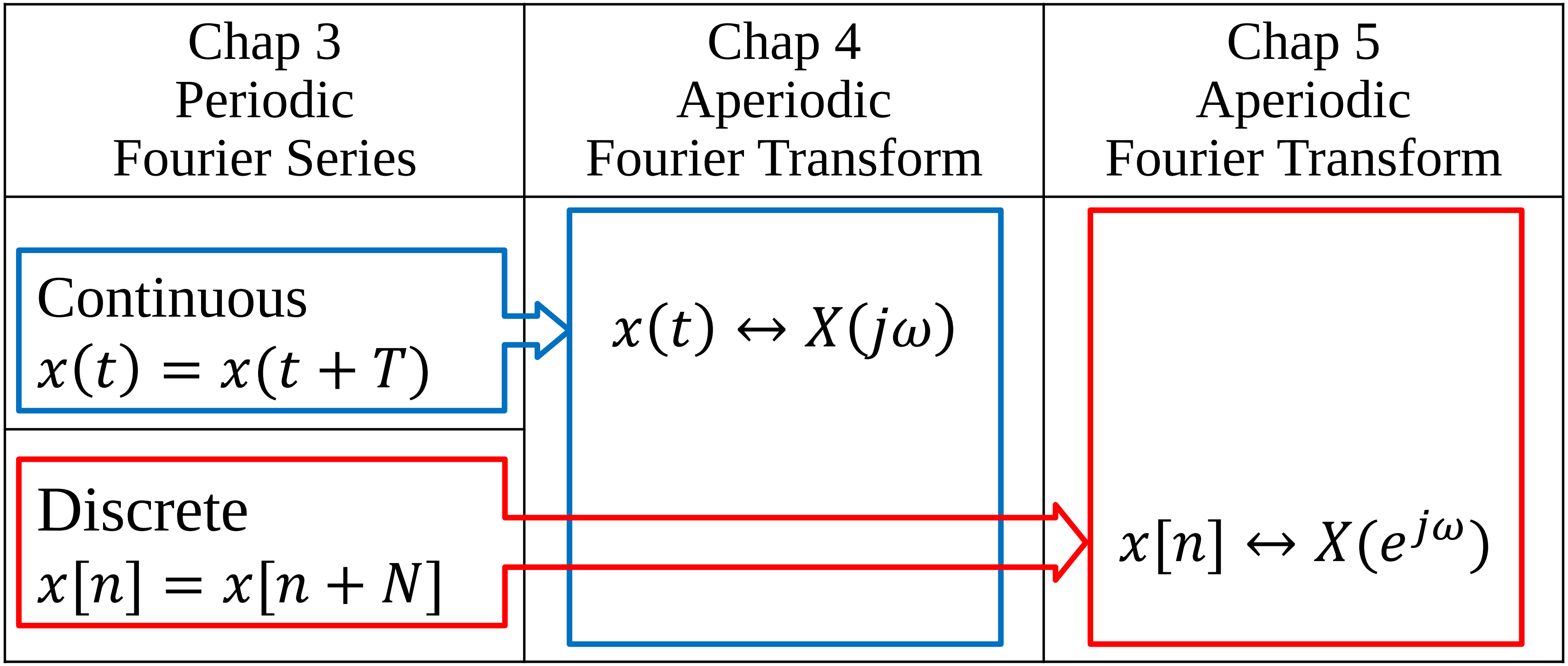
5.0 Discrete-time Fourier Transform

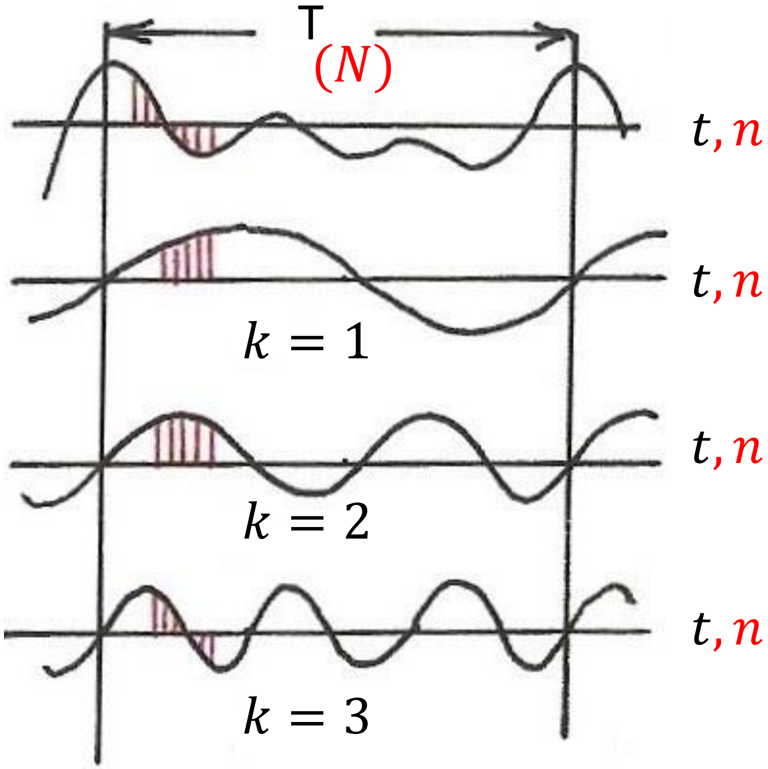
5.1 Discrete-time Fourier Transform Representation for discrete-time signals

Chapters 3, 4, 5



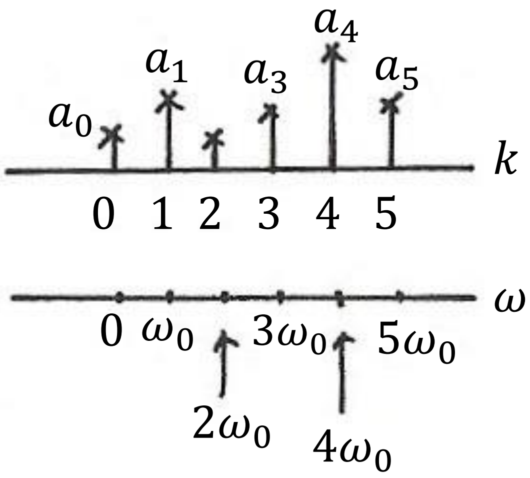
Chap 3 Periodic Fourier Series	Chap 4 Aperiodic Fourier Transform	Chap 5 Aperiodic Fourier Transform
Continuous $x(t) = x(t+T)$	$x(t) \leftrightarrow X(j\omega)$	
Discrete $x[n] = x[n+N]$		$x[n] \leftrightarrow X(e^{j\omega})$

Harmonically Related Exponentials for Periodic Signals (p.11 of 3.0)



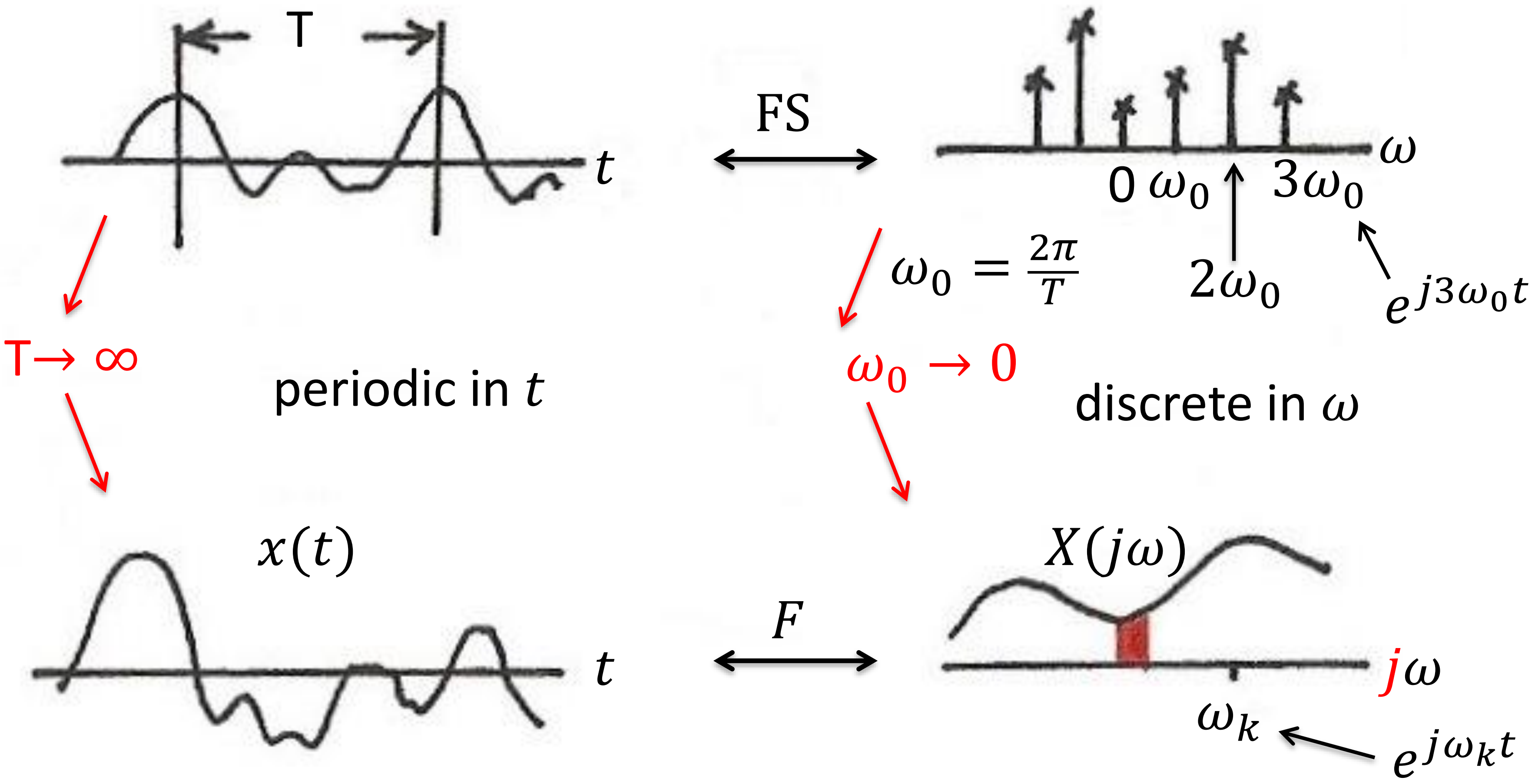
$V = \{x(t)[n] \mid x(t)[n]$ periodic, fundamental period $= T(N)\}$

$\omega_0 = \frac{2\pi}{T(N)}$



- All with period T (N): integer multiples of ω_0
- Discrete in frequency domain

Fourier Transform (p.3 of 4.0)



T

t

FS

ω

$0 \ \omega_0$

$3\omega_0$

$\omega_0 = \frac{2\pi}{T}$

$2\omega_0$

$e^{j3\omega_0 t}$

T→ ∞

periodic in t

$\omega_0 \rightarrow 0$

discrete in ω

$x(t)$

$X(j\omega)$

F

t

$j\omega$

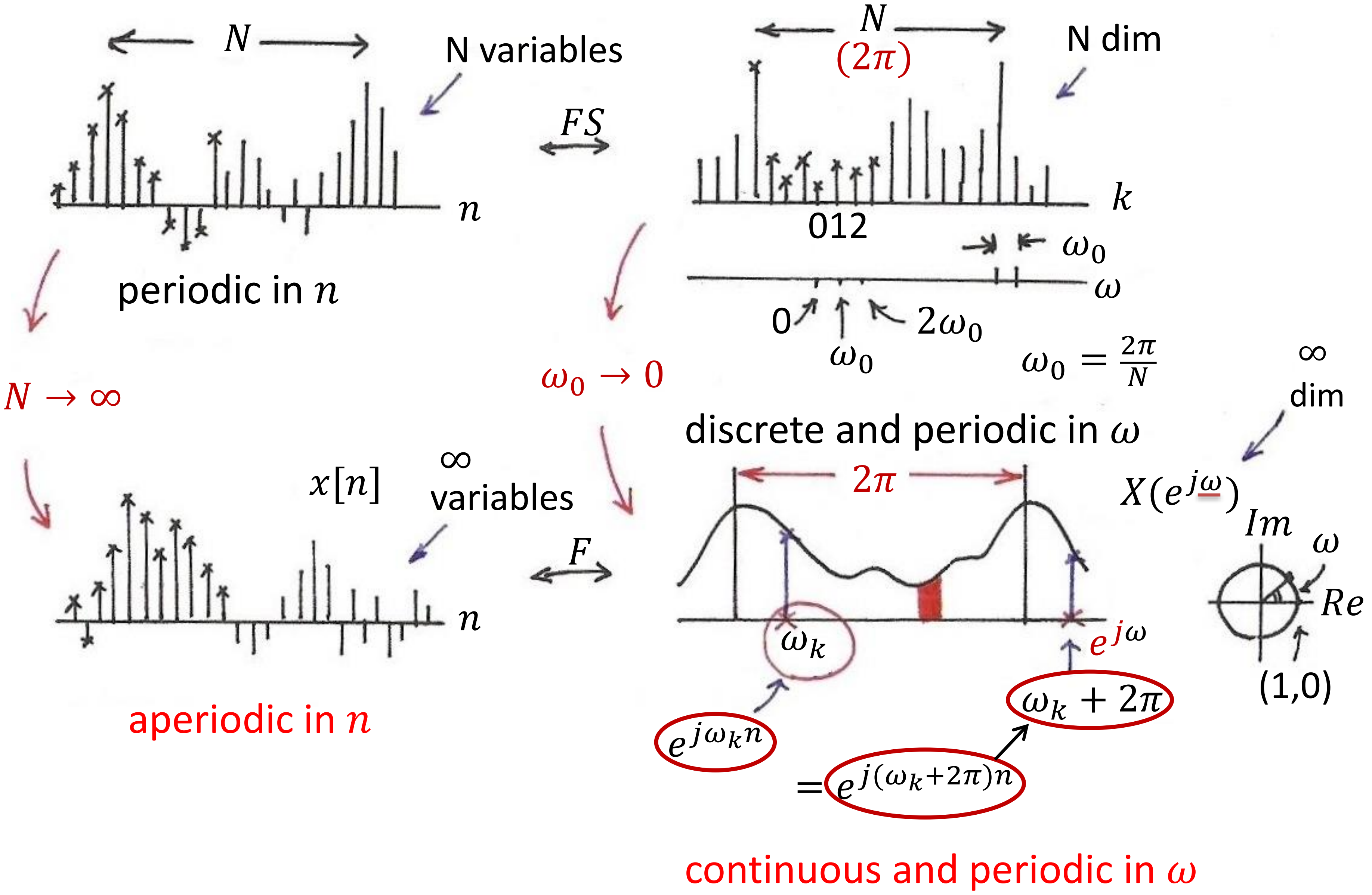
ω_k

$e^{j\omega_k t}$

aperiodic in t

continuous in ω

Discrete-time Fourier Transform



N

N variables

periodic in n

$N \to \infty$

$x[n]$

∞ variables

aperiodic in n

FS

F

$\omega_0 \to 0$

N

(2π)

N dim

k

012

ω_0

ω

0

ω_0

$2\omega_0$

$\omega_0 = \frac{2\pi}{N}$

discrete and periodic in ω

∞

dim

$X(e^{j\omega})$

2π

ω_k

$e^{j\omega_k n}$

$= e^{j(\omega_k+2\pi)n}$

$\omega_k + 2\pi$

$e^{j\omega}$

Im

ω

Re

(1,0)

continuous and periodic in ω

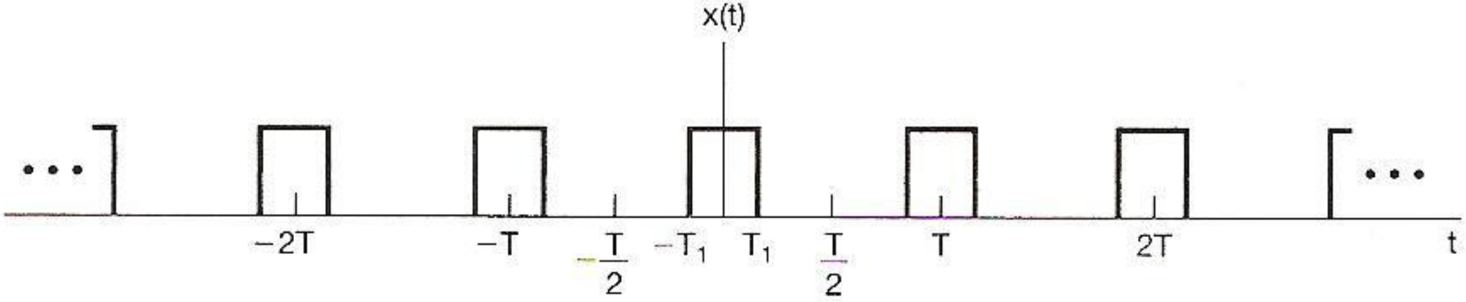


Figure 3.6 Periodic square wave.

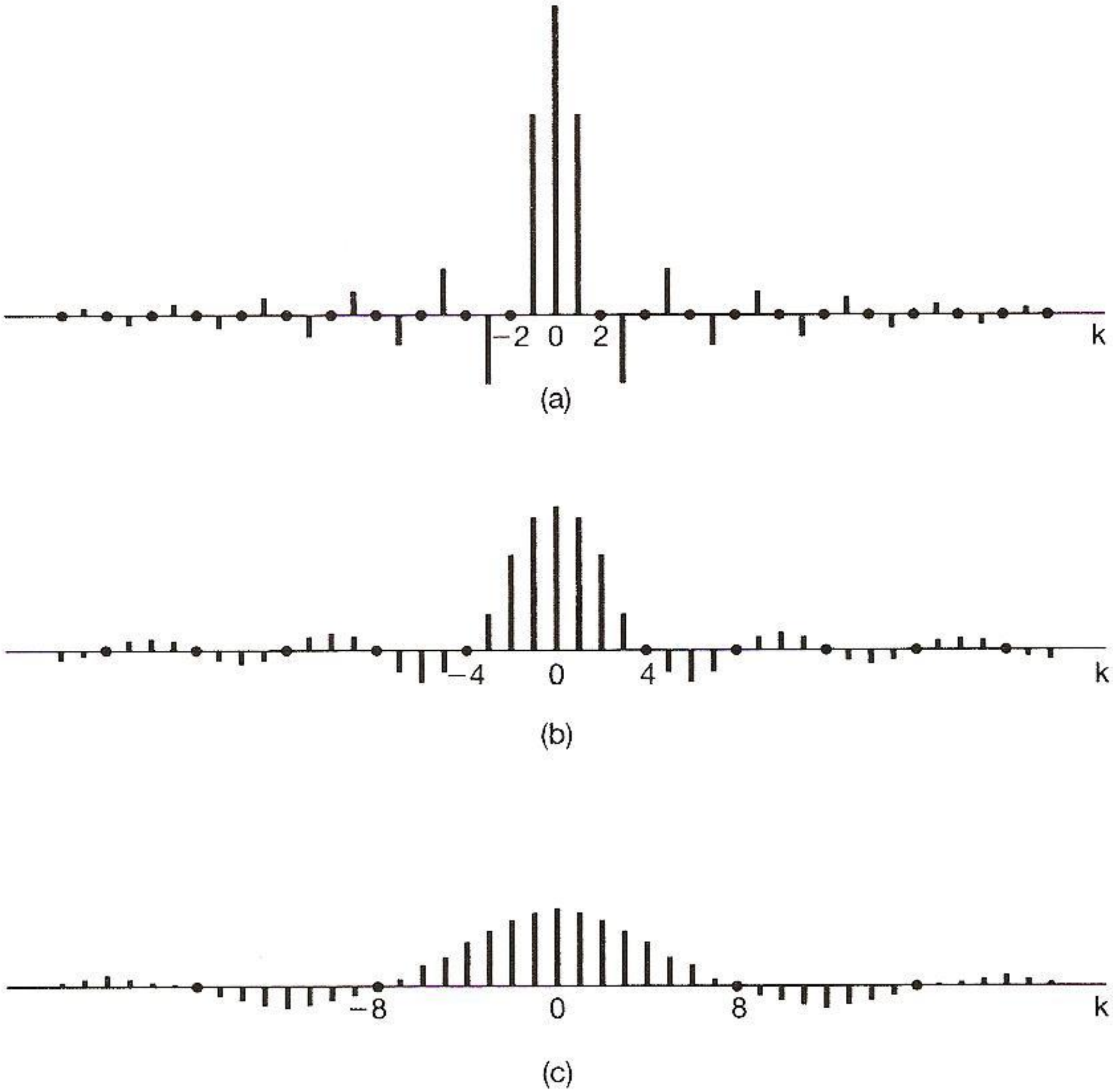


Figure 3.7 Plots of the scaled Fourier series coefficients Ta_k for the periodic square wave with T_1 fixed and for several values of T: (*a*) $T = 4T_1$; (*b*) $T = 8T_1$; (*c*) $T = 16T_1$. The coefficients are regularly spaced samples of the envelope $(2 \sin \omega T_1)/\omega$, where the spacing between samples, $2\pi/T$, decreases as T increases.

