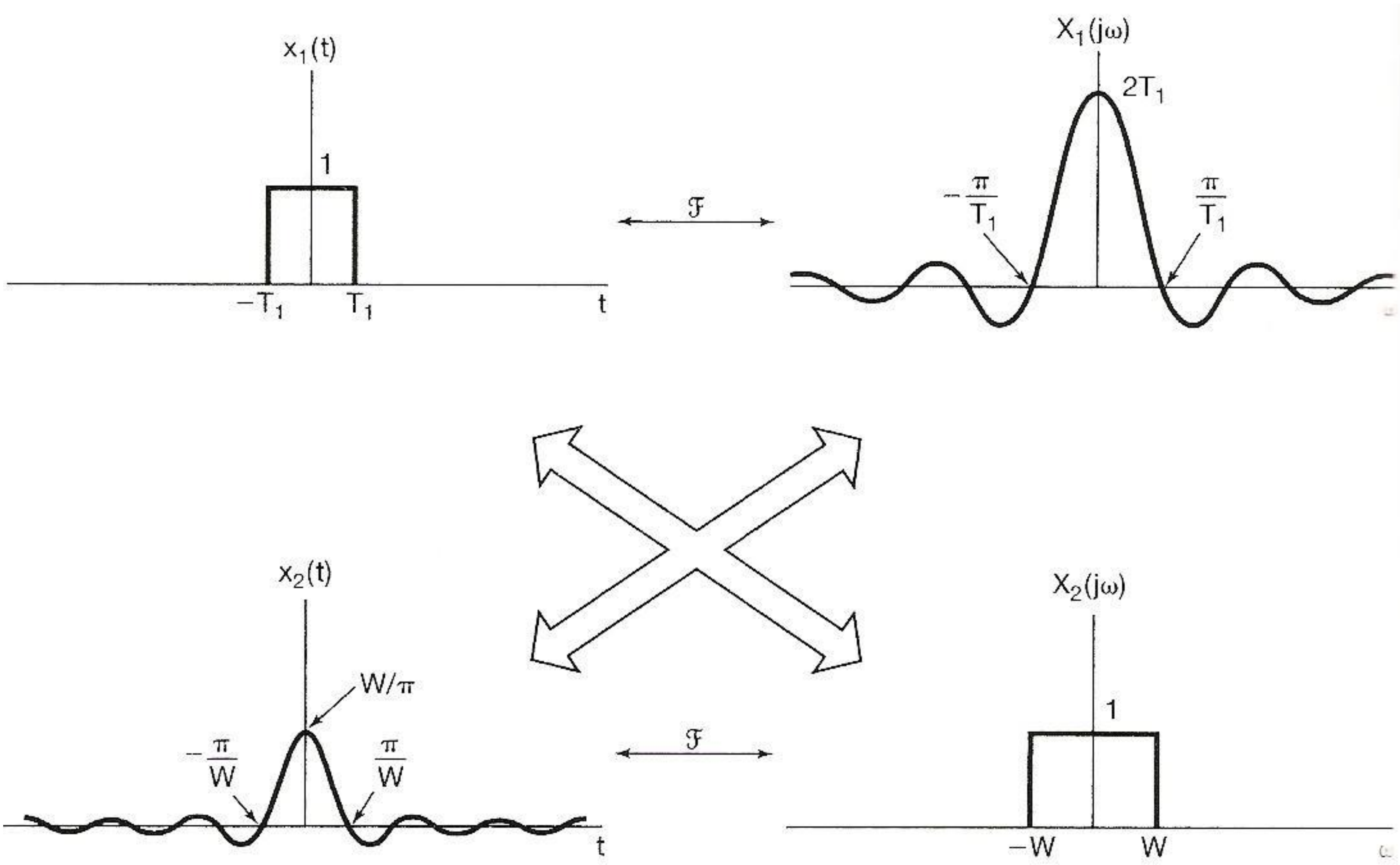


Figure 4.17 Relationship between the Fourier transform pairs of eqs. (4.36) and (4.37).

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt\ :$$ spectrum, frequency domain
Fourier Transform

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega\ :$$ signal, time domain
Inverse Fourier Transform