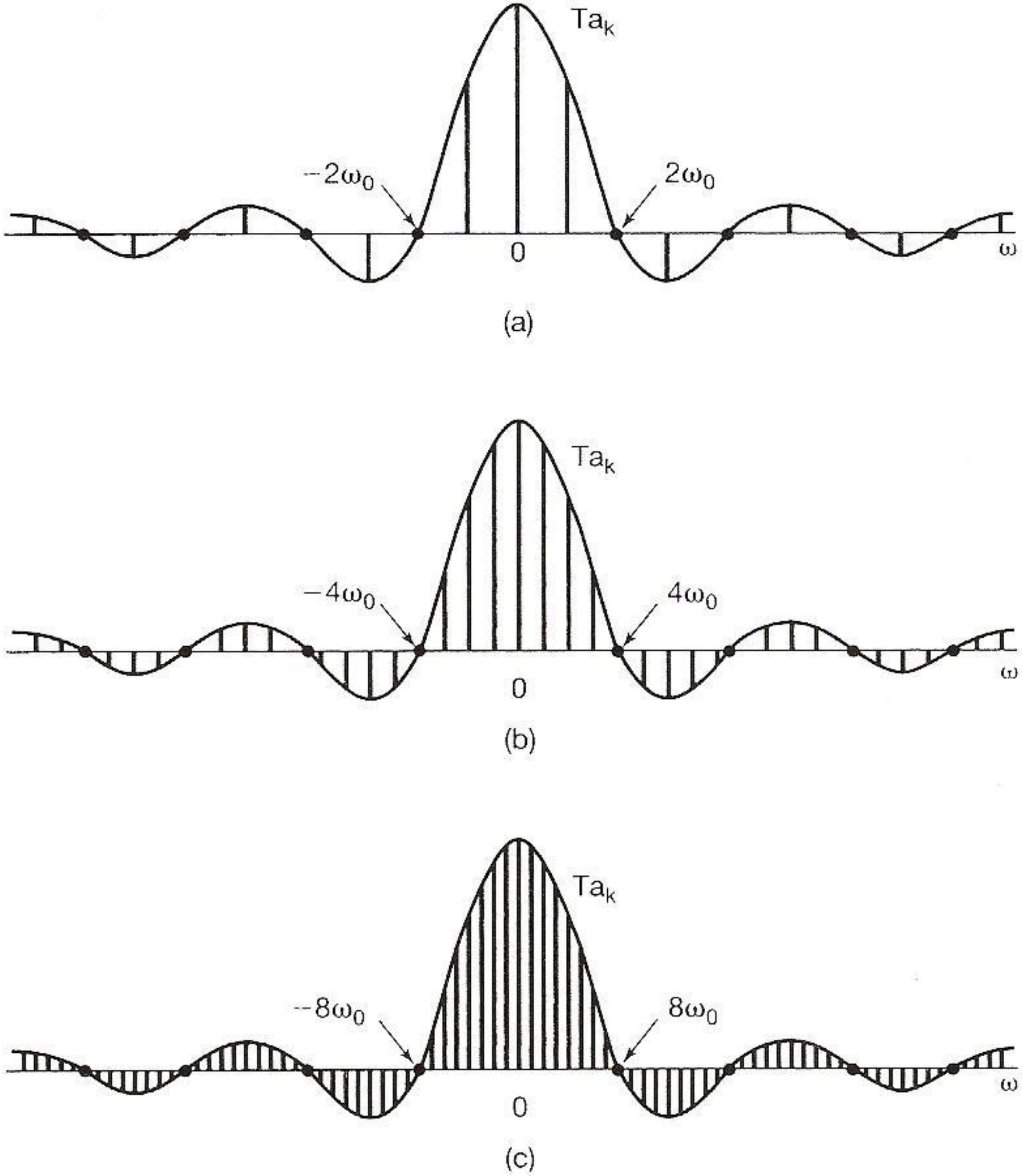


Figure 4.2 The Fourier series coefficients and their envelope for the periodic square wave in Figure 4.1 several values of T (with T_1 fixed) (a) $T = 4T_1$; (b) $T = 8T_1$; (c) $T =$ $16T_1$.

Figure 3.16 Discrete-time periodic square wave.

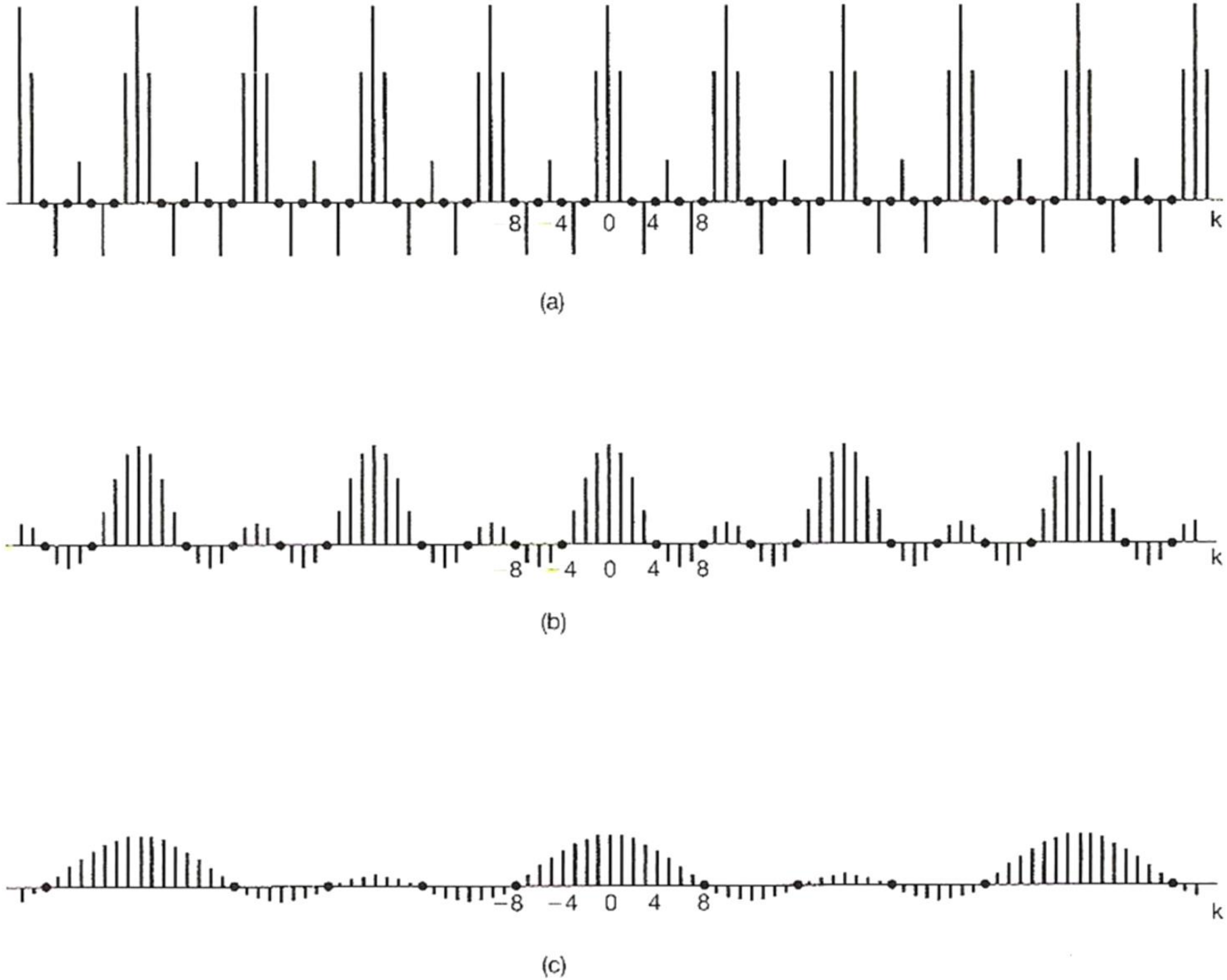


Figure 3.17 Fourier series coefficients for the periodic square wave of Example 3.12; plots of Na_k for $2N_1 + 1 = 5$ and (a) $N = 10$; (b) $N = 20$; and (c) $N = 40$.

From Periodic to Aperiodic

- Considering $x[n]$, $x[n]=0$ for $n > N_2$ or $n < -N_1$
  - Construct $\tilde{x}[n]$ periodic with period $N > N_1 + N_2 + 1$

$$\tilde{x}[n] = x[n] \text{ if } -N_1 \le n \le N_2$$

$$\tilde{x}[n] = x[n] \text{ if } N \to \infty$$

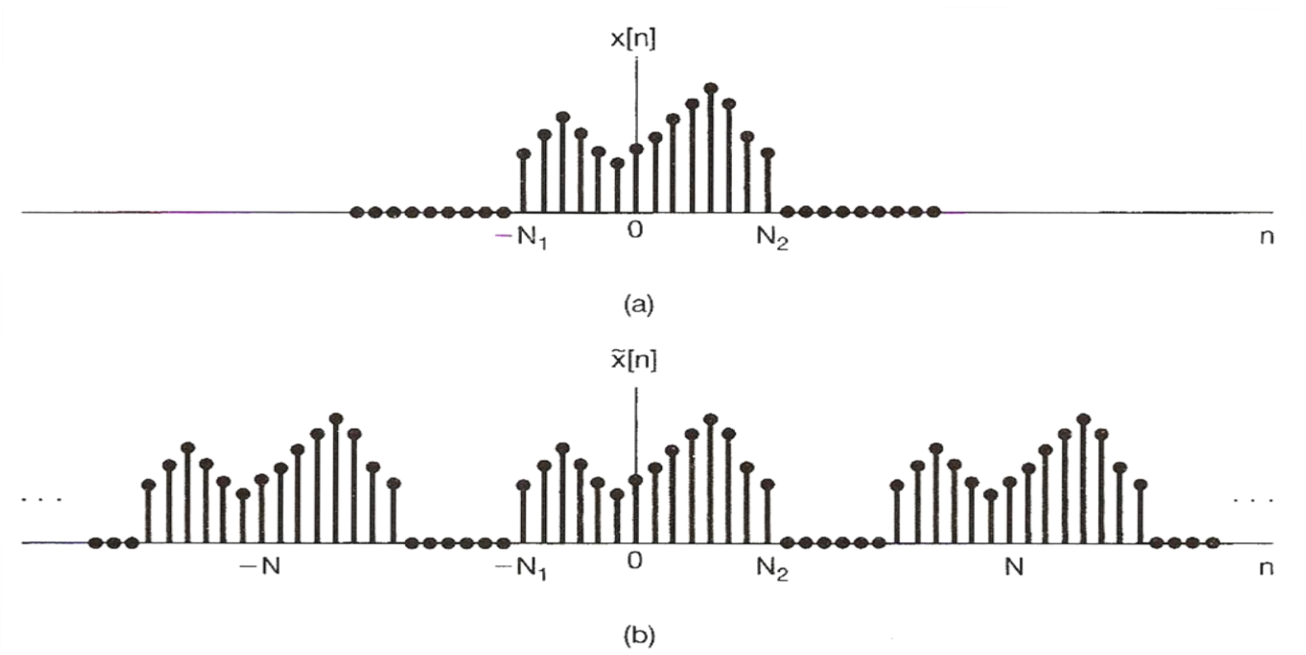


Figure 5.1 (a) Finite-duration signal $x[n]$; (b) periodic signal $\tilde{x}[n]$ constructed to be equal to $x[n]$ over one period.

From Periodic to Aperiodic

- Considering $x[n]$, $x[n]=0$ for $n > N_2$ or $n < -N_1$
  - Fourier series for $\tilde{x}[n]$

$$\tilde{x}[n] = \sum_{k=<N>} \boxed{a_k} e^{jk\omega_0 n} = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$a_k = \frac{1}{N} \sum_{n=<N>} \tilde{x}[n] e^{-jk\left(\frac{2\pi}{N}\right)n} = \frac{1}{N} \boxed{\sum_{n=-\infty}^{\infty} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}}$$

  - Defining envelope of Na_k as $X\left(e^{j\omega}\right)$

$$a_k = \boxed{\frac{1}{N} X\left(e^{jk\left(\frac{2\pi}{N}\right)}\right)} = \frac{1}{N} \boxed{X\left(e^{j\omega}\right)\Big|_{\omega=k\left(\frac{2\pi}{N}\right)=k\omega_0}}$$

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$\tilde{x}[n] = \sum_{k=<N>} \frac{1}{N} X\left(e^{jk\omega_0}\right) e^{jk\omega_0 n}$$

$$= \frac{1}{2\pi} \sum_{k=<N>} X\left(e^{jk\omega_0}\right) e^{jk\omega_0 n} \omega_0, \ \frac{\omega_0}{2\pi} = \frac{1}{N}$$

- As $N \rightarrow \infty, \ \omega'_0 \rightarrow 0, \tilde{x}[n] \rightarrow x[n]$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega$$

signal, time domain, Inverse Discrete-time Fourier Transform

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

spectrum, frequency domain Discrete-time Fourier Transform

- Similar format to all Fourier analysis representations previously discussed

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt:$$ spectrum, frequency domain
Fourier Transform

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega:$$ signal, time domain
Inverse Fourier Transform

Fourier Transform pair, different expressions

$$x(t)\xleftrightarrow{F}X(j\omega)$$

very similar format to Fourier Series for periodic signals

- Note: $X(e^{j\omega})$ is continuous and periodic with period

  2π

  Integration over 2π only

  Frequency domain spectrum is continuous and periodic, while time domain signal is discrete-time and aperiodic

  Frequencies around ω=0 or 2π are low-frequencies, while those around $\omega= \pm\pi$ are high-frequencies, etc.

See Fig. 5.3, p.362 of text
For Examples see Fig. 5.5, 5.6, p.364, 365 of text

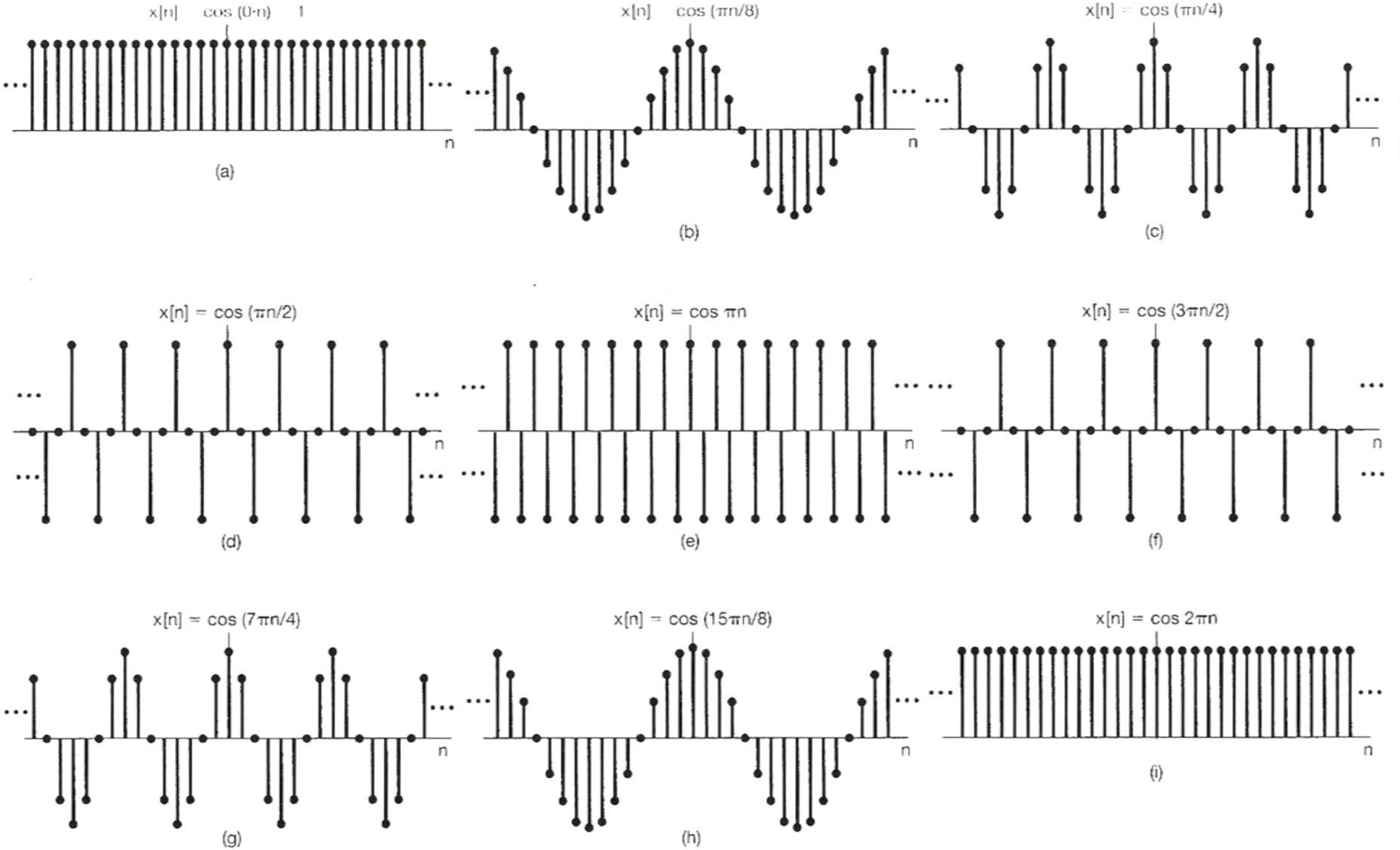


Figure 1.27 Discrete-time sinusoidal sequences for several different frequencies.

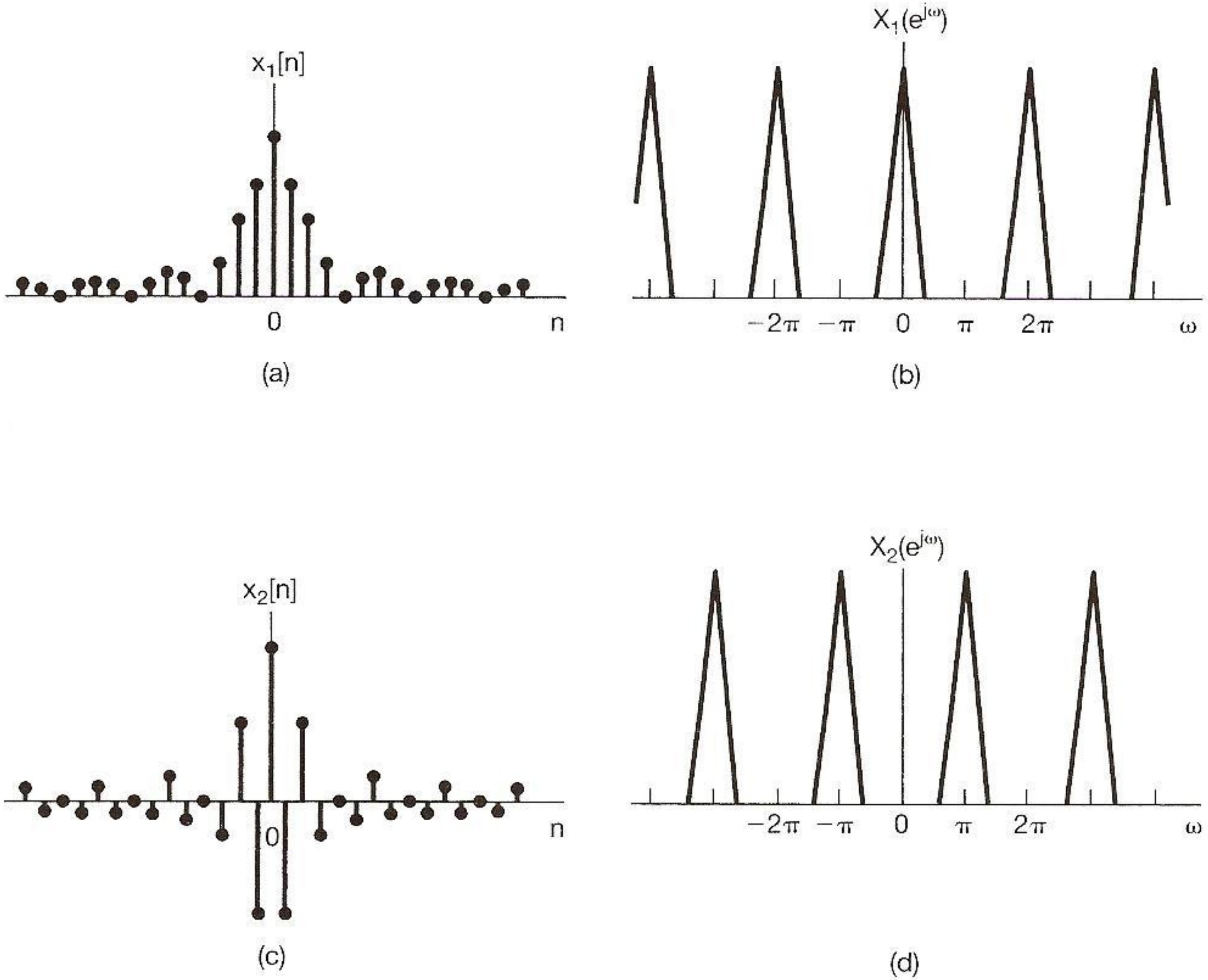


Figure 5.3 (a) Discrete-time signal $x_1[n]$. (b) Fourier transform of $x_1[n]$. Note that $X_1(e^{j\omega})$ is concentrated near $\omega = 0, \pm 2\pi, \pm 4\pi, \ldots$. (c) Discrete-time signal $x_2[n]$. (d) Fourier transform of $x_2[n]$. Note that $X_2(e^{j\omega})$ is concentrated near $\omega = \pm\pi, \pm 3\pi, \ldots$.

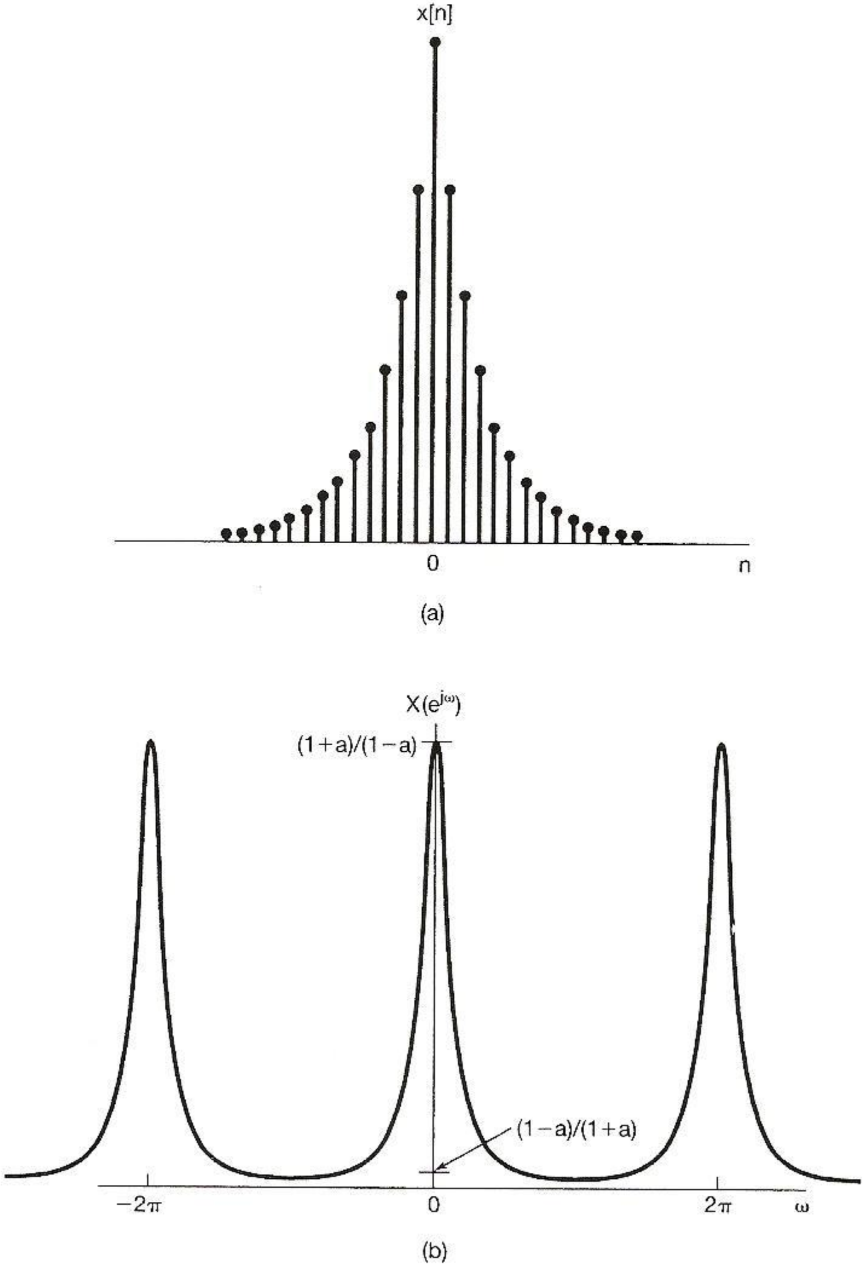


Figure 5.5 (a) Signal $x[n] = a^{|n|}$ of Example 5.2 and (b) its Fourier transform $(0 < a < 1)$.

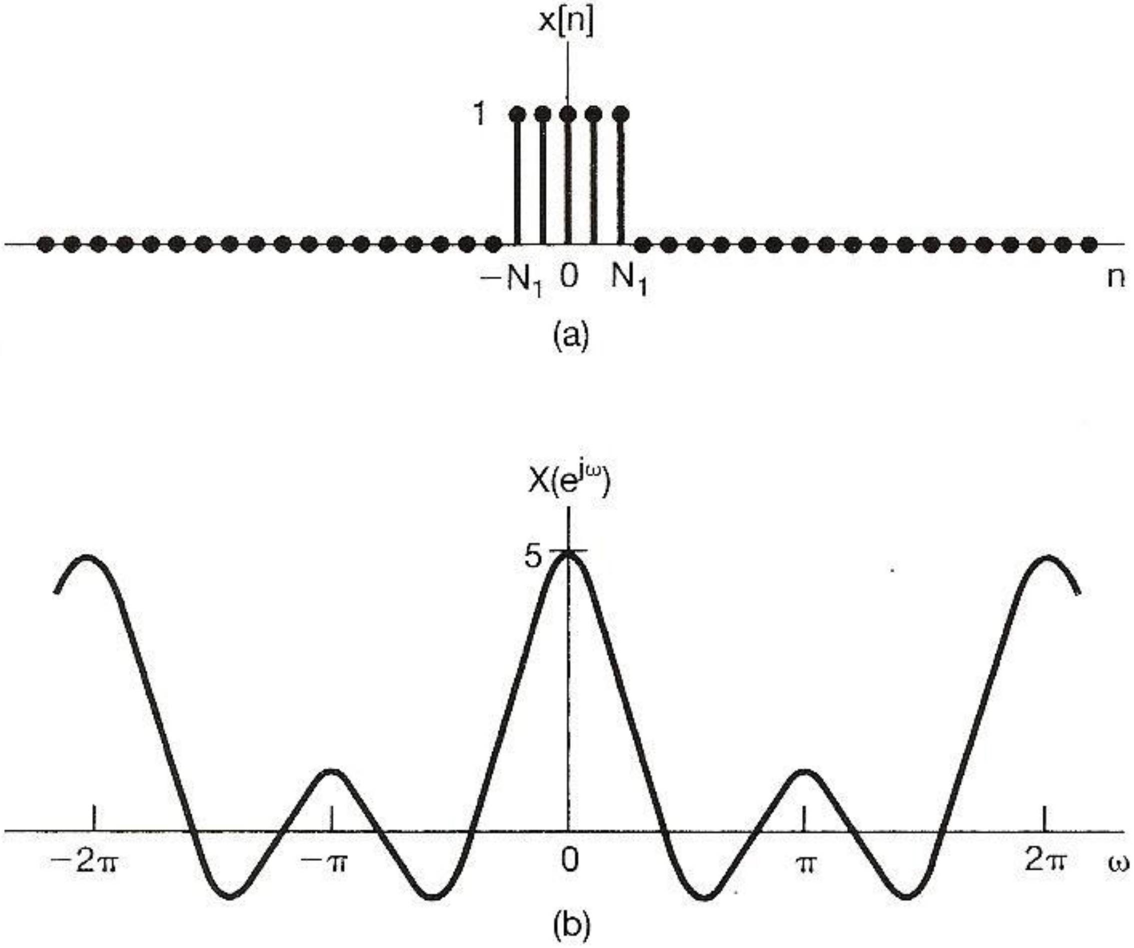


Figure 5.6 (a) Rectangular pulse signal of Example 5.3 for $N_1 = 2$ and (b) its Fourier transform.

From Periodic to Aperiodic

- Convergence Issue

given $x[n]$

$$X\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n}$$

$$\hat{x}[n]=\frac{1}{2\pi}\int_{-W}^{W}X\left(e^{j\omega}\right)e^{j\omega n}d\omega$$

$$\hat{x}[n]=x[n] \text{ when } W=\pi$$

  - No convergence issue since the integration is over an finite interval
  - No Gibbs phenomenon

See Fig. 5.7, p.368 of text

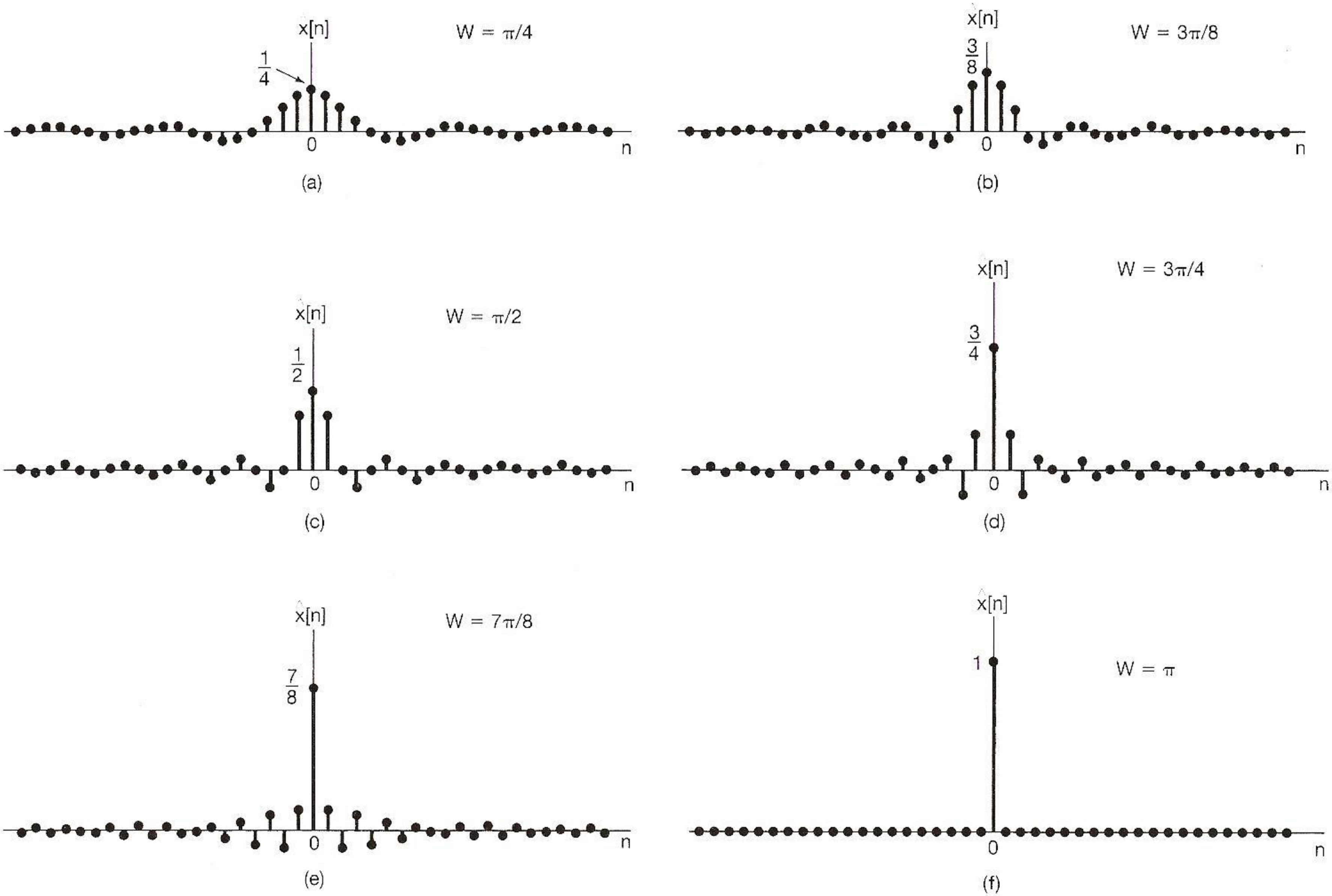


Figure 5.7 Approximation to the unit sample obtained as in eq. (5.16) using complex exponentials with frequencies $|\omega| \leq W$: (a) $W = \pi/4$; (b) $W = 3\pi/8$; (c) $W = \pi/2$; (d) $W = 3\pi/4$; (e) $W = 7\pi/8$; (f) $W = \pi$. Note that for $W = \pi$, $\hat{x}[n] = \delta[n]$.

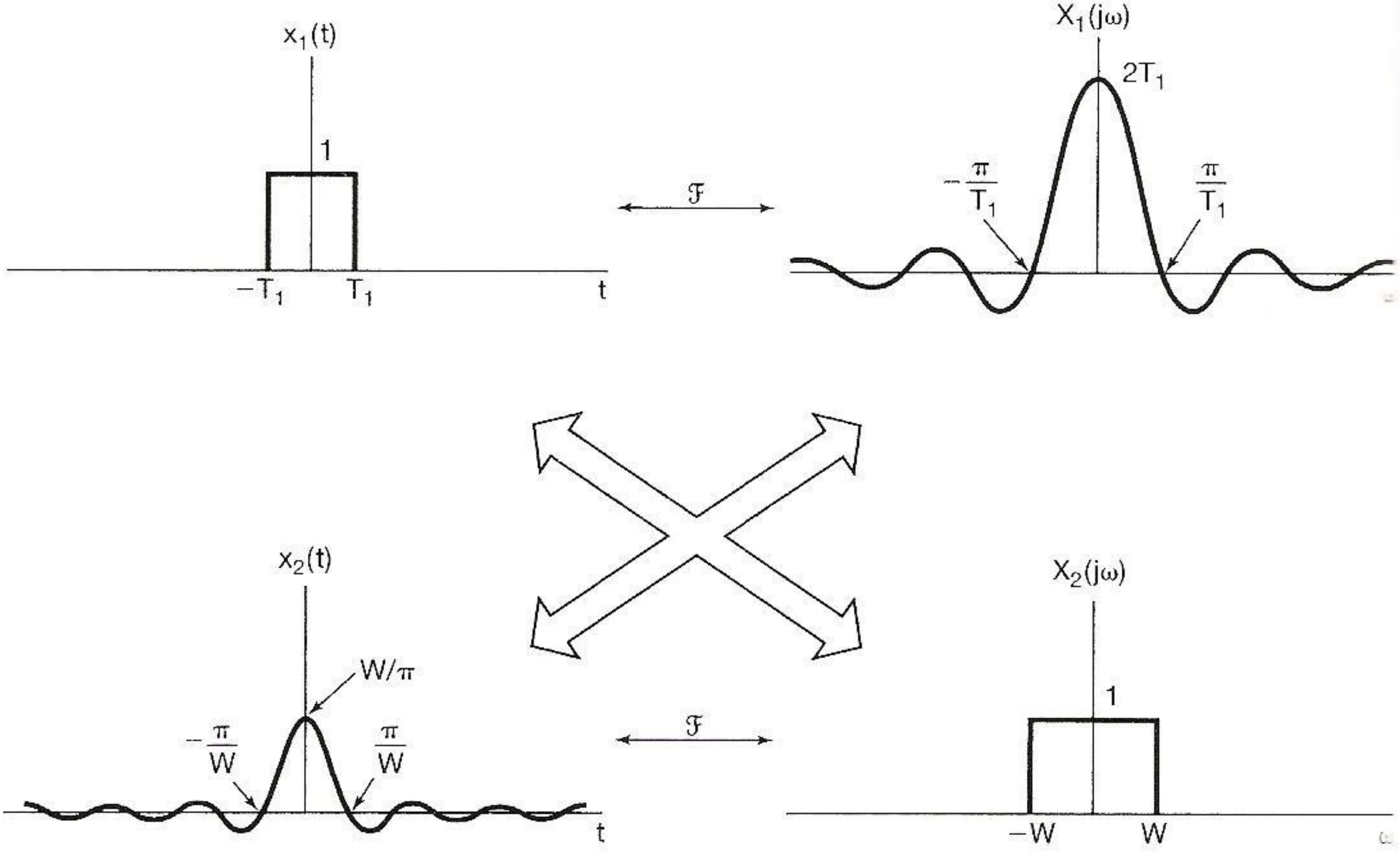


Figure 4.17 Relationship between the Fourier transform pairs of eqs. (4.36) and (4.37).