Fourier Transform pair, different expressions

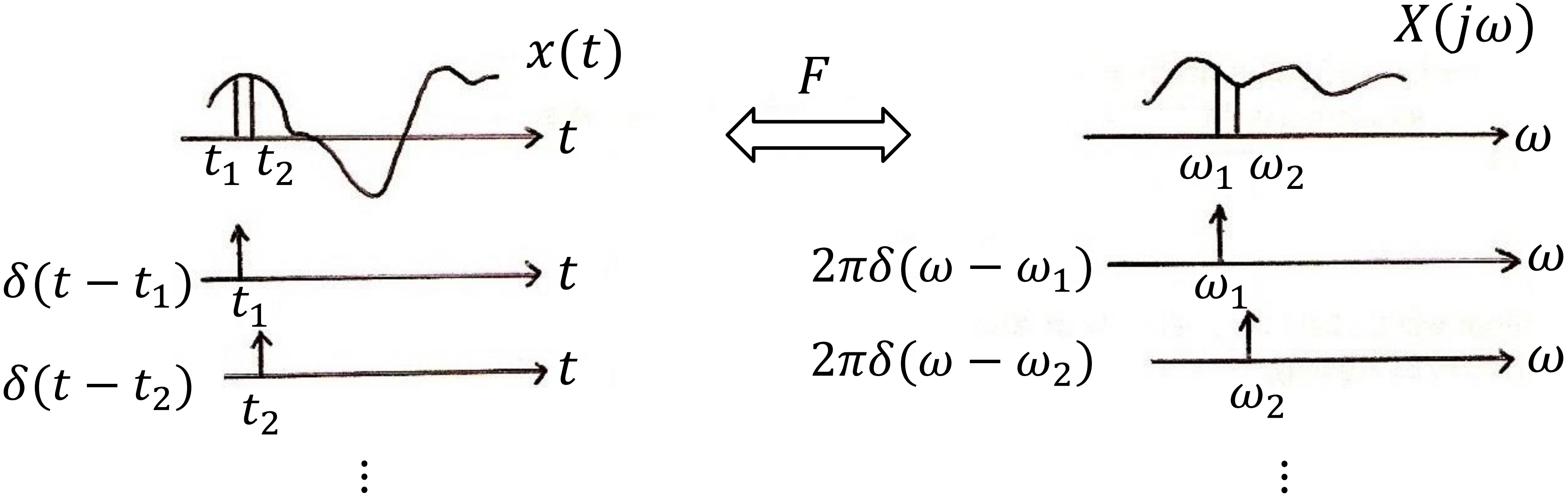
$$x(t)\xleftrightarrow{F}X(j\omega)$$

very similar format to Fourier Series for periodic signals

Signal Representation in Two Domains

Time Domain Basis

Frequency Domain Basis



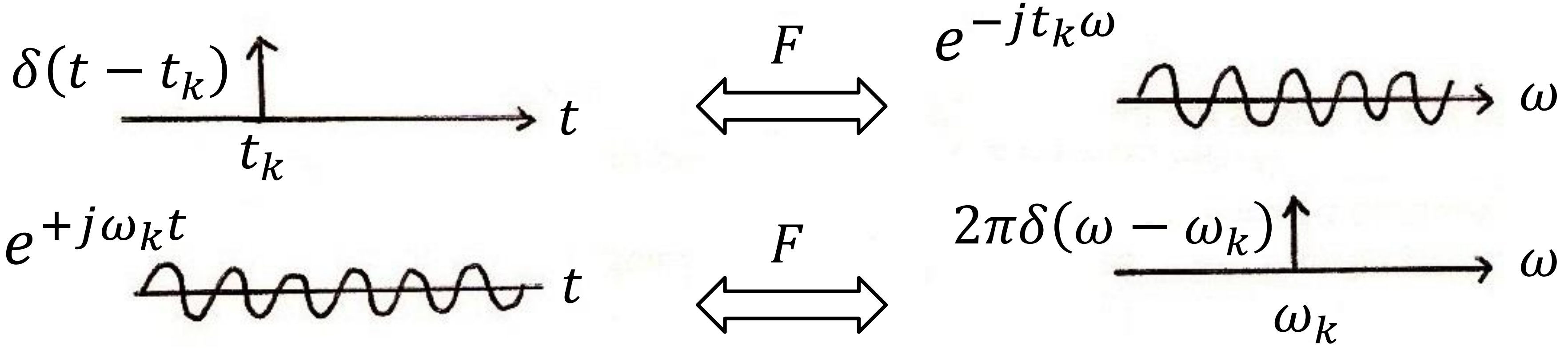
Signal Representation in Two Domains

Time Domain Basis

Frequency Domain Basis

$\{\delta(t - t_k), -\infty < t_k < \infty\}$

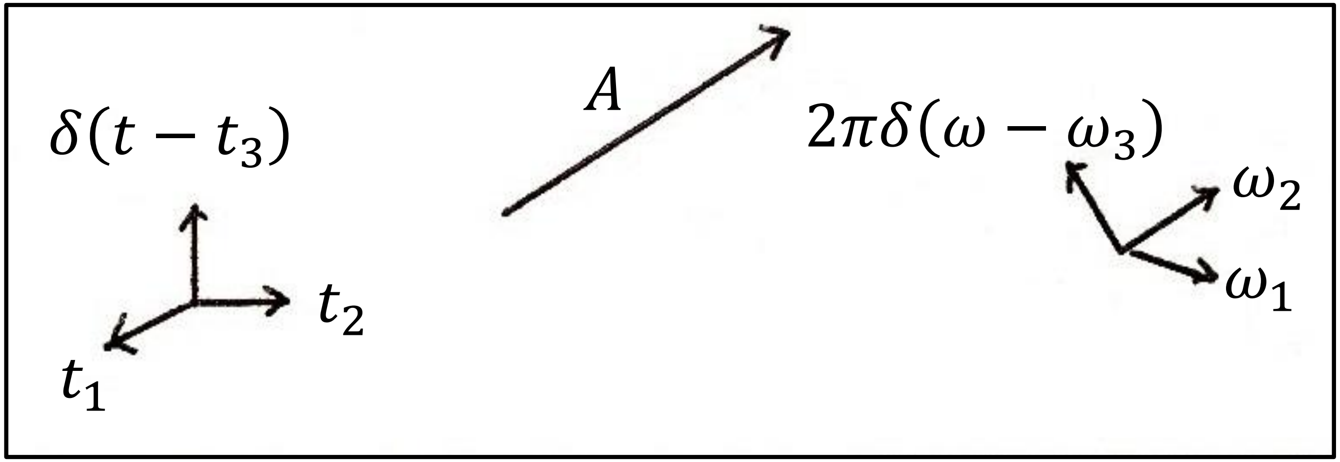
$\{2\pi\delta(\omega - \omega_k), -\infty < \omega_k < \infty\}$



Signal Representation in Two Domains

Time Domain Basis

Frequency Domain Basis



$$\vec{A} = \sum_i a_i \ \vec{v}_i \qquad = \sum_k b_k \vec{u}_k \quad (合成)$$

$$a_i = \vec{A} \cdot \vec{v}_i \qquad b_k = \vec{A} \cdot \vec{u}_k \quad (分析)$$

Signal Representation in Two Domains

Time Domain Basis

$$\vec{A} = \sum_i a_i \; \underset{\Big\downarrow}{\vec{v}_i}$$

$$\Big(\sum_k c_k \; \vec{u}_k\Big)$$

$$= \sum_k \underset{b_k}{(\cdots)} \; \vec{u}_k$$

Frequency Domain Basis

$$\vec{A} = \sum_k b_k \; \underset{\Big\downarrow}{\vec{u}_k}$$

$$\Big(\sum_i d_i \; \vec{v}_i\Big)$$

$$= \sum_i \underset{a_i}{(\cdots)} \; \vec{v}_i$$

Signal Representation in Two Domains

Time Domain Basis $\sum_k c_k \vec{u}_k$

$$\begin{cases} \overset{b_k}{\boxed{X(j\omega)}} = \int_{-\infty}^{\infty} x(t)\, \boxed{e^{-jt\omega}}\, dt & \text{(合成)} \\ x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{jt\omega}\, d\omega & \text{(分析)} \end{cases}$$

$$\overset{\vec{A}}{x(t)} = \int_{-\infty}^{\infty} \overset{a_i}{\boxed{x(\tau)}}\, \overset{\vec{v}_i}{\boxed{\delta(t-\tau)}}\, d\tau$$

(合成/分析)

Frequency Domain Basis

$$\begin{cases} \overset{a_i}{\boxed{x(t)}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overset{b_k}{\boxed{X(j\omega)}}\, \overset{\sum d_i \vec{v}_i}{\boxed{e^{j\omega t}}}\, d\omega & \text{(合成)} \\ X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt & \text{(分析)} \end{cases}$$

$$\overset{\vec{A}}{X(j\omega)} = \int_{-\infty}^{\infty} \overset{b_k}{\boxed{X(j\eta)}}\, \overset{\vec{u}_k}{\boxed{\delta(\omega-\eta)}}\, d\eta$$

(合成/分析)

- Duality
  - If any characteristics of signals in one domain implies some characteristics of signals in the other domain, the inverse is true except for a sign change (dual properties)

$$- \; jtx(t) \xleftrightarrow{F} \frac{dX(j\omega)}{d\omega}$$

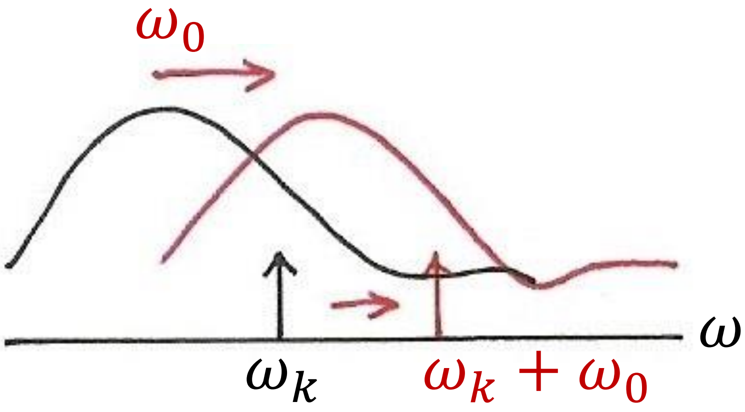
$$- \frac{1}{jt} x(t) + \pi x(0)\delta(t) \xleftrightarrow{F} \int_{-\infty}^{\omega} X(j\eta)\, d\eta$$

$$e^{j\omega_0 t} x(t) \xleftrightarrow{F} X\left(j(\omega - \omega_0)\right)$$

modulation property

Modulation Property

$$e^{j\omega_0 t}x(t) \xleftrightarrow{F} X\left(j(\omega - \omega_0)\right)$$



$$e^{j\omega_0 t}\left(e^{j\omega_k t}\right) = e^{j(\omega_k+\omega_0)t}$$

modulation:
  frequency translation
  shift in frequency

Multiplication Property

$$e^{j\omega_0 t}\cdot x(t) \leftrightarrow \tfrac{1}{2\pi}[2\pi\delta(\omega-\omega_0) * X(j\omega)]$$
$$= X(j(\omega-\omega_0))$$