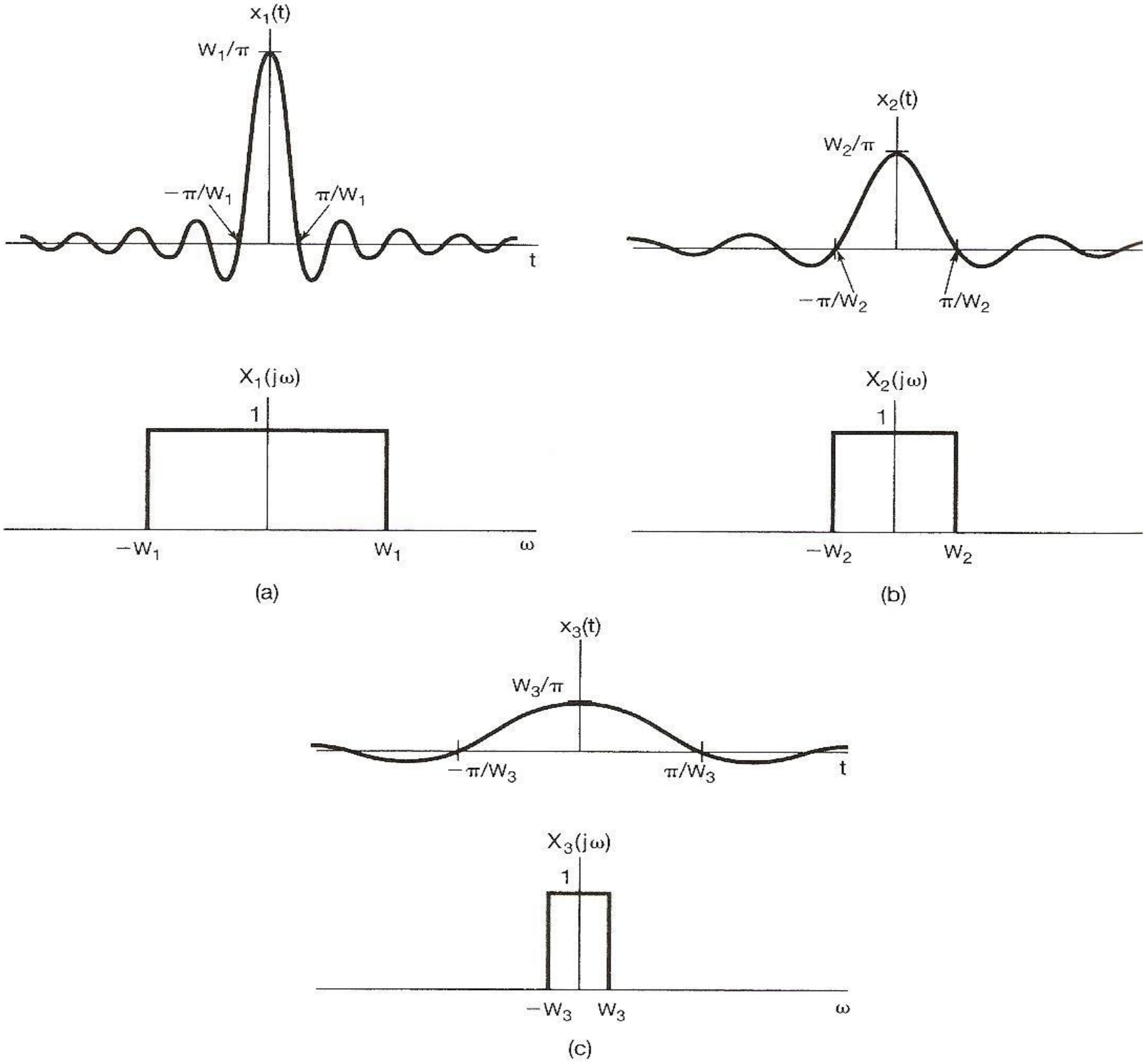
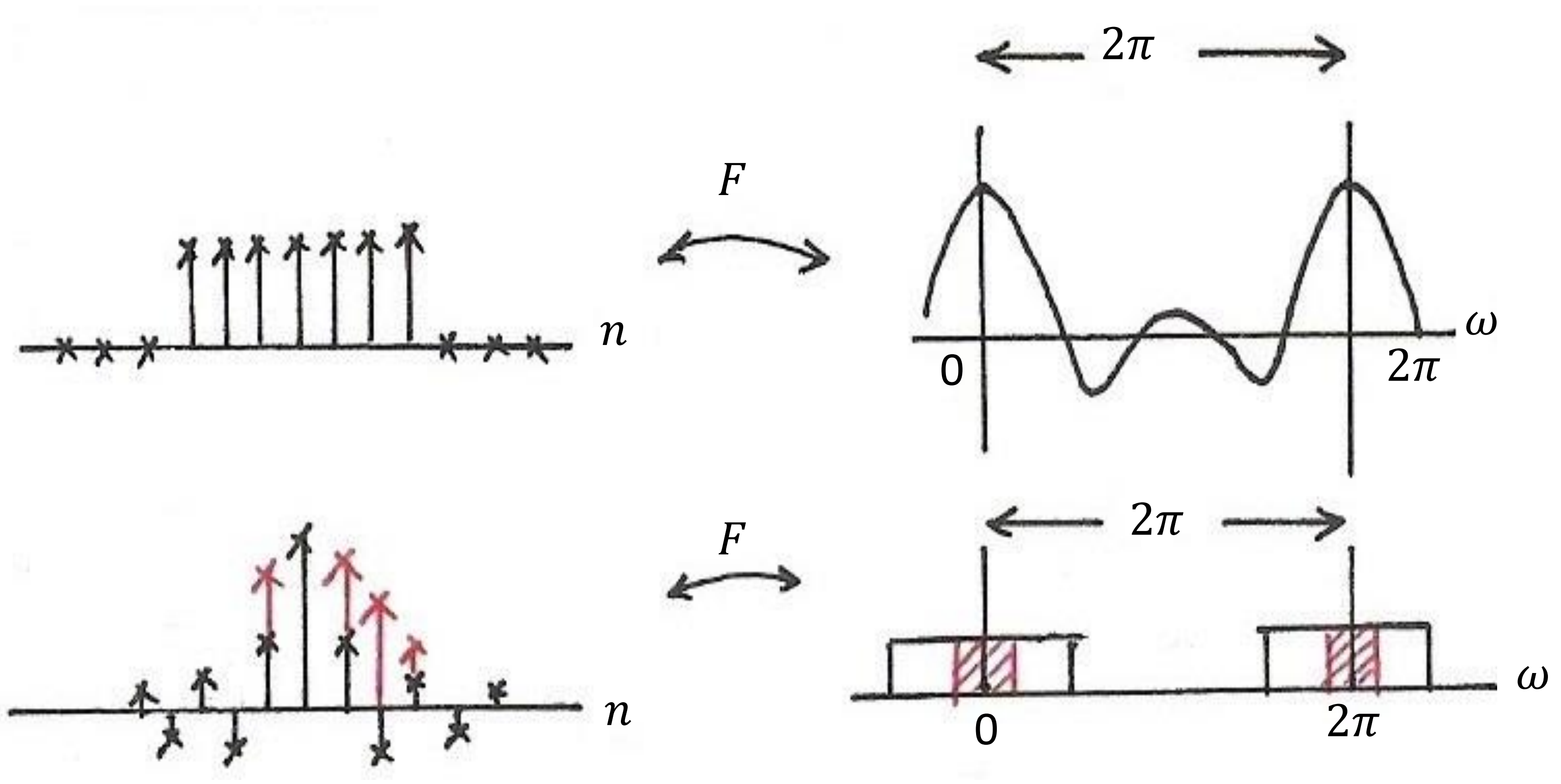


Figure 4.11 Fourier transform pair of Figure 4.9 for several different values of *W*.

Rectangular/Sinc



- Fourier Transform for Periodic Signals – Unified Framework (p.16 of 4.0)
  - Given $x(t)$

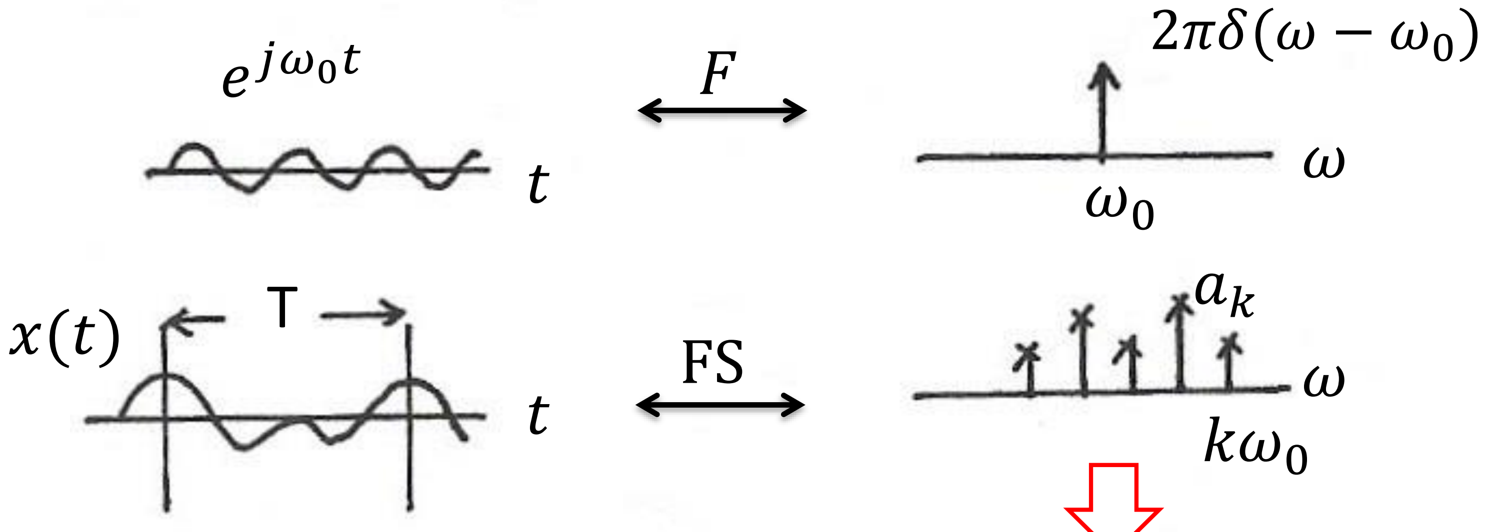
$$\text{assume } X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0) e^{j\omega t} d\omega = e^{j\omega_0 t}$$

$$e^{j\omega_0 t} \xleftrightarrow{F} 2\pi\delta(\omega - \omega_0)$$

(easy in one way)

Unified Framework: Fourier Transform for Periodic Signals (p.17 of 4.0)



- If $x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$$X(j\omega)=\sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega-k\omega_0)$$

$$\begin{tikzcd} x(t) \arrow[r, leftrightarrow, "FS"] \arrow[rr, leftrightarrow, bend right, "F"'] & a_k \arrow[r] & X(j\omega) \end{tikzcd}$$

From Periodic to Aperiodic

- For Periodic Signals – Unified Framework
  - Given $x[n]$

$$\text{assume } X\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} 2\pi\delta\left(\omega - \omega_0 - 2\pi k\right)$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\omega}\right)e^{j\omega n}d\omega = e^{j\omega_0 n}$$

$$e^{j\omega_0 n} \xleftrightarrow{F} \sum_{k=-\infty}^{\infty} 2\pi\delta\left(\omega - \omega_0 - 2\pi k\right)$$

See Fig. 5.8, p.369 of text

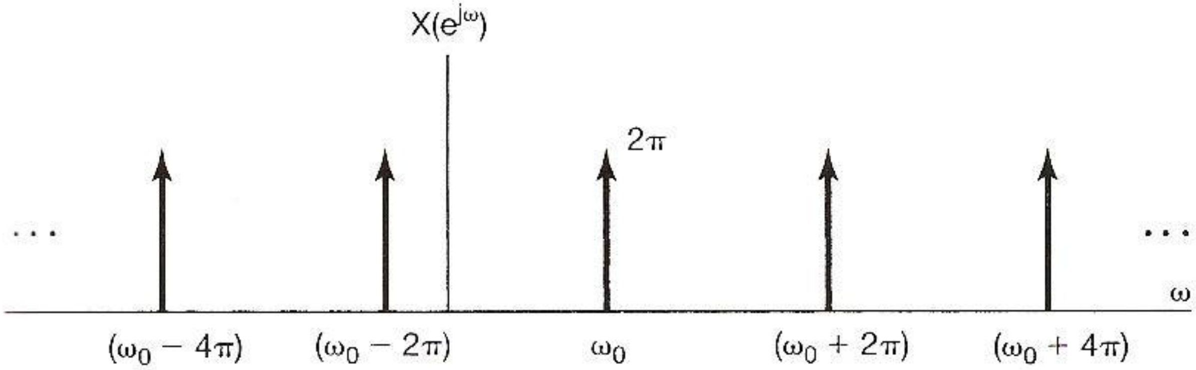


Figure 5.8 Fourier transform of $x[n] = e^{j\omega_0 n}$.

From Periodic to Aperiodic

- For Periodic Signals – Unified Framework
  - If $x[n] = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$

$$X\left(e^{j\omega}\right) = \sum_{k=<N>} a_k \sum_{m=-\infty}^{\infty} 2\pi\delta\left(\omega - \frac{2\pi k}{N} - 2\pi m\right)$$

$$= \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right), a_k = a_k + mN$$

See Fig. 5.9, p.370 of text

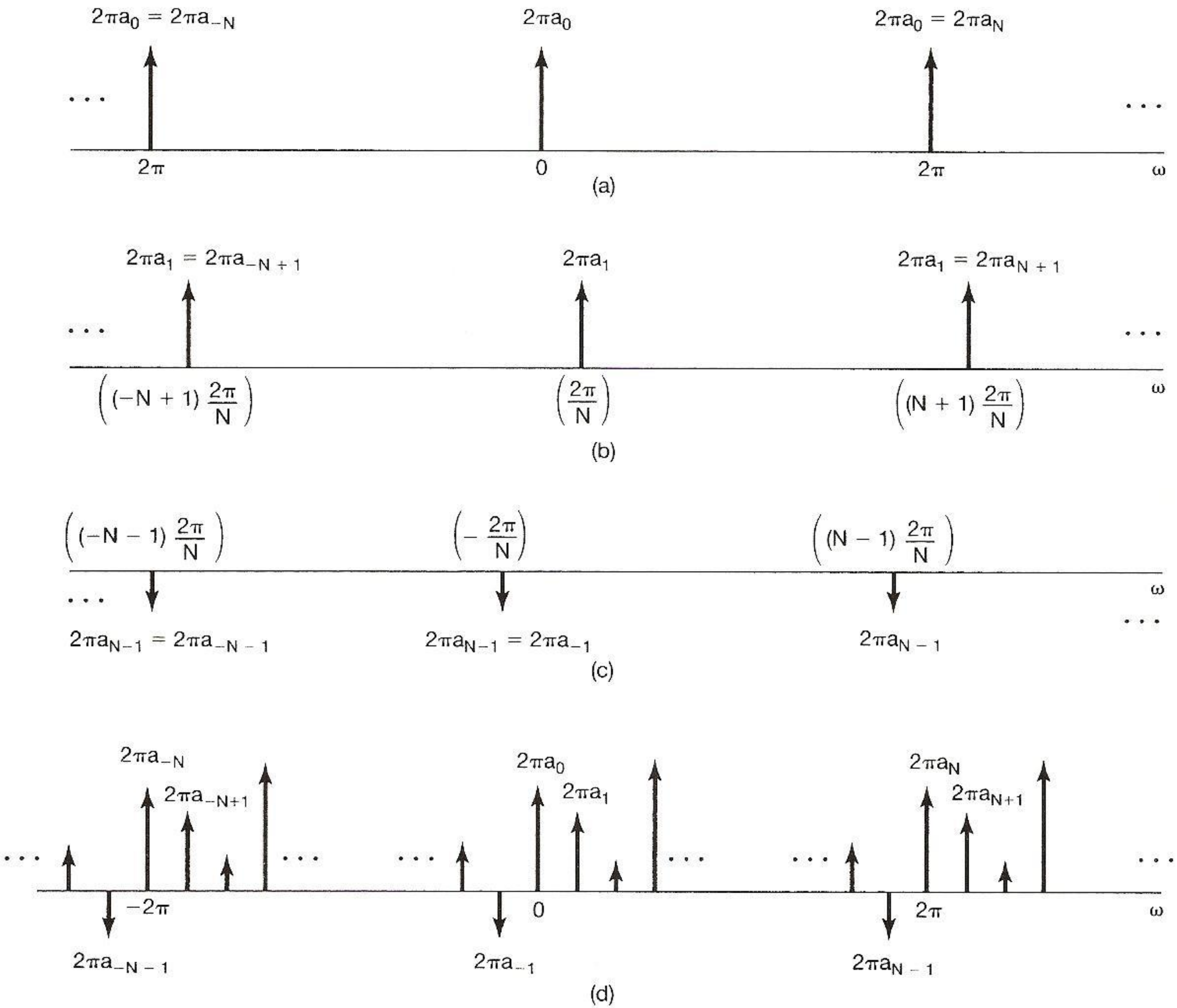
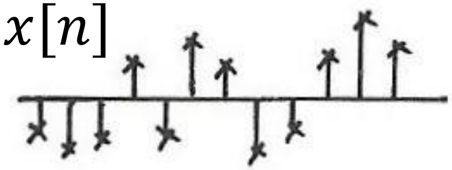


Figure 5.9 Fourier transform of a discrete-time periodic signal: (a) Fourier transform of the first term on the right-hand side of eq. (5.21); (b) Fourier transform of the second term in eq. (5.21); (c) Fourier transform of the last term in eq. (5.21); (d) Fourier transform of $x[n]$ in eq (5.21).

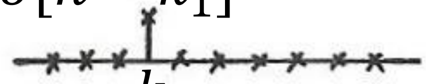
Signal Representation in Two Domains

Time Domain

$x[n]$

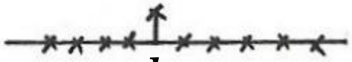


$\delta[n-k_1]$



k_1

$\delta[n-k_2]$



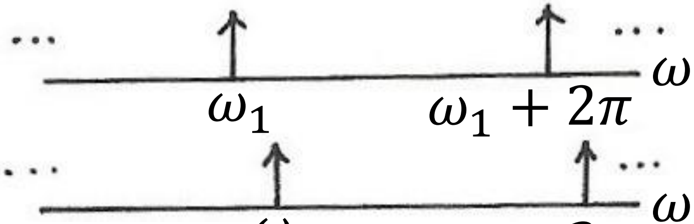
k_2

$\{\delta[n-k]$, k: integer, $-\infty < k < \infty\}$

Frequency Domain

$X(e^{j\omega})$

ω



ω

ω_1 $\quad \omega_1 + 2\pi$

ω

ω_2 $\quad \omega_2 + 2\pi$

$$\{\boxed{\sum_{m=-\infty}^{\infty} 2\pi\delta\,(\omega-\omega_k-2\pi m)},\ 0<\omega_k\le 2\pi\}$$

$\updownarrow F$

n $\qquad e^{j\omega_k n}$

5.2 Properties of Discrete-time Fourier Transform

$$x[n] \xleftrightarrow{F} X\left(e^{j\omega}\right)$$

- Periodicity

$$X\left(e^{j(\omega+2\pi)}\right) = X\left(e^{j\omega}\right)$$

- Linearity

$$x_1[n] \xleftrightarrow{F} X_1\left(e^{j\omega}\right), x_2[n] \xleftrightarrow{F} X_2\left(e^{j\omega}\right)$$

$$ax_1[n] + bx_2[n] \xleftrightarrow{F} aX_1\left(e^{j\omega}\right) + bX_2\left(e^{j\omega}\right)$$

$$x[n] \xleftrightarrow{F} X\left(e^{j\omega}\right)$$

- Time/Frequency Shift

$$x[n-n_0] \xleftrightarrow{F} e^{-j\omega n_0} X\left(e^{j\omega}\right)$$

$$e^{j\omega_0 n} x[n] \xleftrightarrow{F} X\left(e^{j(\omega-\omega_0)}\right)$$

- Conjugation

$$x^*[n] \xleftrightarrow{F} X^*\left(e^{-j\omega}\right)$$

$$X\left(e^{j\omega}\right) = X^*\left(e^{-j\omega}\right) \text{ if } \; x[n] \text{ real}$$