- Convolution Property

$$y(t) = x(t) * h(t) \xleftrightarrow{F} Y(j\omega) = X(j\omega)H(j\omega)$$

  - System Input/Output Relationship

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$= \lim_{\omega_0 \to 0} \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0$$

$$\sum_k a_k X_k(t) \to \sum_k a_k Y_k(t) \text{ superposition property}$$

$$H(jk\omega_0) = \int_{-\infty}^{\infty} h(t)e^{-jk\omega_0 t}dt$$

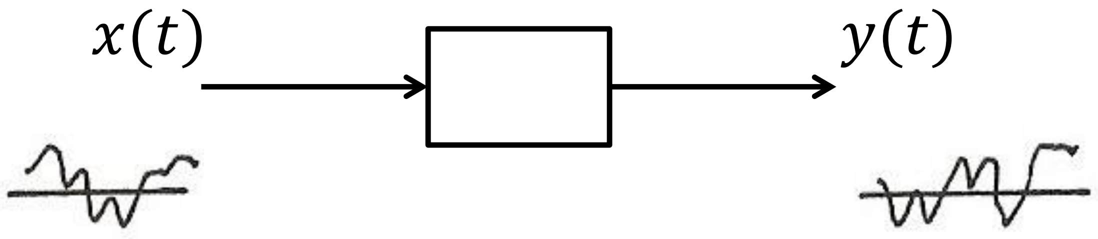
$$y(t) = \lim_{\omega_0 \to 0} \frac{1}{2\pi}\sum_{-\infty}^{\infty} X(jk\omega_0)H(jk\omega_0)e^{jk\omega_0 t}\omega_0$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)H(j\omega)e^{j\omega t}d\omega \quad \text{closed-form solution}$$

$$\therefore Y(j\omega) = X(j\omega)H(j\omega)$$

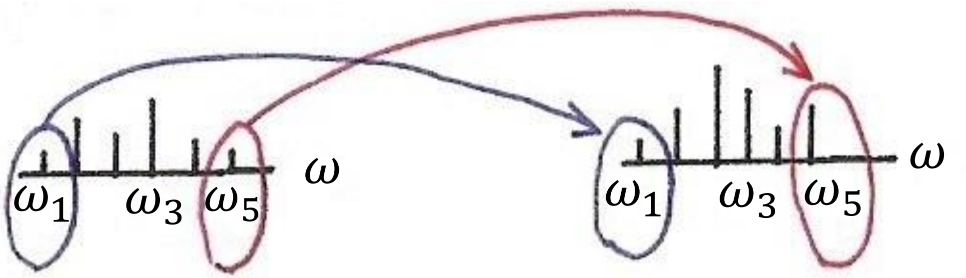
$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt \quad \text{frequency response}$$

Input/Output Relationship (P.5 of 3.0)



- Time Domain

- Frequency Domain



System Characterization (P.9 of 3.0)

- Superposition Property
  - continuous-time

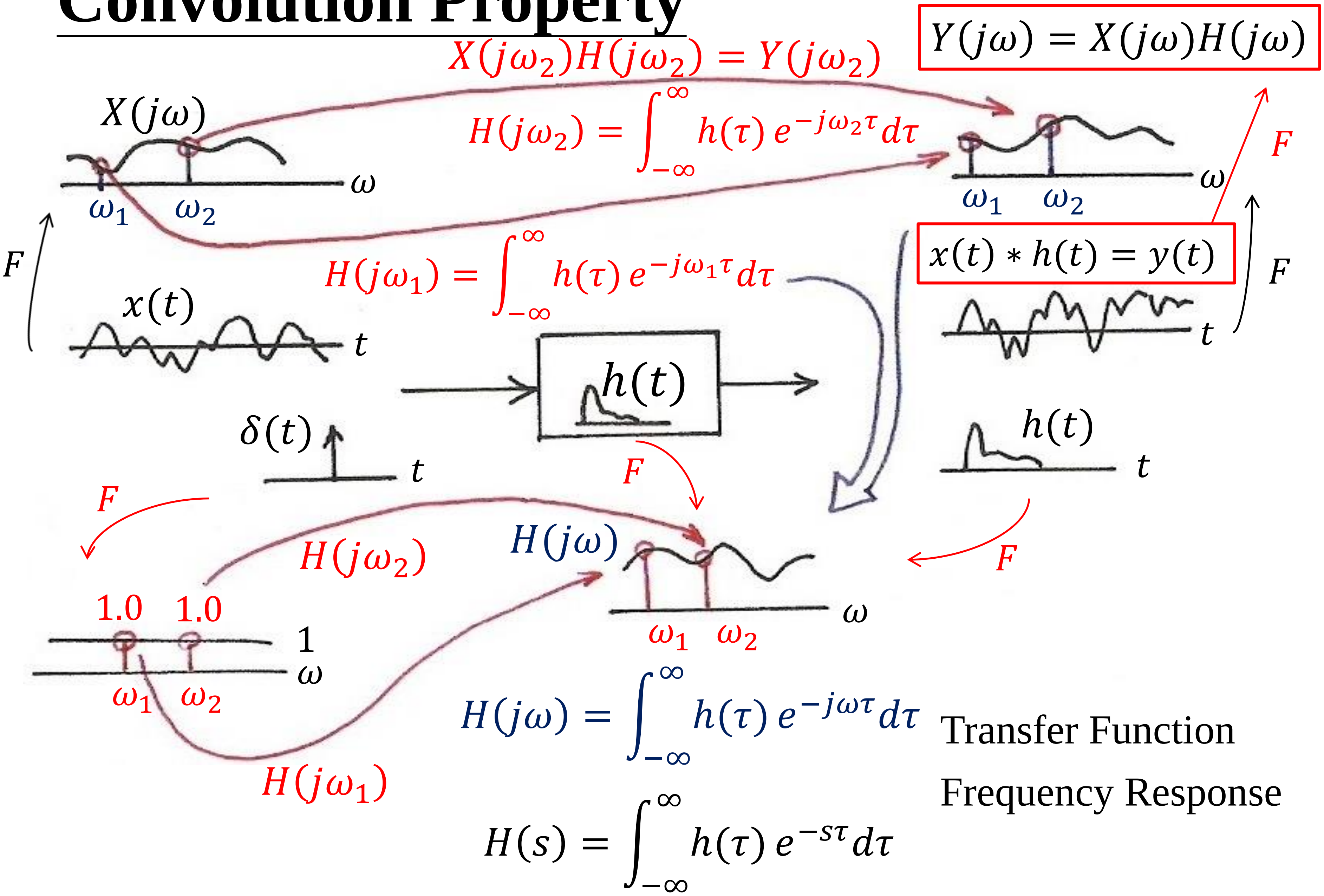
$$x(t)=\sum_k a_k e^{s_k t} \rightarrow y(t)=\sum_k a_k H(s_k) e^{s_k t}$$

  - discrete-time

$$x[n]=\sum_k a_k (z_k)^n \rightarrow y[n]=\sum_k a_k H(z_k)(z_k)^n$$

  - each frequency component never split to other frequency components, no convolution involved
  - desirable to decompose signals in terms of such eigenfunctions

Convolution Property



$$\boxed{Y(j\omega) = X(j\omega)H(j\omega)}$$

$$X(j\omega_2)H(j\omega_2) = Y(j\omega_2)$$

$$H(j\omega_2) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega_2\tau} d\tau$$

$$H(j\omega_1) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega_1\tau} d\tau$$

$$\boxed{x(t) * h(t) = y(t)}$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau} d\tau$$

$$H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau} d\tau$$

Transfer Function

Frequency Response

- Convolution Property

$$y(t) = x(t) * h(t) \xleftrightarrow{F} Y(j\omega) = X(j\omega)H(j\omega)$$

  - unit impulse response $h(t)$
    frequency response or transfer function $H(j\omega)$

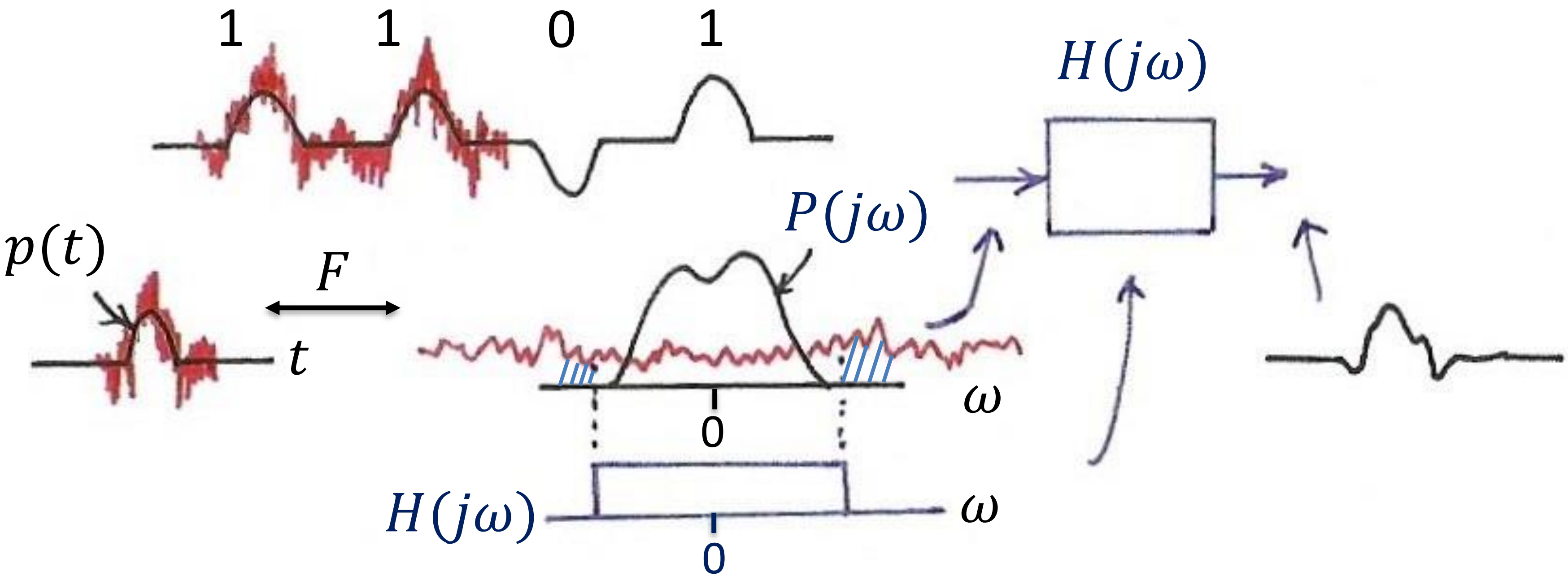
$$h(t) \xleftrightarrow{F} H(j\omega)$$

$$\delta(t) \xleftrightarrow{F} 1$$

  - convolution in time domain reduced to multiplication in frequency domain
  - cascade of two systems implies product of the two frequency responses, independent of the order of the cascade
  - example: filtering of signals

See Fig. 4.20, 4.21, p.318, 319 of text

Filtering of Signals



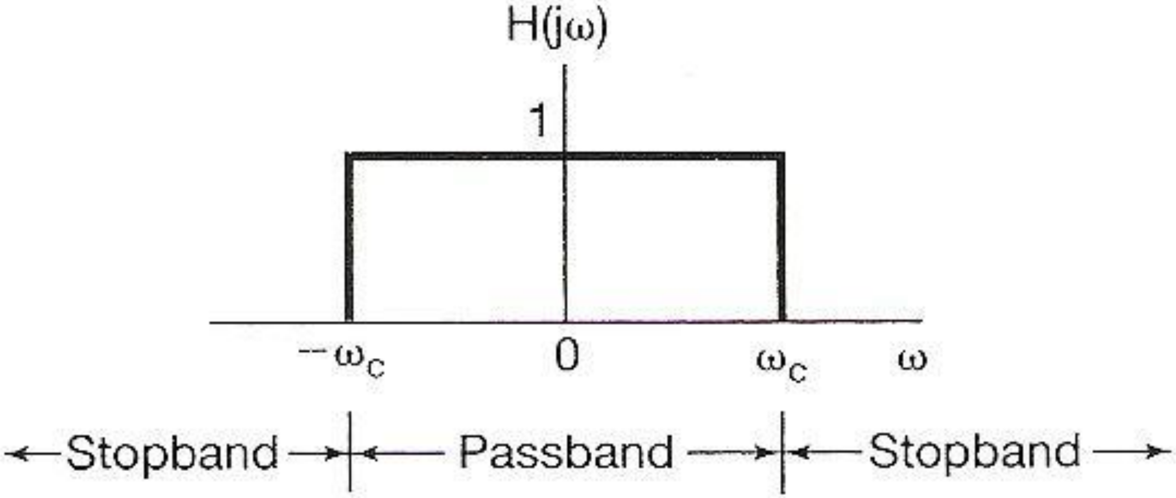


Figure 4.20 Frequency response of an ideal lowpass filter.