Even/Odd Relations

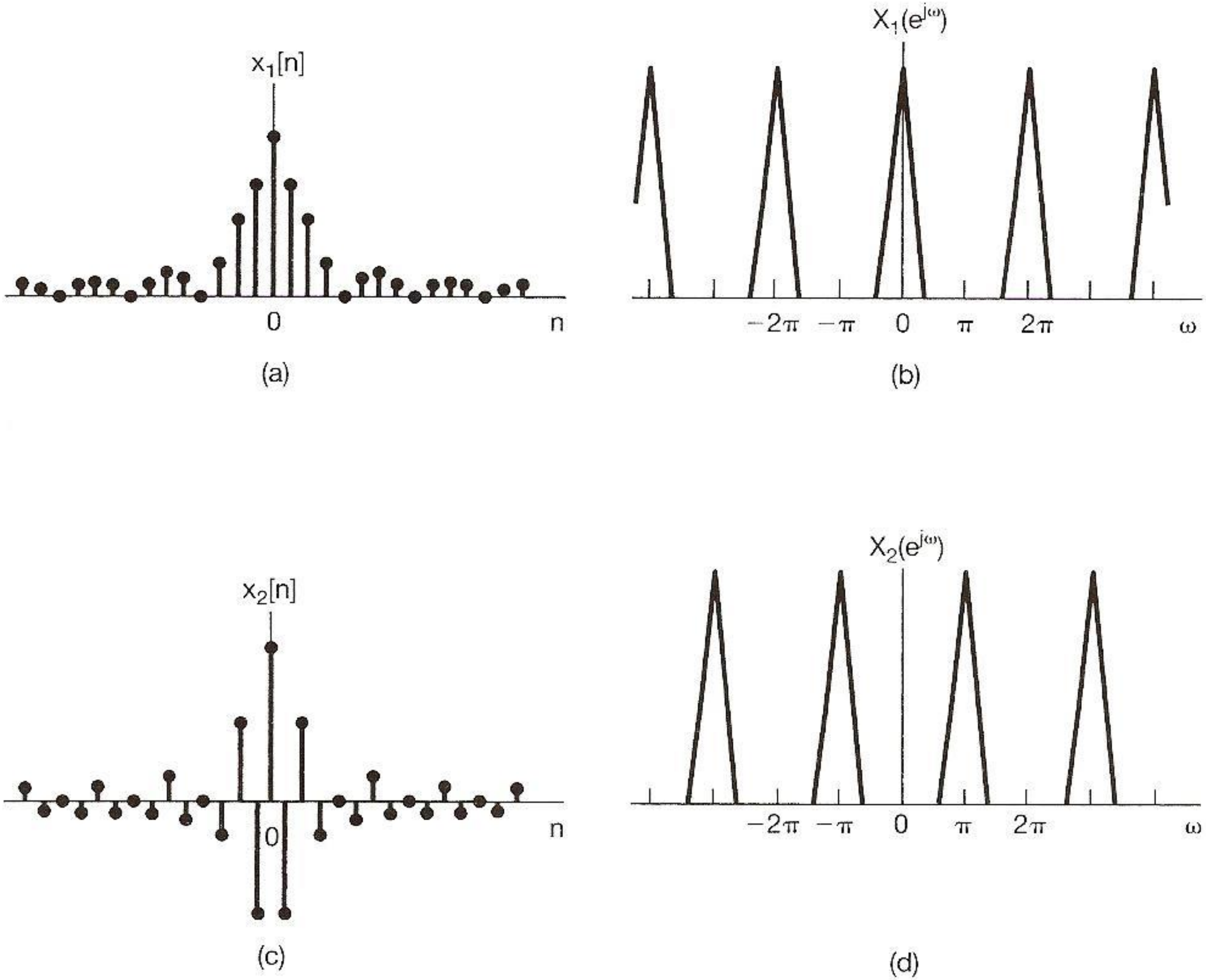


Figure 5.3 (a) Discrete-time signal $x_1[n]$. (b) Fourier transform of $x_1[n]$. Note that $X_1(e^{j\omega})$ is concentrated near $\omega = 0, \pm 2\pi, \pm 4\pi, \ldots$. (c) Discrete-time signal $x_2[n]$. (d) Fourier transform of $x_2[n]$. Note that $X_2(e^{j\omega})$ is concentrated near $\omega = \pm\pi, \pm 3\pi, \ldots$.

$$x[n] \xleftrightarrow{F} X\left(e^{j\omega}\right)$$

- Differencing/Accumulation

$$x[n] - x[n-1] \xleftrightarrow{F} \left(1 - e^{-j\omega}\right) X\left(e^{j\omega}\right)$$

$$\sum_{m=-\infty}^{n} x[m] \xleftrightarrow{F} \frac{1}{1 - e^{-j\omega}} X\left(e^{j\omega}\right) + \pi X\left(e^{j0}\right) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

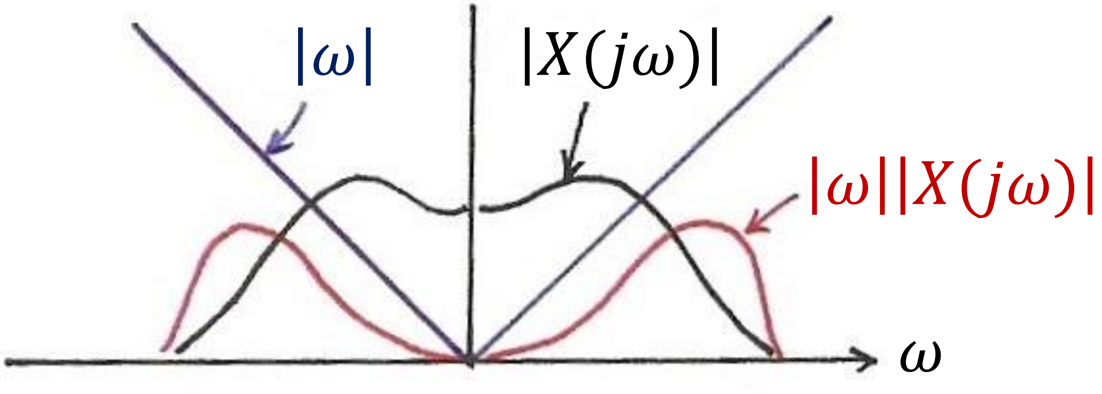
- Time Reversal

$$x[-n] \xleftrightarrow{F} X\left(e^{-j\omega}\right)$$

Differentiation (p.35 of 4.0)

$$\frac{dx(t)}{dt} \xleftrightarrow{F} j\omega X(j\omega)$$

$$|j\omega \cdot X(j\omega)| = |\omega| \cdot |X(j\omega)|$$



Enhancing higher frequencies

De-emphasizing lower frequencies

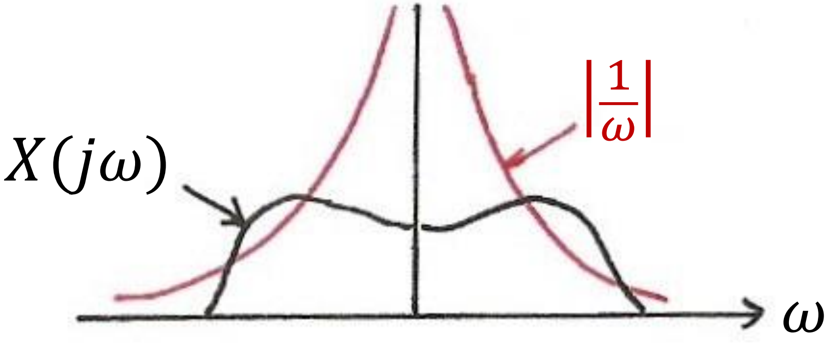
Deleting DC term (=0 for ω=0)

Integration (p.36 of 4.0)

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{F} \frac{1}{j\omega}X(j\omega)+\pi \underset{\substack{\uparrow \\ \text{dc term}}}{X(j0)}\delta(\omega)$$

$$\frac{1}{j} = e^{-j90^\circ}$$

$$\left|\frac{1}{j\omega}\right| \cdot |X(j\omega)| = \left|\frac{1}{\omega}\right| \cdot |X(j\omega)|$$



Enhancing lower frequencies (accumulation effect)

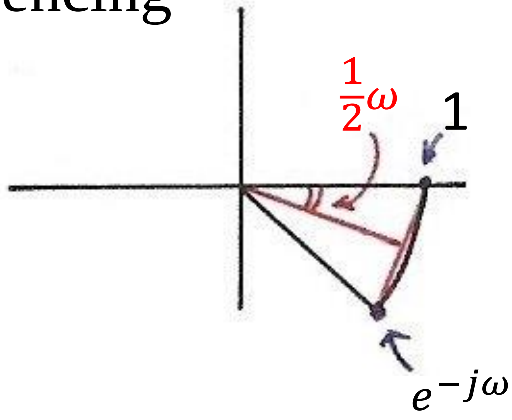
De-emphasizing higher frequencies

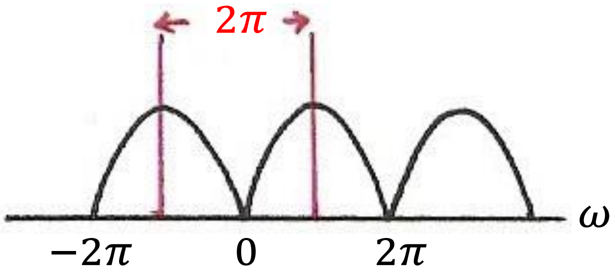
(smoothing effect)

Undefined for ω=0

Differencing/Accumulation

Differencing

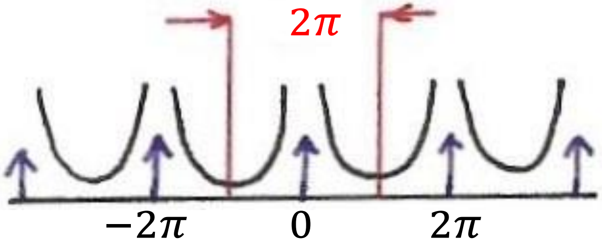




Enhancing higher frequencies
De-emphasizing lower freq
Deleting DC term

$$|1-e^{-j\omega}| = 2\left|\sin\left(\frac{\omega}{2}\right)\right|$$

Accumulation



- Differencing/Accumulation

$$x[n]-x[n-1] \xleftrightarrow{F} \left(1-e^{-j\omega}\right)X\left(e^{j\omega}\right)$$

$$\sum_{m=-\infty}^{n} x[m] \xleftrightarrow{F} \frac{1}{1-e^{-j\omega}}X\left(e^{j\omega}\right)$$

$$+\pi X\left(e^{j0}\right)\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$$

- Time Reversal (p.29 of 3.0)

$$x(-t) \xleftrightarrow{FS} a_{-k}$$

the effect of sign change for $x(t)$ and a_k are identical

$$\cdots a_{-1} e^{-j\omega_0 t} + a_0 + a_1 e^{j\omega_0 t} + \cdots = x(t)$$

$$\cdots a_{-1} e^{j\omega_0 t} \cdots = x(-t)$$

unique representation for orthogonal basis

$$x[n] \xleftrightarrow{F} X\left(e^{j\omega}\right)$$

- Time Expansion

define $x_{(k)}[n] = x[n/k]$, If n/k is an integer,
k: positive integer

$= 0$, else

See Fig. 5.13, p.377 of text

$$x_{(k)}[n] \xleftrightarrow{F} X\left(e^{jk\omega}\right)$$

See Fig. 5.14, p.378 of text

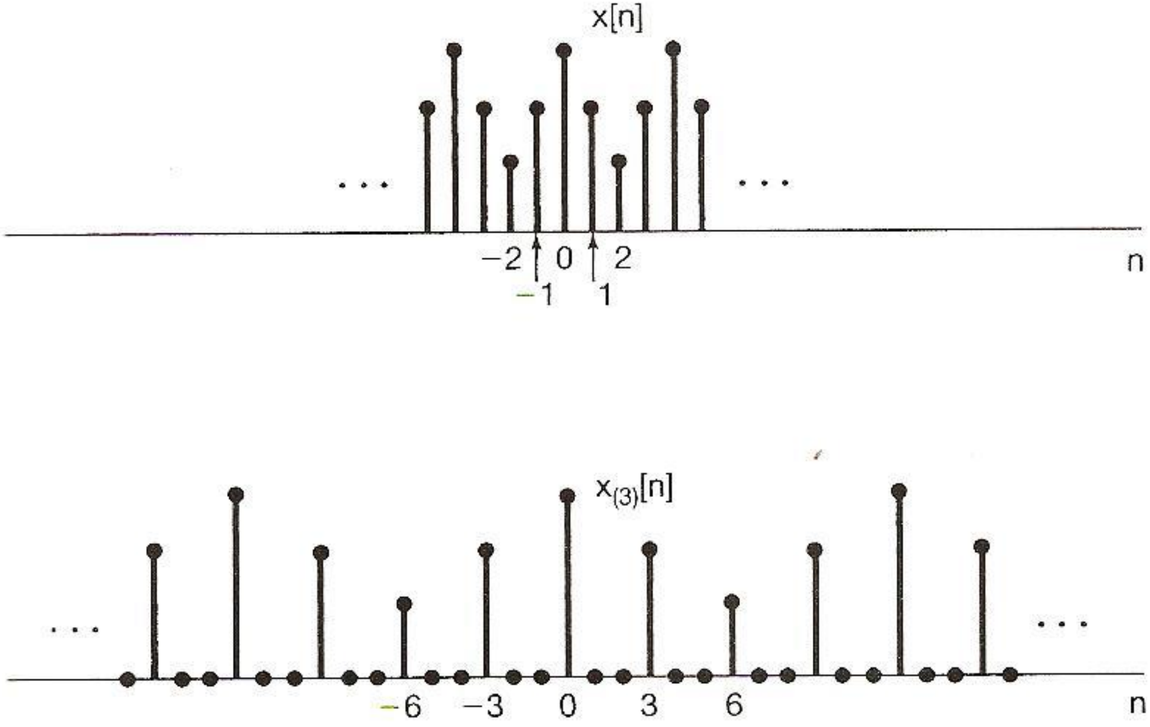


Figure 5.13 The signal $x_{(3)}[n]$ obtained from $x[n]$ by inserting two zeros between successive values of the original signal.

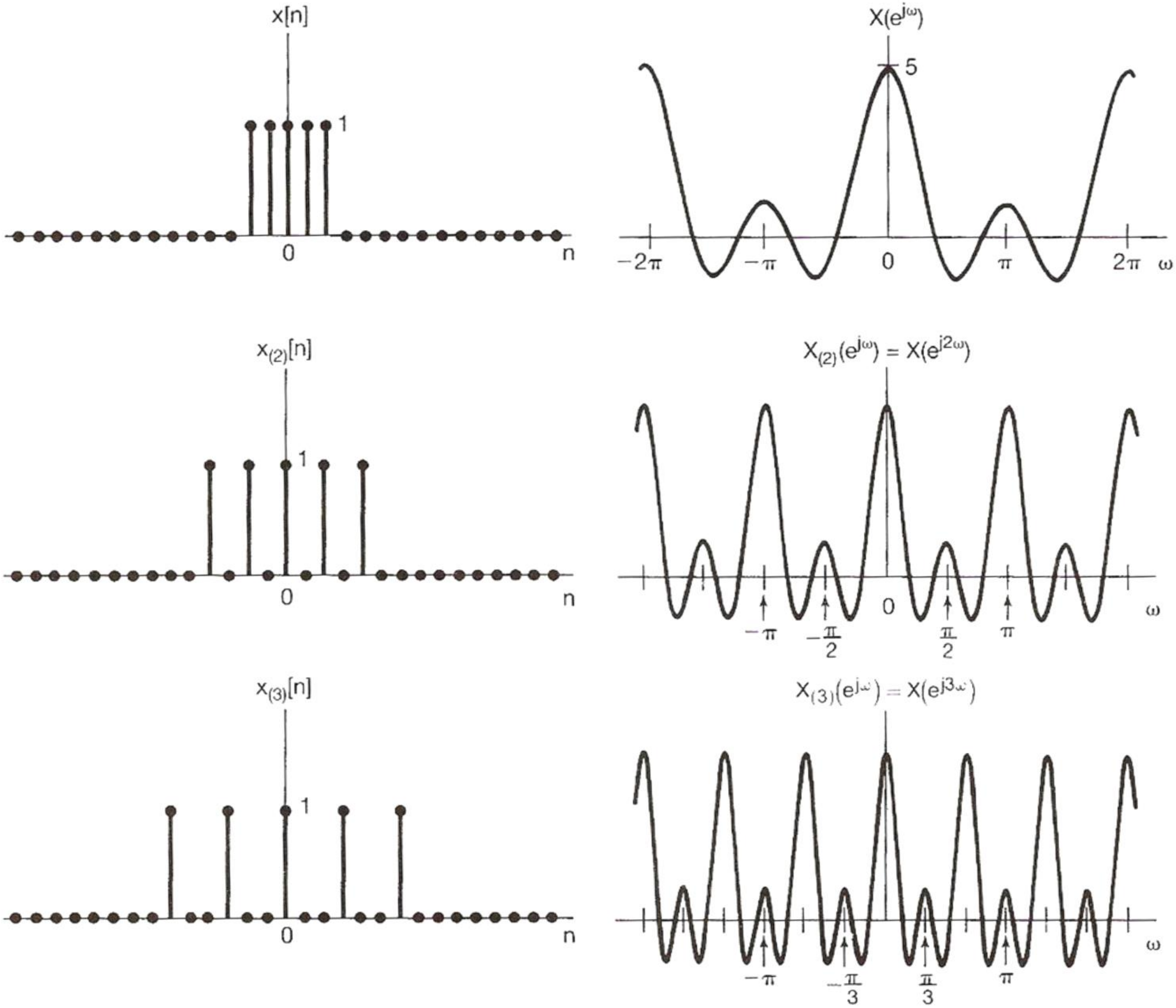
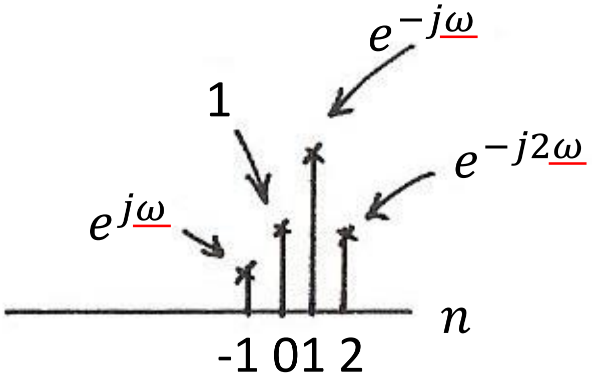
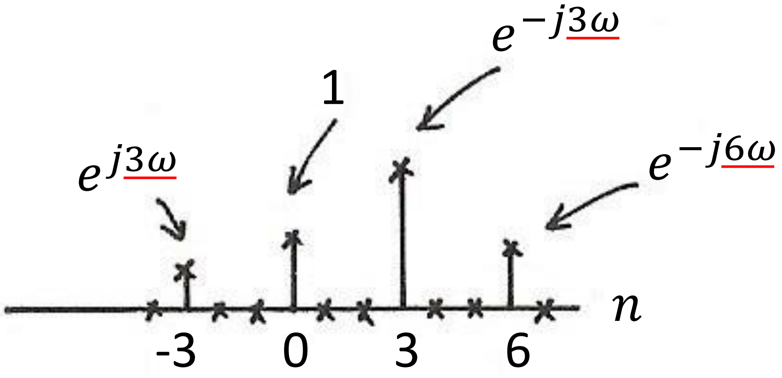


Figure 5.14 Inverse relationship between the time and frequency domains: As k increases, $x_{(k)}[n]$ spreads out while its transform is compressed.