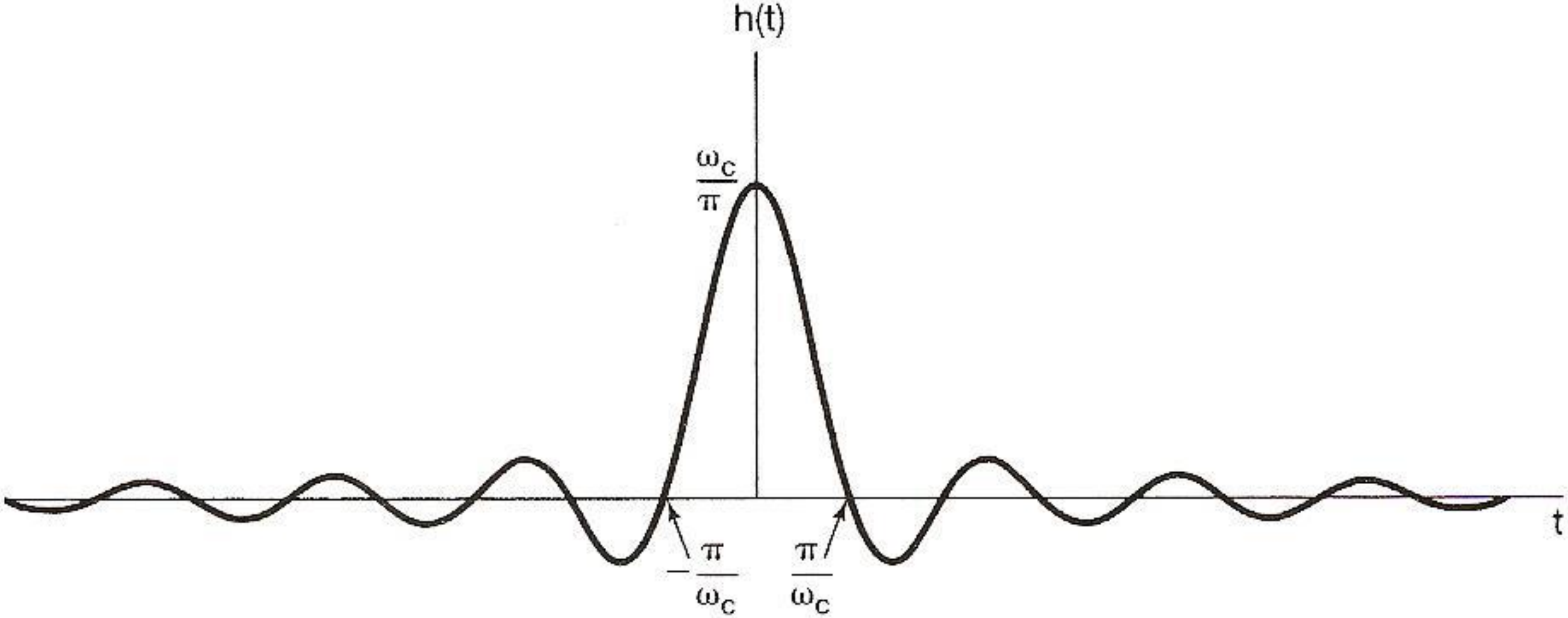


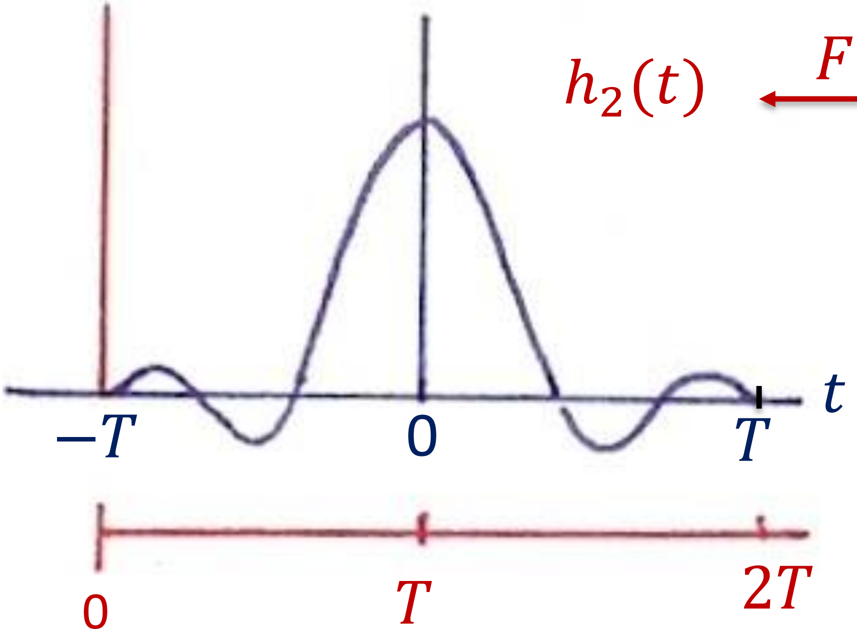
Figure 4.21 Impulse response of an ideal lowpass filter.

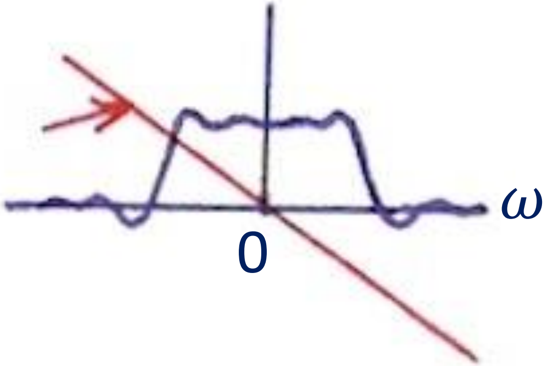
Realizable Lowpass Filter

$$h_1(t) \xleftrightarrow{F} |H_1(j\omega)|$$

$$\Vert$$

$$h_2(t) \xleftrightarrow{F} \begin{cases} |H_2(j\omega)| \\ \angle H_2(j\omega) \end{cases}$$





- Differentiation/Integration (P.33 of 4.0)

$$\frac{dx(t)}{dt} \xleftrightarrow{F} j\omega X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{F} \frac{1}{j\omega} X(j\omega) + \pi X(j0)\delta(\omega)$$

$$\underset{\text{dc term}}{\uparrow}$$

Integration

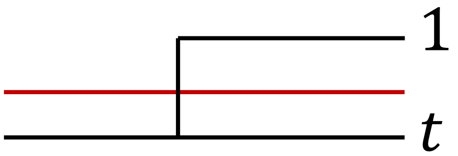
$$u(t)=\int_{-\infty}^{t}\delta(\tau)d\tau$$

$$\nearrow\ \frac{1}{2}\cdot 2\pi\delta(\omega)=\pi\delta(\omega)$$

$$U(j\omega)=\frac{1}{j\omega}+(\text{dc term})$$

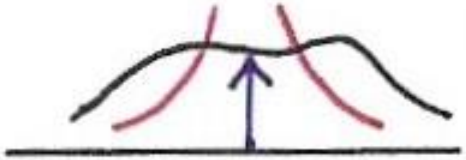
$$\|\ \frac{1}{2}$$

$$=\frac{1}{j\omega}+\pi\delta(\omega)$$



$$\int_{-\infty}^{t}x(\tau)d\tau=x(t)*u(t)\overset{F}{\leftrightarrow}X(j\omega)\cdot[\frac{1}{j\omega}+\pi\delta(\omega)]$$

$$=\frac{1}{j\omega}X(j\omega)+\pi X(jo)\delta(\omega)$$



- Multiplication Property

$$r(t) = s(t)p(t) \overset{F}{\longleftrightarrow} R(j\omega) = \frac{1}{2\pi}\left[S(j\omega) * P(j\omega)\right]$$

dual property of the convolution property

- example: frequency-selective filtering with variable center frequency

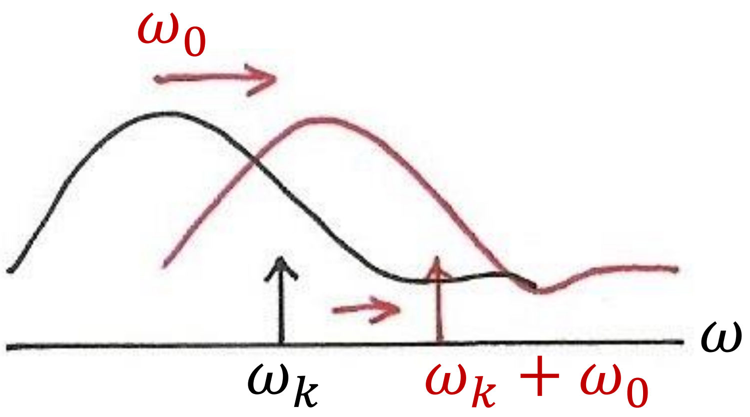
See Fig. 4.26, 4.27, p.326 of text

- Tables of Properties and Pairs

See Tables. 4.1, 4.2, p.328, 329 of text

Modulation Property (P.53 of 4.0)

$$e^{j\omega_0 t}x(t) \xleftrightarrow{F} X(j(\omega-\omega_0))$$



$$e^{j\omega_0 t}\left(e^{j\omega_k t}\right)=e^{j(\omega_k+\omega_0)t}$$

modulation:

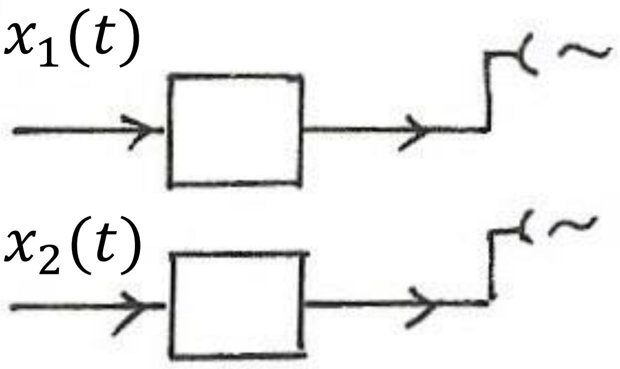
frequency translation

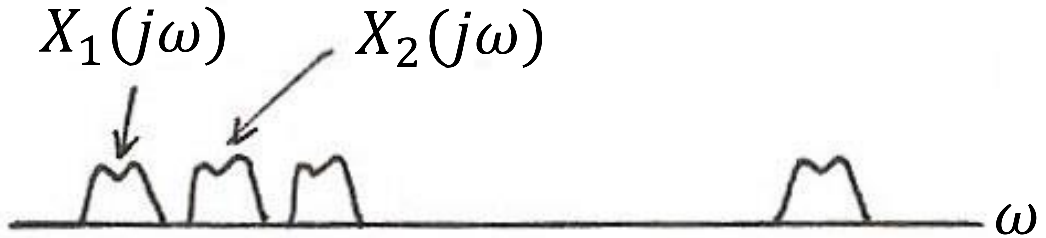
shift in frequency

Multiplication Property

$$e^{j\omega_0 t}\cdot x(t) \longleftrightarrow \frac{1}{2\pi}[2\pi\delta(\omega-\omega_0)*X(j\omega)]$$
$$=X(j(\omega-\omega_0))$$

Frequency Division Multiplexing





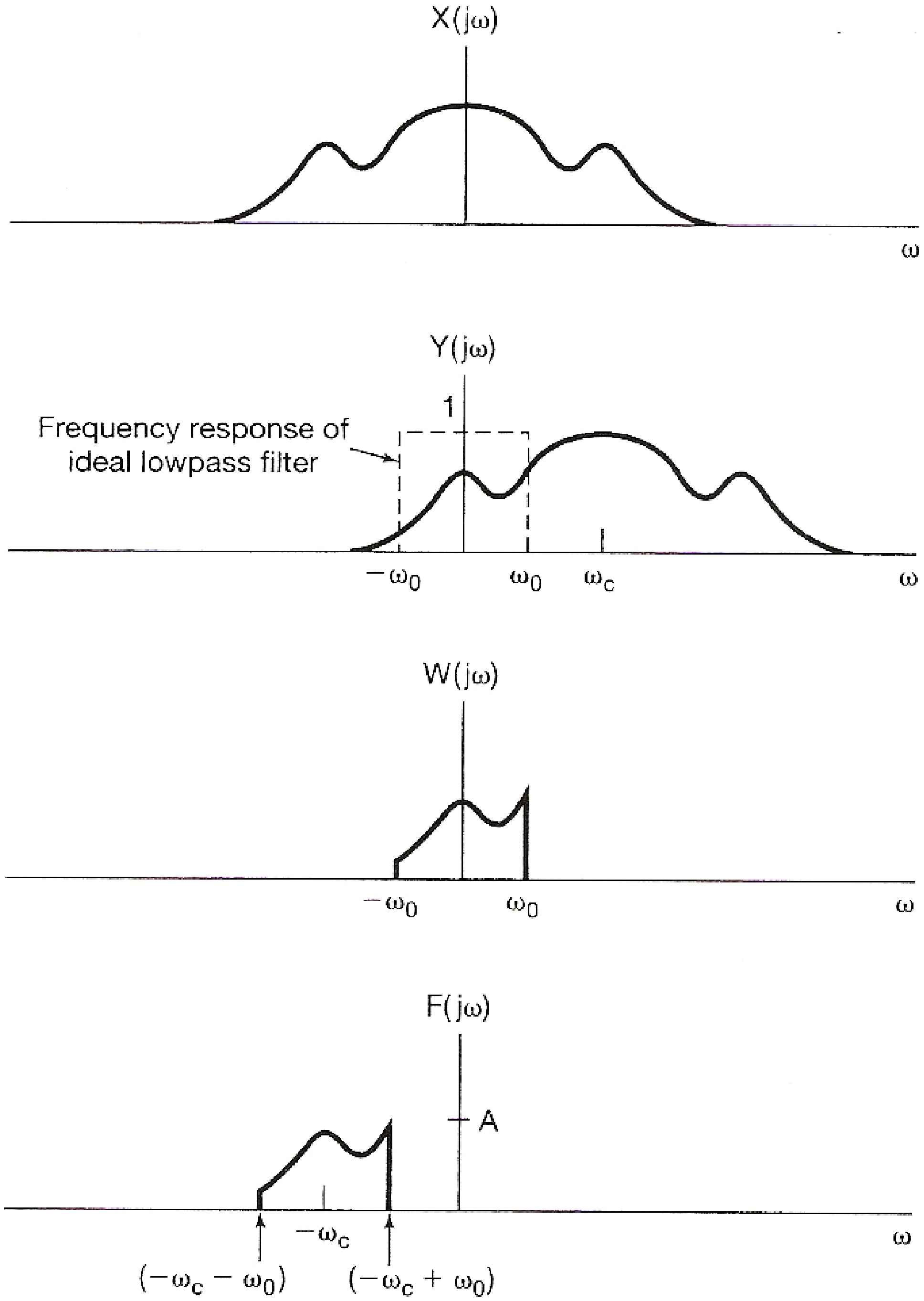


Figure 4.27 Spectra of the signals in the system of Figure 4.26.