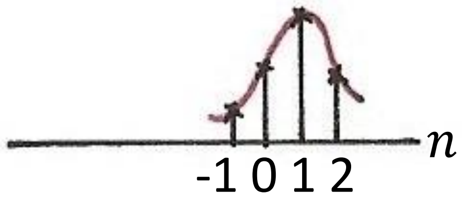
Time Expansion

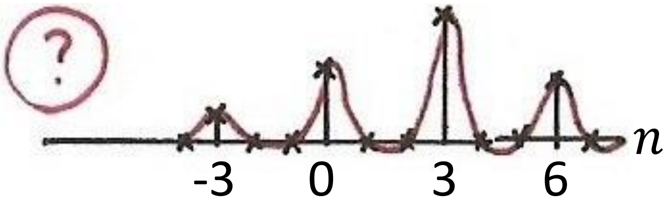




$$X\left(e^{j\overset{\overset{(k\omega)3\omega}{\uparrow}}{\underline{\omega}}}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\overset{\overset{(k\omega)3\omega}{\uparrow}}{\underline{\omega}}n}$$

Time Expansion





Discrete-time

$x[n]$

F (chap5)

$$\boxed{\sum_{n} x[n]\, e^{-j\omega n}}$$

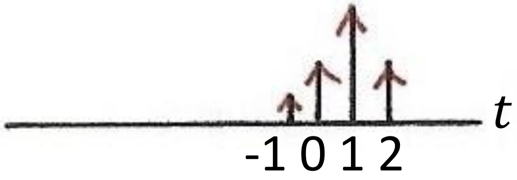
Continuous-time

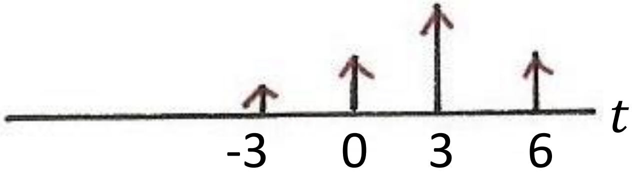
$$x(t) = \sum_{n} x[n]\, \delta(t-n)$$

F (chap4)

$$\int_{-\infty}^{\infty}\{\sum_{n} x[n]\, \delta\,(t-n)\}e^{-j\omega t}dt$$

$$= \boxed{\sum_{n} x[n]\, e^{-j\omega n}}$$





?

$$x(a t) \overset{F}{\leftrightarrow} \frac{1}{|a|} X\,(\frac{j\omega}{a})$$

$a = \frac{1}{k}$, k=integer

$$x[n] \xleftrightarrow{F} X\left(e^{j\omega}\right)$$

- Differentiation in Frequency

$$nx[n] \xleftrightarrow{F} j\,\frac{dX\left(e^{j\omega}\right)}{d\omega}$$

- Parseval's Relation

$$\sum_{n=-\infty}^{\infty} \left| x[n] \right|^2 = \frac{1}{2\pi} \int_{2\pi} \left| X\left(e^{j\omega}\right) \right|^2 d\omega$$

$$x[n] \xleftrightarrow{F} X\left(e^{j\omega}\right)$$

- Convolution Property

$$y[n] = x[n] * h[n] \xleftrightarrow{F} Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right)H\left(e^{j\omega}\right)$$

$H\left(e^{j\omega}\right)$: frequency response or transfer function

$$h[n] \xleftrightarrow{F} H\left(e^{j\omega}\right)$$

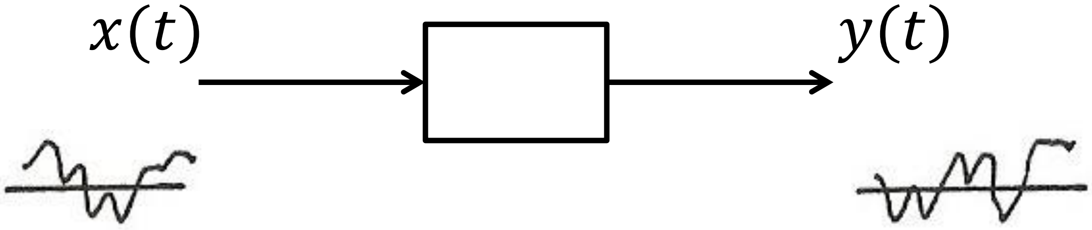
$$\delta[n] \xleftrightarrow{F} 1, \quad 0 < \omega \leq 2\pi$$

- Multiplication Property

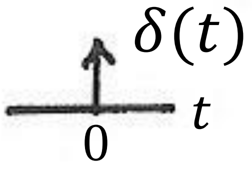
$$y[n] = x_1[n]x_2[n] \xleftrightarrow{F} Y\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{2\pi} X_1\left(e^{j\theta}\right)X_2\left(e^{j(\omega-\theta)}\right)d\theta$$

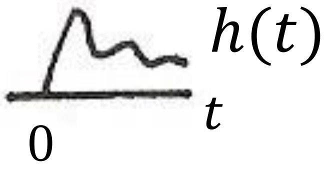
periodic convolution

Input/Output Relationship (P.55 of 4.0)

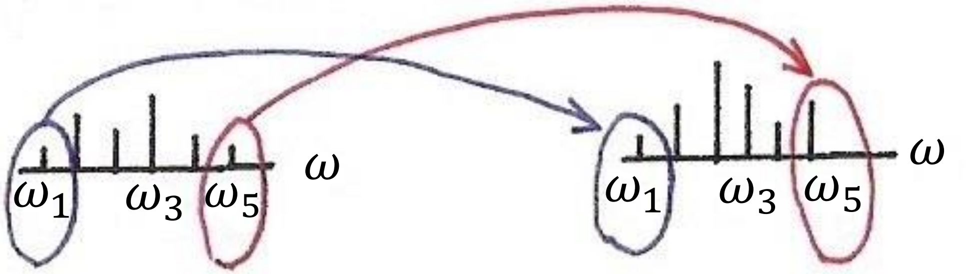


- Time Domain

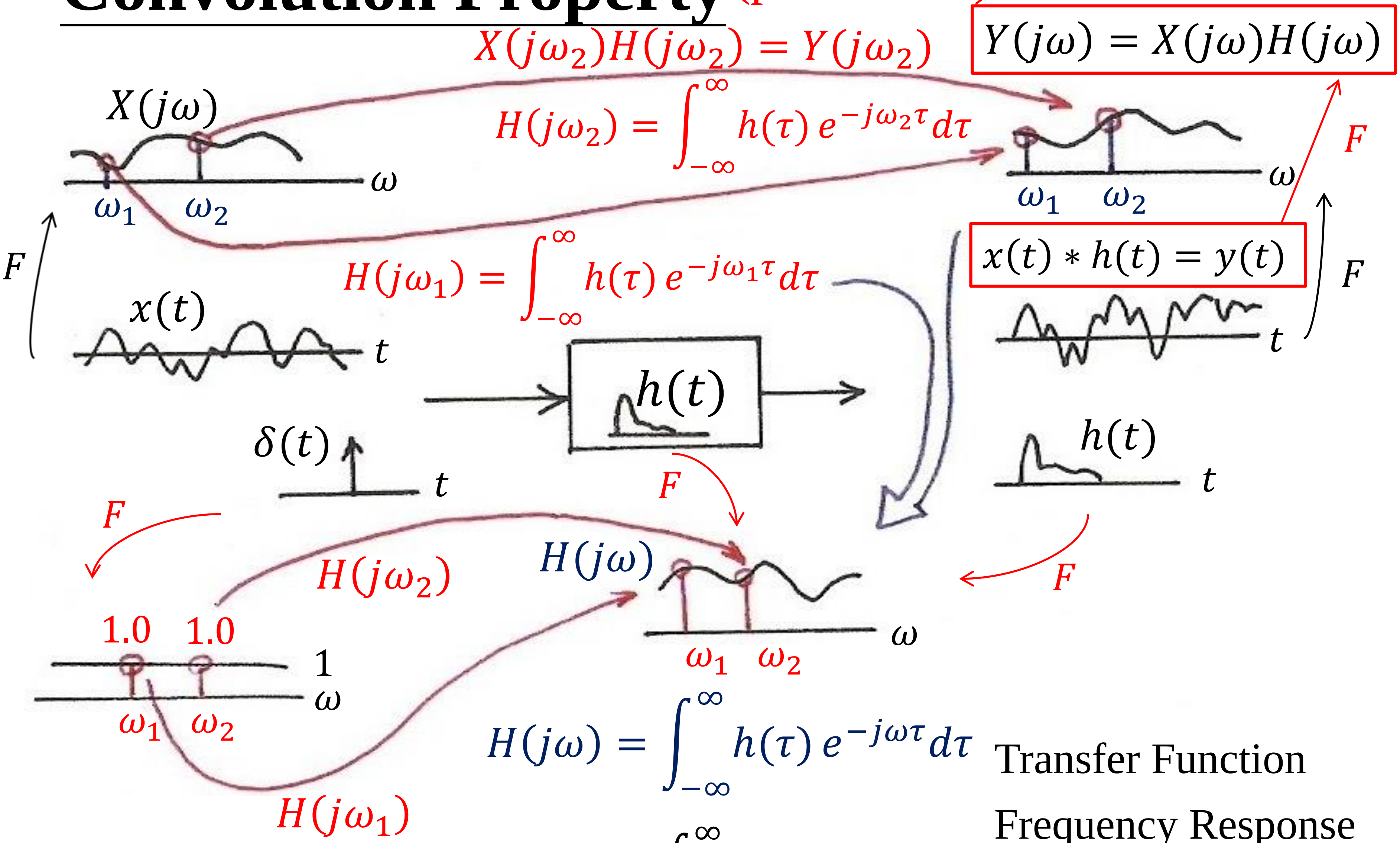




- Frequency Domain



Convolution Property (p.57 of 4.0)



$$\boxed{Y(j\omega) = X(j\omega)H(j\omega)}$$

$$X(j\omega_2)H(j\omega_2) = Y(j\omega_2)$$

$$H(j\omega_2) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega_2\tau} d\tau$$

$$H(j\omega_1) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega_1\tau} d\tau$$

$$\boxed{x(t) * h(t) = y(t)}$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau} d\tau$$

Transfer Function

Frequency Response

$$H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau} d\tau$$

$$x[n] \xleftrightarrow{F} X\left(e^{j\omega}\right)$$

- System Characterization

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y\left(e^{j\omega}\right) = \sum_{k=0}^{M} b_k e^{-jk\omega} X\left(e^{j\omega}\right)$$

$$h[n] \xleftrightarrow{F} H\left(e^{j\omega}\right) = \frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$

- Tables of Properties and Pairs

See Table 5.1, 5.2, p.391, 392 of text

TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{matrix}X(e^{j\omega})\\ Y(e^{j\omega})\end{matrix}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals	$\sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$	

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \leq N_1 \\ 0, & N_1 < \lvert n\rvert \leq N/2 \end{cases}$ and $x[n + N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \quad k \neq 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}, \quad k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] = \begin{cases} 1, & \lvert n\rvert \leq N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \leq \lvert\omega\rvert \leq W \\ 0, & W < \lvert\omega\rvert \leq \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi \delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n + 1)a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n + r - 1)!}{n!(r - 1)!} a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

- Vector Space Interpretation

  $\{x[n]$, aperiodic defined on $-\infty < n < \infty\} = V$

  is a vector space

$$\left(x_1[n]\right)\cdot\left(x_2[n]\right) = \sum_{k=-\infty}^{\infty} x_1[k]\,x_2^*[k]$$

  - basis signal sets

$$\left\{\phi_\omega[n] = e^{j\omega n},\ -\infty < \omega < \infty\right\}$$

$$\phi_\omega[n] = \phi_{\omega+2\pi k}[n]$$

repeats itself for very 2π

- Vector Space Interpretation
  - Generalized Parseval's Relation

$$\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n] = \frac{1}{2\pi}\int_{2\pi} X_1\left(e^{j\omega}\right)X_2^*\left(e^{j\omega}\right)d\omega$$

$\{X(e^{j\omega})$, with period 2π defined on $-\infty < \omega < \infty\} = V$: a vector space

inner-product can be evaluated in either domain

- Vector Space Interpretation
  - Orthogonal Bases

$$e^{j\omega_k n} \xleftrightarrow{F} \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega - \omega_k - 2\pi m)$$

$$e^{j\omega_j n} \xleftrightarrow{F} \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_j - 2\pi l)$$

$$\begin{aligned}
&\left[\phi_{\omega_k}(n)\right]\cdot\left[\phi_{\omega_j}(n)\right] \\
&= 2\pi\int_{2\pi}\left[\sum_{m=-\infty}^{\infty}\delta(\omega-\omega_k-2\pi m)\right]\left[\sum_{l=-\infty}^{\infty}\delta(\omega-\omega_j-2\pi l)\right]d\omega \\
&= 0,\quad \omega_k \neq \omega_j \\
&\neq 1,\quad \omega_k = \omega_j
\end{aligned}$$

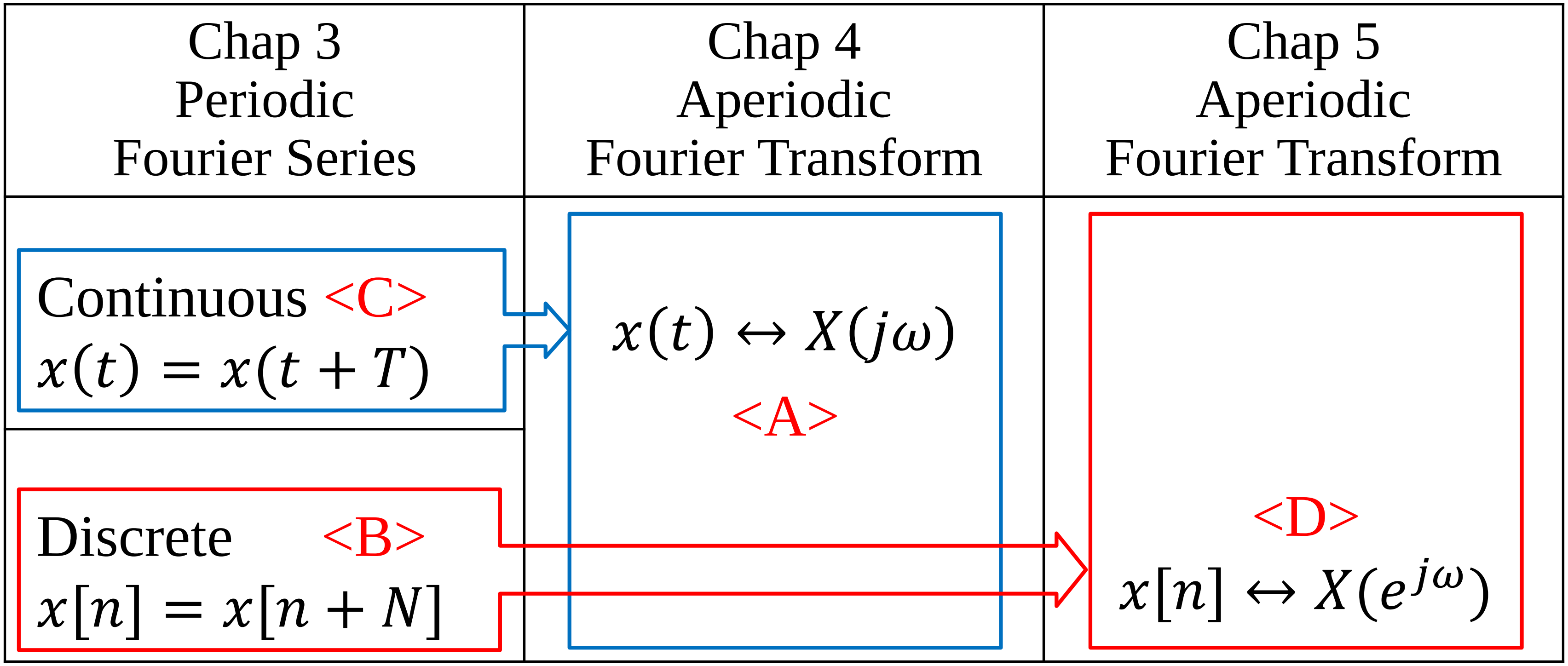
- Vector Space Interpretation
  - Orthogonal Bases

    Similar to the case of continuous-time Fourier transform. Orthogonal bases but not normalized, while makes sense with operational definition.

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\omega}\right)e^{j\omega n}d\omega$$

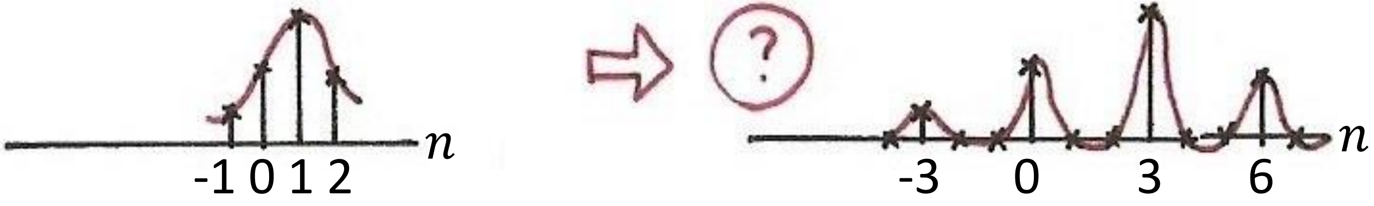
$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = (x[n])\cdot(\phi_{\omega}[n])$$

Summary and Duality (p.1 of 5.0)



Chap 3 Periodic Fourier Series	Chap 4 Aperiodic Fourier Transform	Chap 5 Aperiodic Fourier Transform
Continuous <C> $x(t) = x(t + T)$	$x(t) \leftrightarrow X(j\omega)$ <A>	<D> $x[n] \leftrightarrow X(e^{j\omega})$
Discrete <B> $x[n] = x[n + N]$		

Time Expansion (p.41 of 5.0)



Discrete-time

$x[n]$

F (chap5)

$$\boxed{\sum_{n} x[n]\, e^{-j\omega n}}$$

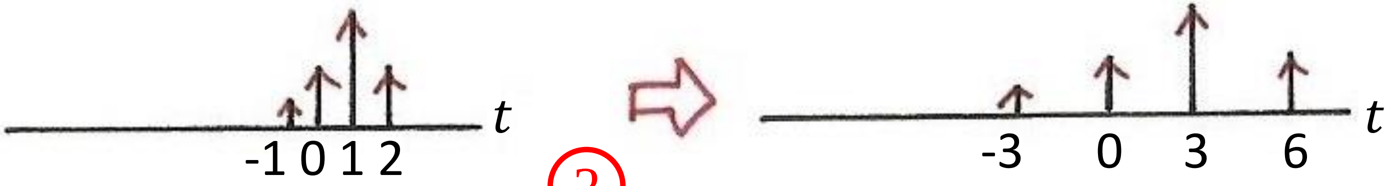
Continuous-time

$$x(t) = \sum_{n} x[n]\,\delta(t-n)$$

F (chap4)

$$\int_{-\infty}^{\infty}\{\sum_{n} x[n]\,\delta\,(t-n)\}e^{-j\omega t}dt$$

$$= \boxed{\sum_{n} x[n]\, e^{-j\omega n}}$$



?

$$x(a\,t) \overset{F}{\leftrightarrow} \frac{1}{|a|}X\,(\frac{j\omega}{a})$$

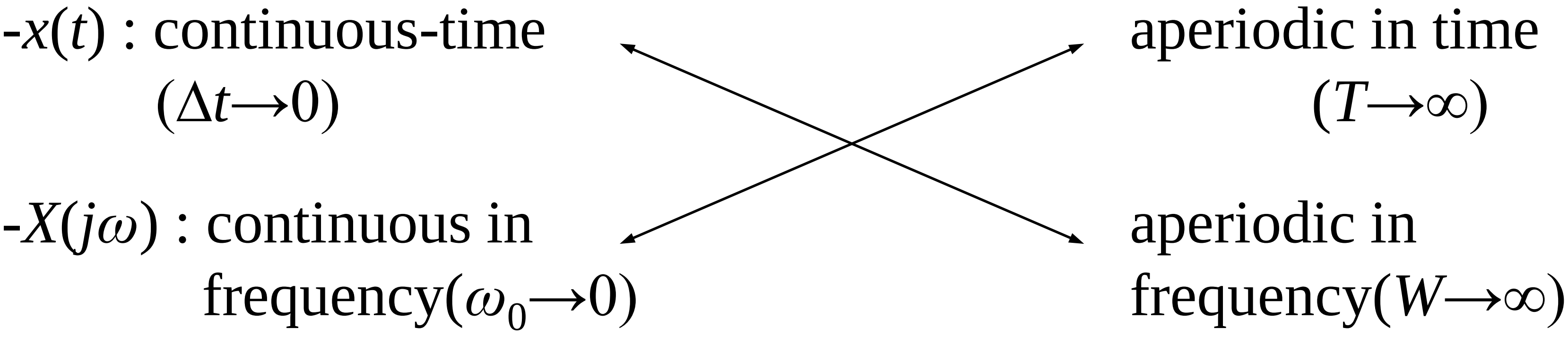
$a = \frac{1}{k}$, k=integer

5.3 Summary and Duality

<A> Fourier Transform for Continuous-time Aperiodic Signals

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \qquad \text{(Synthesis)} \qquad (4.8)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad \text{(Analysis)} \qquad (4.9)$$



-$x(t)$: continuous-time ($\Delta t \to 0$)

-$X(j\omega)$: continuous in frequency($\omega_0 \to 0$)

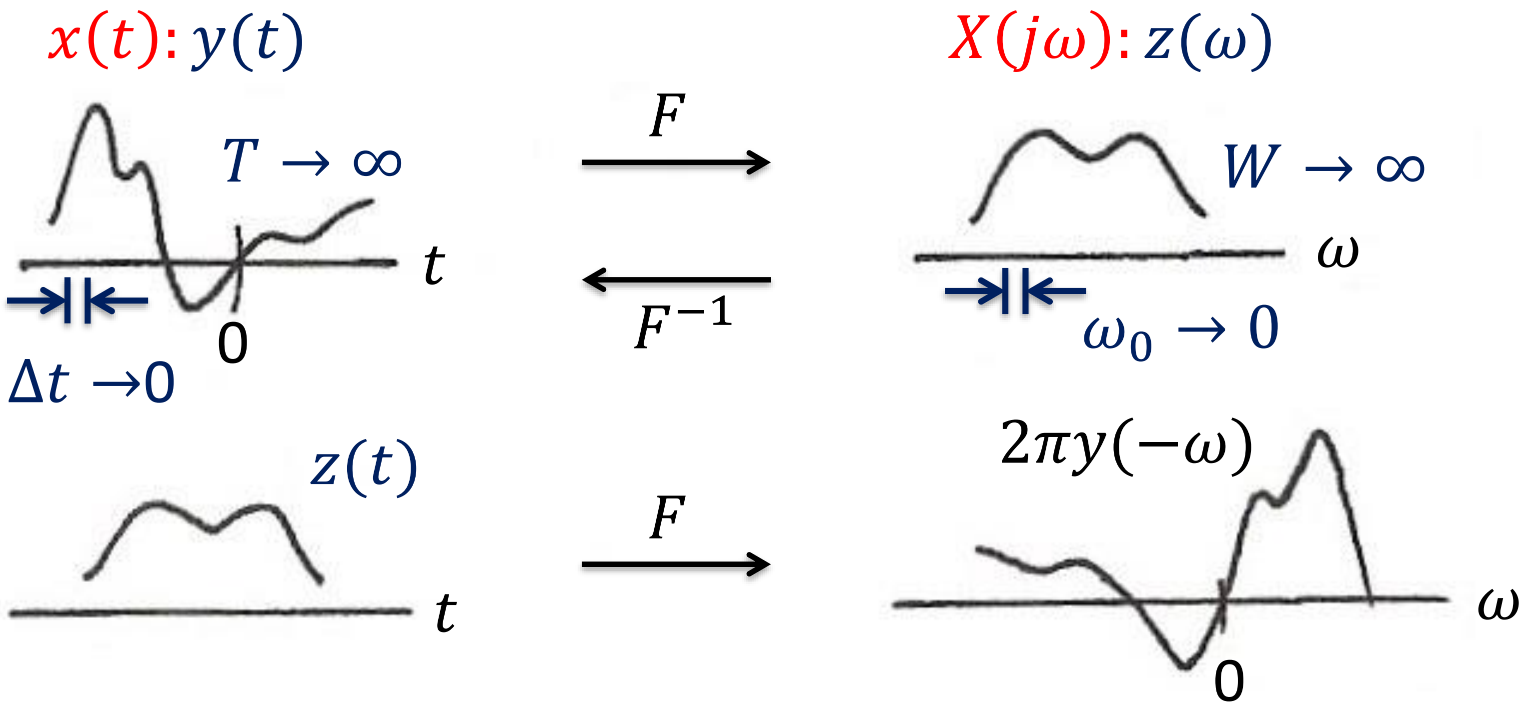
aperiodic in time ($T \to \infty$)

aperiodic in frequency($W \to \infty$)

Duality<A> : $x(t) \xleftrightarrow{F} X(j\omega) \Rightarrow y(t) \xleftrightarrow{F} z(\omega)$

$$z(t) \xleftrightarrow{F} 2\pi y(-\omega)$$

Case <A> (p.44 of 4.0)



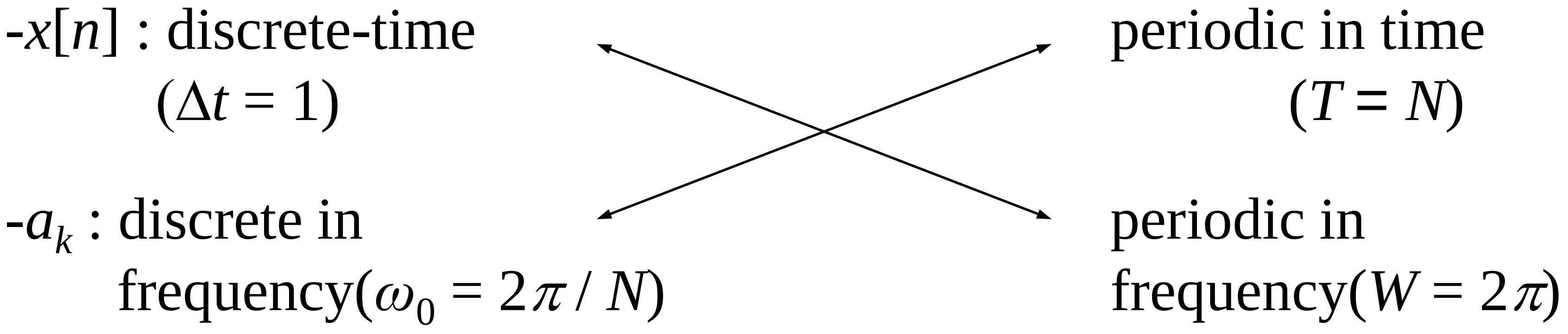
$$x(t) \xleftrightarrow{F} X(j\omega) \Rightarrow y(t) \xleftrightarrow{F} z(\omega)$$

$$z(t) \xleftrightarrow{F} 2\pi y(-\omega)$$

<B> Fourier Series for Discrete-time Periodic Signals

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}, \quad \omega_0 = \frac{2\pi}{N} \qquad \text{(Synthesis)} \qquad (3.94)$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n}, \quad \omega_0 = \frac{2\pi}{N} \qquad \text{(Analysis)} \qquad (3.95)$$



-$x[n]$: discrete-time
($\Delta t = 1$)

periodic in time
($T = N$)

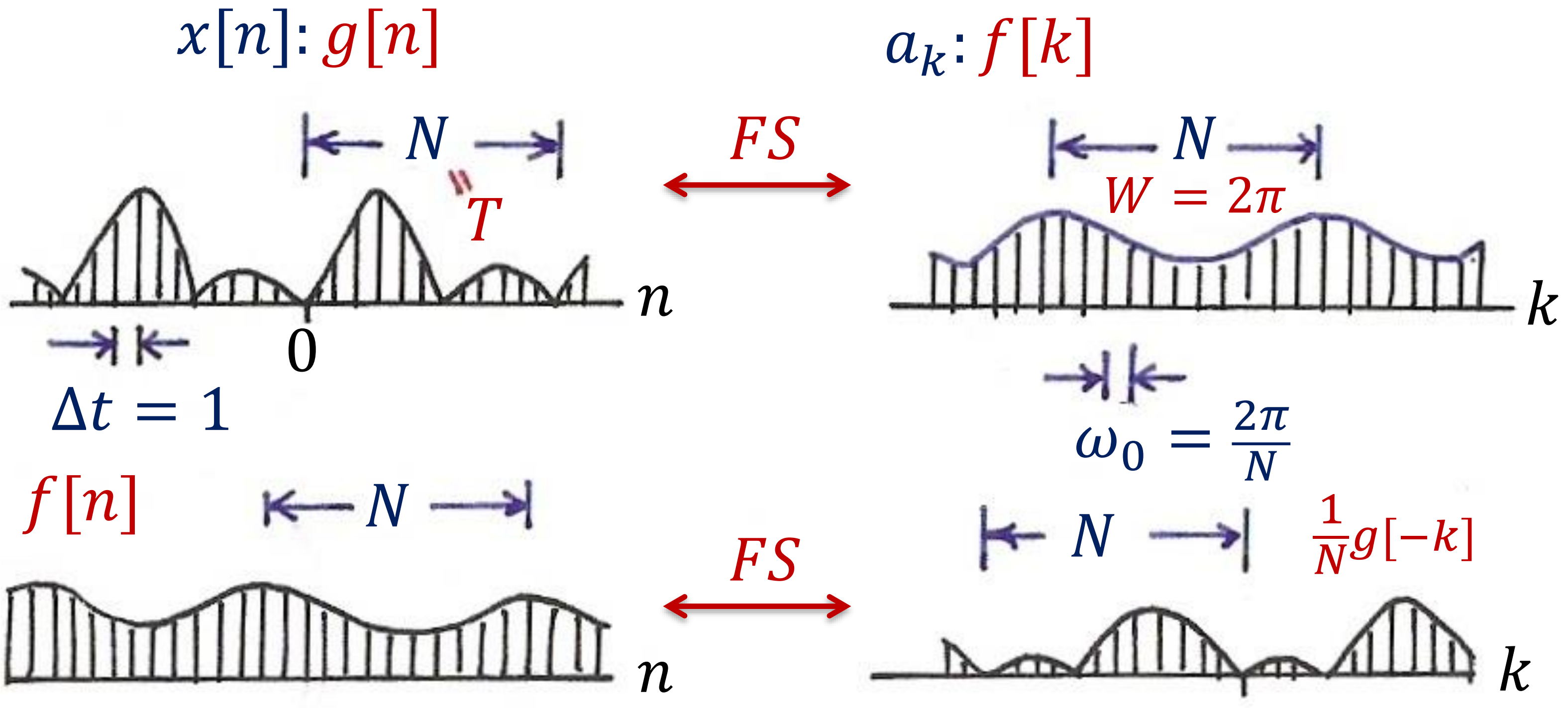
-a_k : discrete in
frequency($\omega_0 = 2\pi / N$)

periodic in
frequency($W = 2\pi$)

$$\text{Duality<B>} : x[n] \xleftrightarrow{FS} a_k \Rightarrow g[n] \xleftrightarrow{FS} f[k]$$

$$f[n] \xleftrightarrow{FS} \frac{1}{N} g[-k]$$

Case <B> Duality



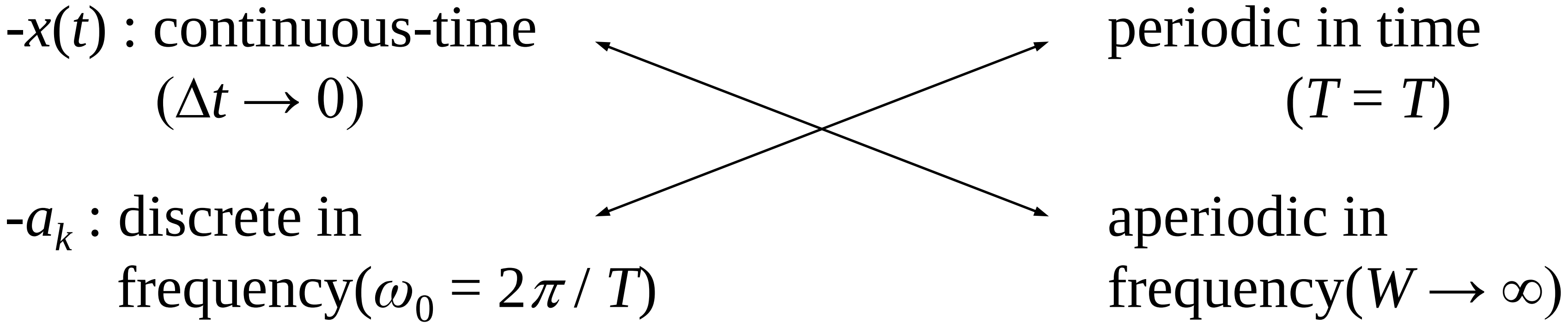
$$x[n] \xleftrightarrow{FS} a_k \Rightarrow g[n] \xleftrightarrow{FS} f[k]$$

$$f[n] \xleftrightarrow{FS} \frac{1}{N} g[-k]$$

<C> Fourier Series for Continuous-time Periodic Signals

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T} \qquad \text{(Synthesis)} \qquad (3.38)$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt, \quad \omega_0 = \frac{2\pi}{T} \qquad \text{(Analysis)} \qquad (3.39)$$



-$x(t)$: continuous-time
($\Delta t \rightarrow 0$)

periodic in time
($T = T$)

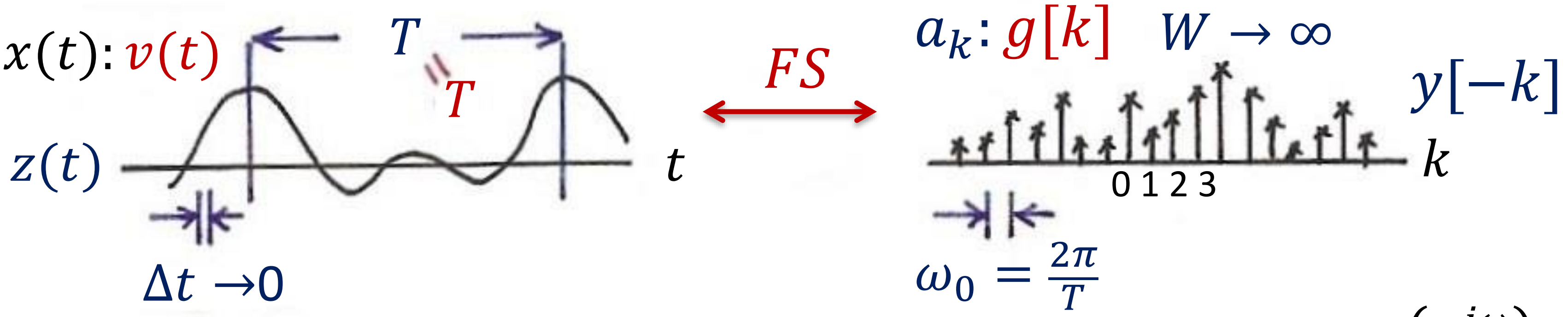
-a_k : discrete in
frequency($\omega_0 = 2\pi / T$)

aperiodic in
frequency($W \rightarrow \infty$)

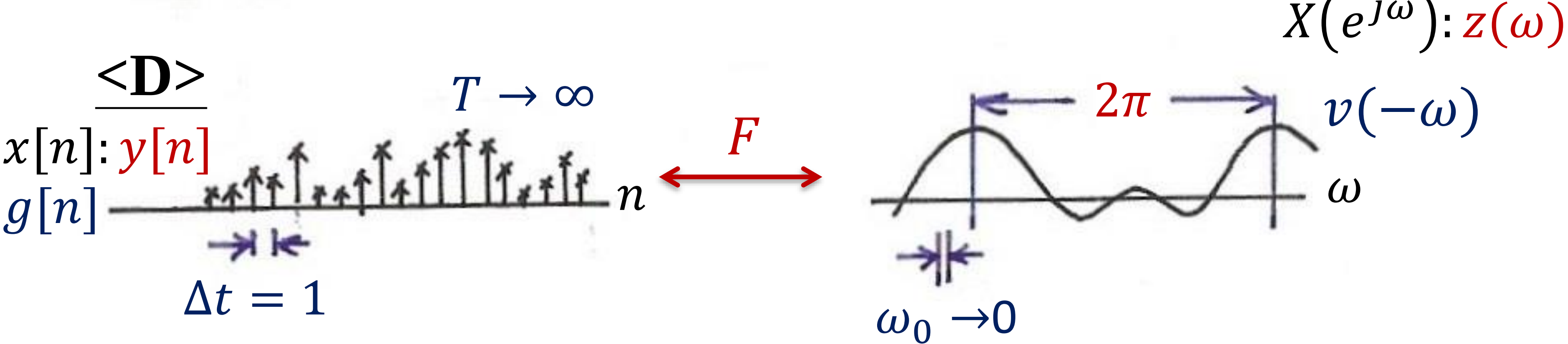
$$x(t) \xleftrightarrow{FS} a_k \Rightarrow v(t) \xleftrightarrow{FS} g[k]$$

Case <C> <D> Duality

<C>



<D>



For <C> $x(t) \xleftrightarrow{FS} a_k \Rightarrow v(t) \xleftrightarrow{FS} g[k]$

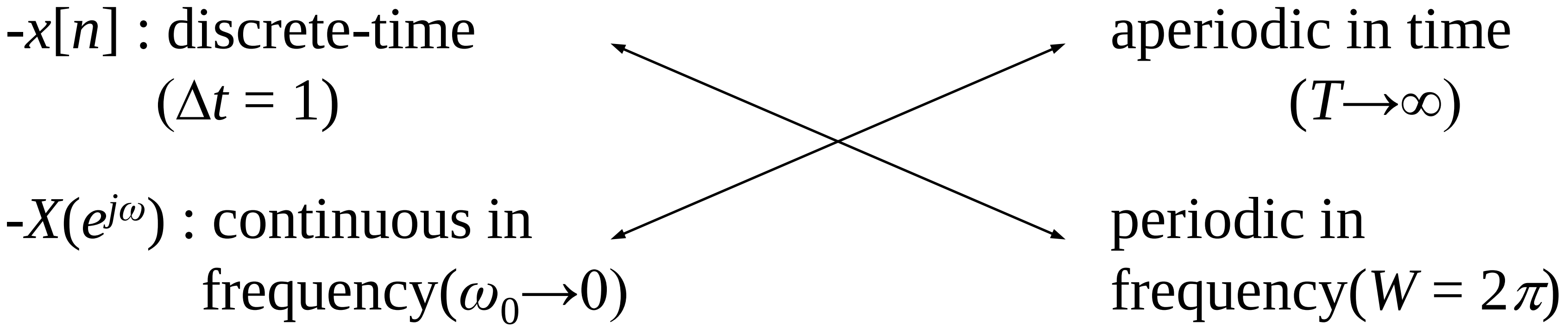
For <D> $x[n] \xleftrightarrow{F} X(e^{j\omega}) \Rightarrow y[n] \xleftrightarrow{F} z(\omega)$

Duality $z(t) \xleftrightarrow{FS} y[-k],\ g[n] \xleftrightarrow{F} v(-\omega)$

<D> Discrete-time Fourier Transform for Discrete-time Aperiodic Signals

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\omega}\right)e^{j\omega n}d\omega$$ (Synthesis) (5.8)

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$ (Analysis) (5.9)



-$x[n]$: discrete-time ($\Delta t = 1$)

aperiodic in time ($T \to \infty$)

-$X(e^{j\omega})$: continuous in frequency($\omega_0 \to 0$)

periodic in frequency($W = 2\pi$)

$$x[n] \xleftrightarrow{F} X\left(e^{j\omega}\right) \Rightarrow y[n] \xleftrightarrow{F} z(\omega)$$

Duality<C> / <D>

For <C> $x(t) \xleftrightarrow{FS} a_k \quad \Rightarrow \quad v(t) \xleftrightarrow{FS} g[k]$

For <D> $x[n] \xleftrightarrow{F} X\left(e^{j\omega}\right) \Rightarrow \quad y[n] \xleftrightarrow{F} z(\omega)$

Duality $z(t) \xleftrightarrow{FS} y[-k], \quad g[n] \xleftrightarrow{F} v(-\omega)$

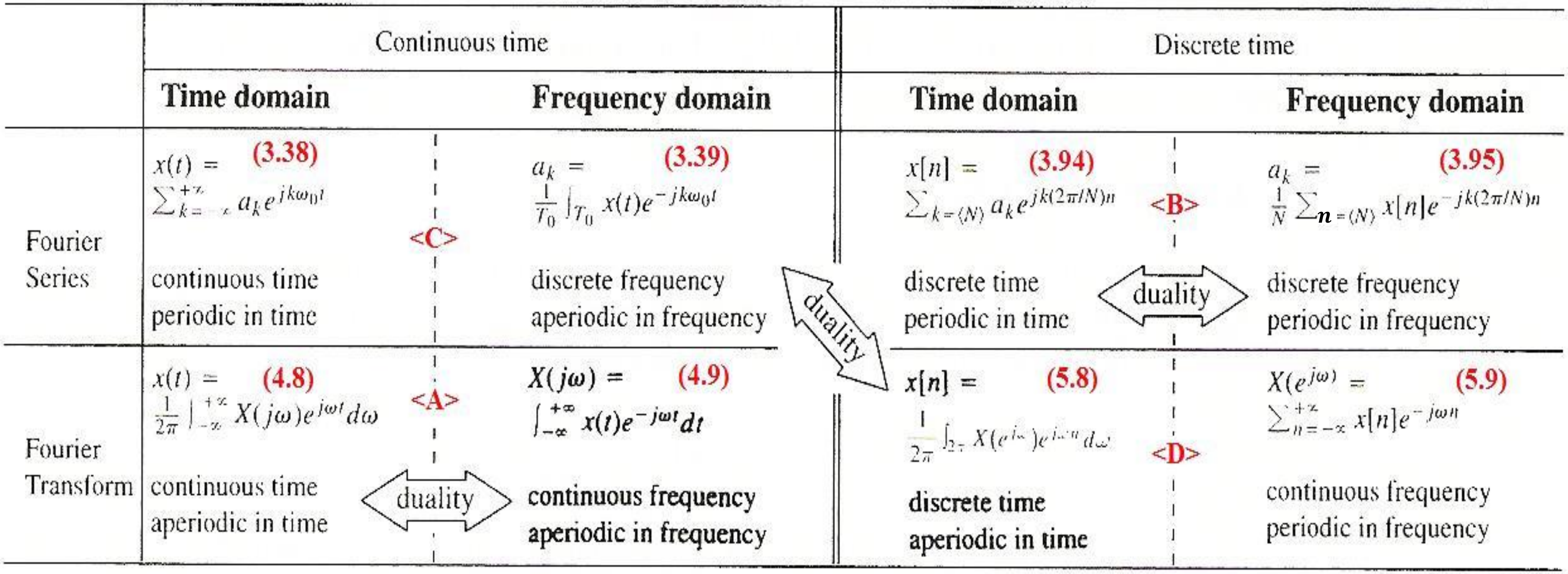
- taking $z(t)$ as a periodic signal in time with period 2π, substituting into (3.38), $\omega_0 = 1$

  $$z(t) = \sum_{n=-\infty}^{\infty} a_k e^{jkt}$$

  which is of exactly the same form of (5.9) except for a sign change, (3.39) indicates how the coefficients a_k are obtained, which is of exactly the same form of (5.8) except for a sign change, etc.