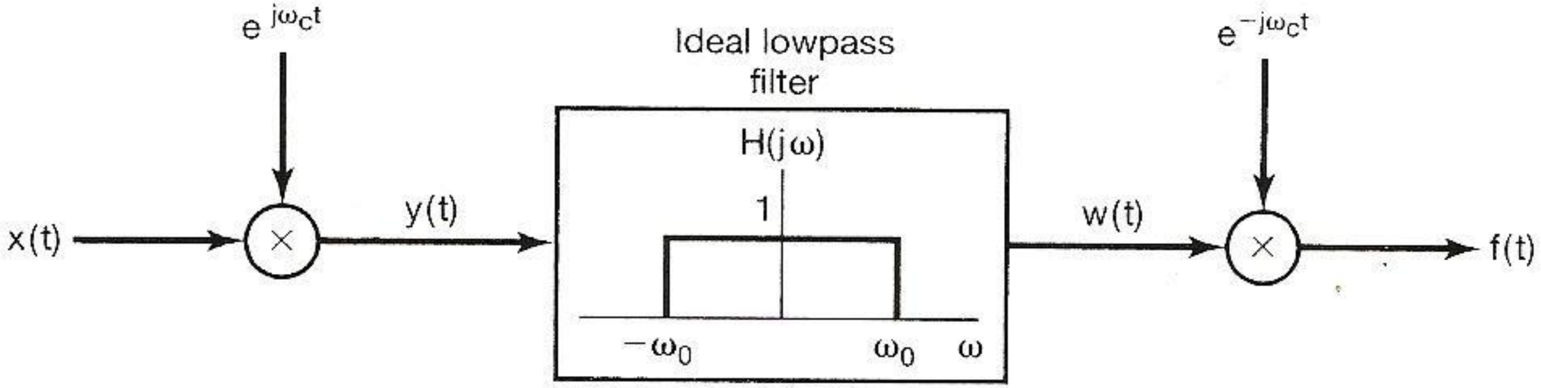


Figure 4.26 Implementation of a bandpass filter using amplitude modulation with a complex exponential carrier.

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega - \theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{R}e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$
4.3.7	Parseval's Relation for Aperiodic Signals $\int_{-\infty}^{+\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \lvert X(j\omega)\rvert^2 d\omega$		

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \; k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \mathcal{Re}\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \mathcal{Re}\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\mathcal{Re}\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

- Another Application Example

systems described by differential equations:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$Y(j\omega)\left[\sum_{k=0}^{N} a_k (j\omega)^k\right] = X(j\omega)\left[\sum_{k=0}^{M} b_k (j\omega)^k\right]$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

– closed-form solution

- Vector Space Interpretation of Fourier Transform
  - generalized Parseval’s Relation

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)Y^*(j\omega)d\omega$$

$$[x(t)]\cdot[y(t)] = \frac{1}{2\pi}[X(j\omega)]\cdot[Y(j\omega)]$$

$\{X(j\omega)$ defined on $-\infty < \omega < \infty\}=V$: a vector space

inner-product of two vectors(signals) can be evaluated in either the time domain or the frequency domain

Parseval’s relation is a special case here: the magnitude (norm) of a vector can be evaluated in either the time domain or the frequency domain

- Vector Space Interpretation of Fourier Transform
  - considering the basis signal set

$$\left\{ \phi_{\omega}(t) = e^{j\omega t}, -\infty < \omega < \infty \right\}$$

$$\phi_{\omega_k}(t) = e^{j\omega_k t} \xleftrightarrow{F} 2\pi\delta(\omega - \omega_k)$$

$$\left[\phi_{\omega_k}(t)\right] \cdot \left[\phi_{\omega_j}(t)\right]$$

$$= \frac{1}{2\pi}\left[2\pi\delta(\omega - \omega_k)\right] \cdot \left[2\pi\delta(\omega - \omega_j)\right]$$

$$= 2\pi\left[\delta(\omega - \omega_k)\right] \cdot \left[\delta(\omega - \omega_j)\right]$$

$$= 0, \omega_k \neq \omega_j$$

$$\neq 1, \omega_k = \omega_j$$

- Vector Space Interpretation of Fourier Transform
  - considering the basis signal set

    similar to the vector space of continuous-time signals
  - orthogonal bases but not normalized, while makes sense considering operational definition

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\, d\omega \qquad \text{(合成)}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = [x(t)] \cdot [\phi_{\omega}(t)] \quad \text{(分析)}$$

Examples

- Example 4.8, p.299 of text

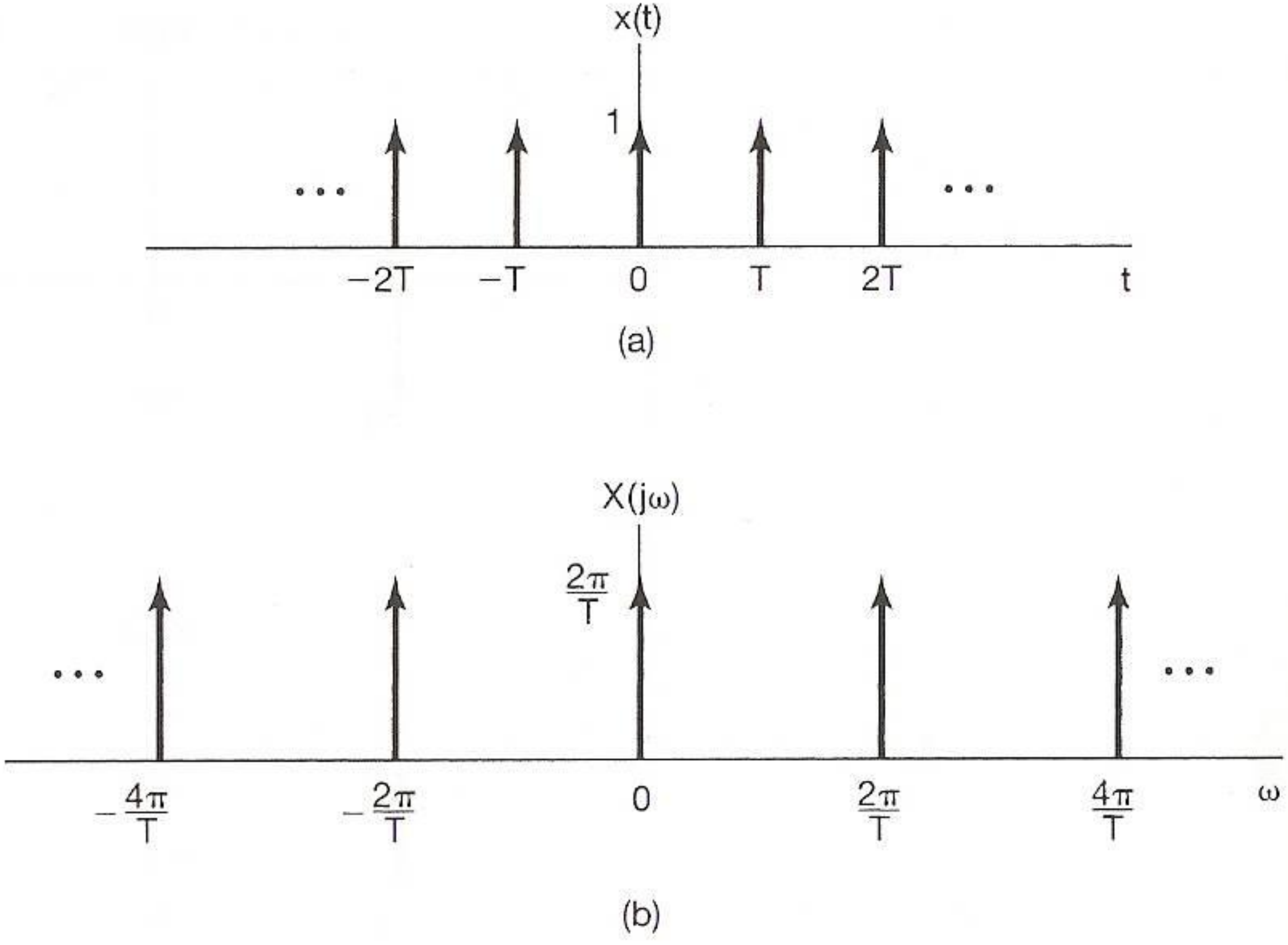


Figure 4.14 (a) Periodic impulse train; (b) its Fourier transform.

Examples

- Example 4.13, p.310 of text
  - From Example 4.2

$$x(t) = e^{-2|t|} \quad \overset{F}{\leftrightarrow} \quad X(j\omega) = \frac{2}{1+\omega^2}$$

by duality

$$x(t) = \frac{2}{1+t^2} \quad \overset{F}{\leftrightarrow} \quad 2\pi e^{-2|\omega|}$$

Examples

- Example 4.19, p.320 of text

$$h(t) = e^{-at}u(t),\ a > 0$$

$$x(t) = e^{-bt}u(t),\ b > 0$$

$$X(j\omega) = \frac{1}{b+j\omega}, \qquad H(j\omega) = \frac{1}{a+j\omega}$$

$$Y(j\omega) = \frac{1}{(a+j\omega)(b+j\omega)} = \frac{1}{b-a}\left[\frac{1}{a+j\omega} - \frac{1}{b+j\omega}\right]$$

$$y(t) = \frac{1}{b-a}[e^{-at}\,u(t) - e^{-bt}u(t)], b \neq a$$

$$b = a : \quad Y(j\omega) = \frac{1}{(a+j\omega)^2} = j\,\frac{d}{d\omega}\left[\frac{1}{a+j\omega}\right]$$

Since $-jtx(t) \leftrightarrow \frac{d}{d\omega}\,X(j\omega)$

$$y(t) = te^{-at}\,u(t), b = a$$

Problem 4.12, p.336 of text

- (a) Given $e^{-|t|} \overset{F}{\longleftrightarrow} \frac{2}{1+\omega^2}$

$$te^{-|t|} \overset{F}{\longleftrightarrow} j\frac{d}{d\omega}\left[\frac{2}{1+\omega^2}\right] = -\frac{4j\omega}{(1+\omega^2)^2}$$

by differentiation in frequency domain

- (b) By duality $-\frac{4jt}{(1+t^2)^2} \overset{F}{\longleftrightarrow} -2\pi\omega e^{-|\omega|}$

$$\therefore \frac{4t}{(1+t^2)^2} \overset{F}{\longleftrightarrow} -j2\pi\omega e^{-|\omega|}$$

Problem 4.13, p.336 of text

- (a) $X(j\omega) = \delta(\omega) + \delta(\omega - \pi) + \delta(\omega - 5)$

  Is $x(t)$ periodic ?

  $$x(t) = \frac{1}{2\pi} + \frac{1}{2\pi} e^{j\pi t} + \frac{1}{2\pi} e^{j5t}$$

  π and 5 are not integer multiples of any common fundamental frequency

  $\therefore x(t)$ Not periodic

- (b) $h(t) = u(t) - u(t - 2)$

  Is $x(t) * h(t)$ periodic ?

  $$H(j\omega) = e^{-j\omega}\left[\frac{2\sin\omega}{\omega}\right], H(j\pi) = 0$$

  $X(j\omega)H(j\omega) = H(j0)\delta(\omega) + H(j5)\delta(\omega - 5)$

  $\therefore x(t) * h(t)$ is periodic

Problem 4.33, p.345 of text

$$\frac{d^2}{dt^2}y(t)+6\frac{d}{dt}y(t)+8y(t)=2x(t)$$

- (a) find impulse response

$$H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{2}{-\omega^2+6j\omega+8}=\frac{1}{j\omega+2}-\frac{1}{j\omega+4}$$

$$\therefore h(t)=e^{-2t}u(t)-e^{-4t}u(t)$$

- (b) For $x(t)=te^{-2t}u(t)$

$$X(j\omega)=\frac{1}{(2+j\omega)^2}$$

$$Y(j\omega)=X(j\omega)H(j\omega)=\frac{1/4}{j\omega+2}-\frac{1/2}{(j\omega+2)^2}+\frac{1}{(j\omega+2)^3}-\frac{1/4}{j\omega+4}$$

$$\therefore y(t)=\frac{1}{4}e^{-2t}u(t)-\frac{1}{2}te^{-2t}u(t)+t^2e^{-2t}u(t)-\frac{1}{4}e^{-4t}u(t)$$

Problem 4.35, p.346 of text

- (a) $H(j\omega) = \frac{a-j\omega}{a+j\omega}$

  $|H(j\omega)| = \sqrt{a^2+\omega^2}/\sqrt{a^2+\omega^2} = 1$

  $\measuredangle H(j\omega) = -\tan^{-1}\frac{\omega}{a} - \tan^{-1}\frac{\omega}{a} = -2\tan^{-1}\frac{\omega}{a}$

  $H(j\omega) = -1 + \frac{2a}{a+j\omega}, h(t) = -\delta(t) + 2ae^{-at}u(t)$

- (b) $x(t) = \cos t + \cos\sqrt{3}\, t,\ \ a = 1$

  $X(j\omega) = \pi[\delta(\omega-1) + \delta(\omega+1) + \delta\left(\omega-\sqrt{3}\right) + \delta(\omega+\sqrt{3})]$

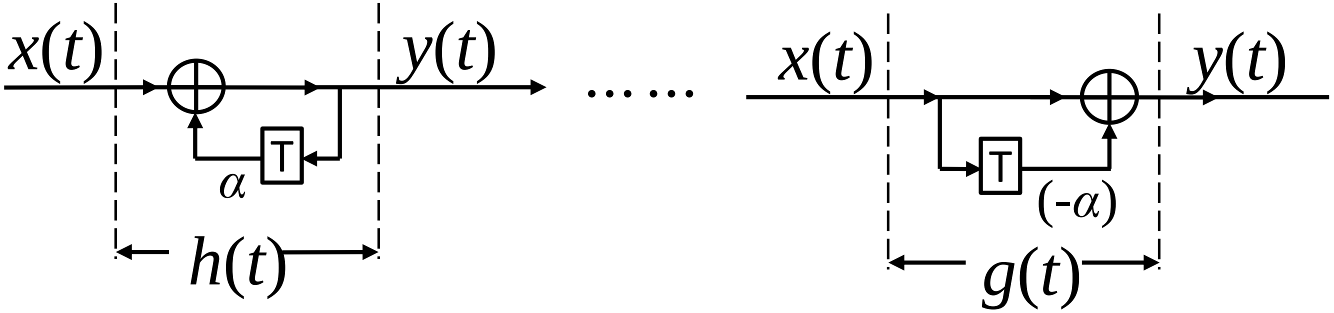
  $H(j\omega) = 1\cdot e^{-j\frac{\pi}{2}}$ at $\omega = 1,$ $H(j\omega) = 1\cdot e^{-j\frac{2}{3}\pi}$ at $\omega = \sqrt{3},$ etc

  $y(t) = \frac{1}{2}\left[e^{j\left(t-\frac{\pi}{2}\right)} + e^{-j\left(t-\frac{\pi}{2}\right)} + e^{j\left(\sqrt{3}t-\frac{2}{3}\pi\right)} + e^{-j\left(\sqrt{3}t-\frac{2}{3}\pi\right)}\right]$

  $\therefore y(t) = \cos(t-\frac{\pi}{2}) + \cos(\sqrt{3}t - \frac{2}{3}\pi)$

Problem 4.51, p.354 of text, part (c)

- An echo system



$$h(t) = \sum_{k=0}^{\infty} \alpha^k \delta(t - kT)$$

$$H(j\omega) = \sum_{k=0}^{\infty} \left(\alpha^k e^{-j\omega kT}\right) = \frac{1}{1 - \alpha e^{-j\omega T}}$$

$$G(j\omega) = 1 - \alpha e^{-j\omega T}$$