See Table 5.3, p.396 of text

TABLE 5.3 SUMMARY OF FOURIER SERIES AND TRANSFORM EXPRESSIONS



	Continuous time				Discrete time		
	Time domain		**Frequency domain**		**Time domain**		**Frequency domain**
Fourier Series	$x(t) =$ **(3.38)** $\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ continuous time periodic in time	<C>	$a_k =$ **(3.39)** $\frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}$ discrete frequency aperiodic in frequency	duality	$x[n] =$ **(3.94)** $\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ discrete time periodic in time	<B> duality	$a_k =$ **(3.95)** $\frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$ discrete frequency periodic in frequency
Fourier Transform	$x(t) =$ **(4.8)** $\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$ continuous time aperiodic in time	<A> duality	$X(j\omega) =$ **(4.9)** $\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$ continuous frequency aperiodic in frequency		$x[n] =$ **(5.8)** $\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$ discrete time aperiodic in time	<D>	$X(e^{j\omega}) =$ **(5.9)** $\sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$ continuous frequency periodic in frequency

- More Duality
  - Discrete in one domain with Δ between two values

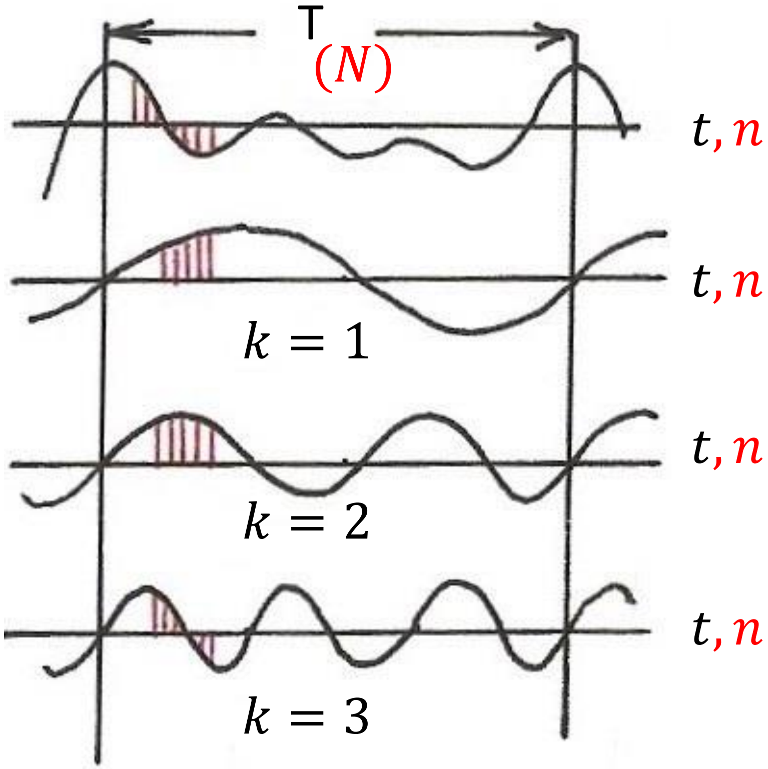
    → periodic in the other domain with period $\frac{2\pi}{\Delta}$

  - Continuous in one domain ($\Delta \to 0$)

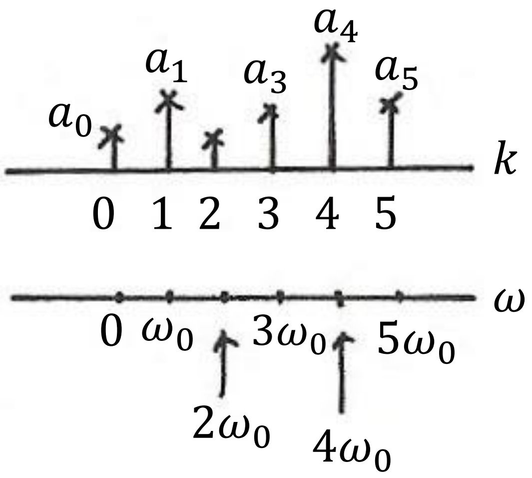
    → aperiodic in the other domain $\frac{2\pi}{\Delta} \to \infty$

Harmonically Related Exponentials for Periodic Signals (P.11 of 3.0)

$V = \{x(t)[n] \mid x(t)[n]$ periodic, fundamental period $= T(N)\}$



$\omega_0 = \frac{2\pi}{T(N)}$



- All with period T (N): integer multiples of ω_0
- Discrete in frequency domain

- Extra Properties Derived from Duality
  - examples for Duality <B>

$$x[n-n_0] \xleftrightarrow{FS} a_k e^{-jk\left(\frac{2\pi}{N}\right)n_0}$$

$$e^{jm\left(\frac{2\pi}{N}\right)n} x[n] \xleftrightarrow{FS} a_{k-m}$$

duality

$$x[n]y[n] \xleftrightarrow{FS} \sum_{l=<N>} a_l b_{k-l}$$

$$\sum_{l=<N>} x[r]y[n-r] \xleftrightarrow{FS} N a_k b_k$$

duality

Unified Framework

- Fourier Transform : case <A>

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \qquad (4.8)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad (4.9)$$

Unified Framework

- Discrete frequency components for signals periodic in time domain: case <C>

$$e^{j\omega_0 t} \xleftrightarrow{F} 2\pi\delta(\omega-\omega_0)$$

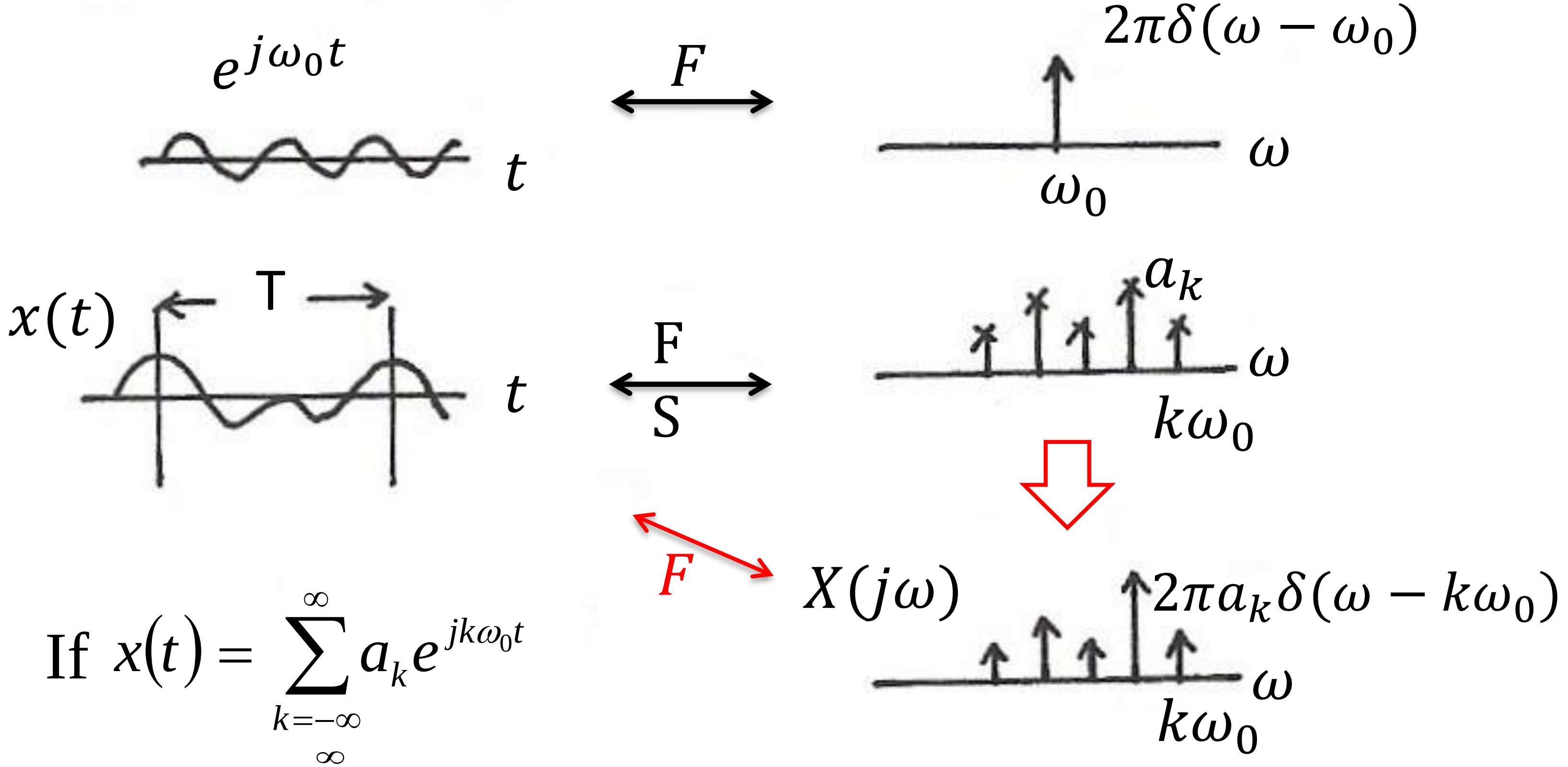
$$x(t) \xleftrightarrow{FS} a_k$$

$$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \xleftrightarrow{F} X(j\omega)=2\pi\sum_{k=-\infty}^{\infty} a_k\delta(\omega-k\omega_0)$$

you get (3.38) (applied on (4.8))

Case <C> is a special case of Case <A>

Unified Framework: Fourier Transform for Periodic Signals (p.17 of 4.0)



– If $x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$$X(j\omega)=\sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega-k\omega_0)$$

$$\begin{tikzcd} x(t) \arrow[r, leftrightarrow, "FS"] \arrow[rr, leftrightarrow, bend right, "F"'] & a_k \arrow[r] & X(j\omega) \end{tikzcd}$$

Unified Framework

- Discrete time values with spectra periodic in frequency domain: case <D>

$$\delta(t - t_0) \xleftrightarrow{F} e^{-j\omega t_0}$$

$$x[n] \xleftrightarrow{F} X\left(e^{j\omega}\right)$$

$$x[n] \to x(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - n)$$

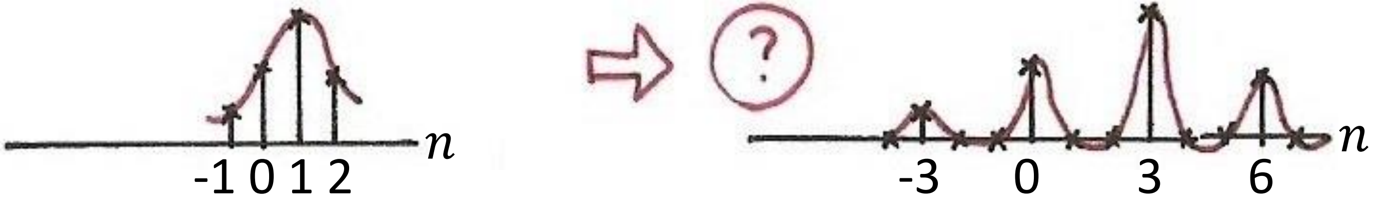
(4.9) becomes

$$X(j\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad (5.9)$$

Case <D> is a special case of Case <A>

Note : ω in rad/sec for continuous-time but in rad for discrete-time

Time Expansion (p.41 of 5.0)



Discrete-time

$x[n]$

F (chap5)

$$\boxed{\sum_{n} x[n]\, e^{-j\omega n}}$$

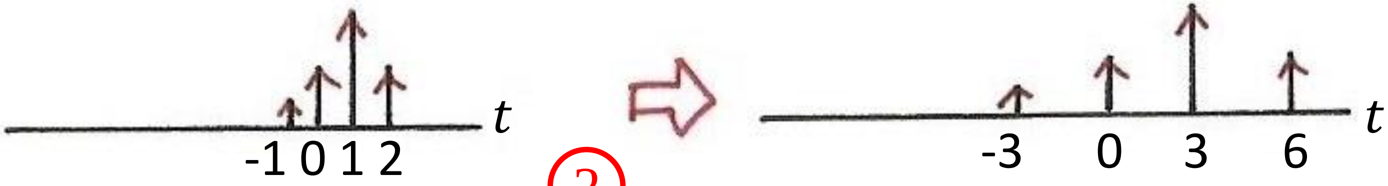
Continuous-time

$$x(t) = \sum_{n} x[n]\,\delta(t-n)$$

F (chap4)

$$\int_{-\infty}^{\infty} \{\sum_{n} x[n]\,\delta\,(t-n)\} e^{-j\omega t} dt$$

$$= \boxed{\sum_{n} x[n]\, e^{-j\omega n}}$$



?

$$x(a t) \overset{F}{\leftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

$a = \frac{1}{k}$, k=integer

Unified Framework

- Both discrete/periodic in time/frequency domain: case <B> -- case <C> plus case <D>

periodic and discrete, summation over a period of *N*

$$x(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t-n)$$

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

(4.8) becomes

$$x[n] = \sum_{n=<N>} a_k e^{jk\omega_0 n}$$

(3.94)

(4.9) becomes

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n}$$

(3.95)

Unified Framework

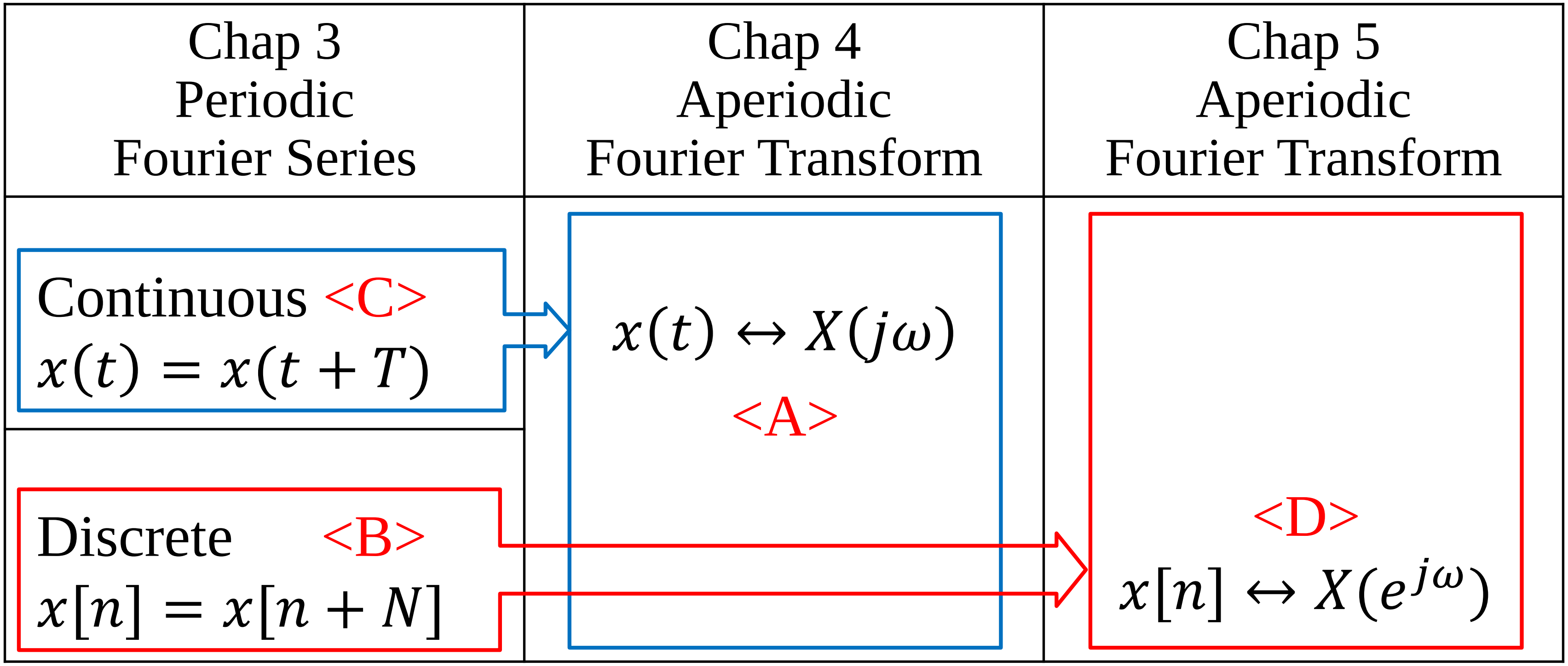
- Cases <B> <C> <D> are special cases of case <A>

  Dualities <B>, <C>/<D> are special case of Duality <A>

- Vector Space Interpretation

  ----similarly unified

Summary and Duality (p.1 of 5.0)



Chap 3 Periodic Fourier Series	Chap 4 Aperiodic Fourier Transform	Chap 5 Aperiodic Fourier Transform
Continuous <C> $x(t) = x(t+T)$	$x(t) \leftrightarrow X(j\omega)$ <A>	<D> $x[n] \leftrightarrow X(e^{j\omega})$
Discrete <B> $x[n] = x[n+N]$		

An Example across Cases <A><B><C><D>

<A> $x(at) \overset{F}{\leftrightarrow} \frac{1}{\enclose{circle}{|a|}} X(\frac{j\omega}{a})$ (4.34)

<D> $x_{(k)}[n] \overset{F}{\leftrightarrow} (\quad)X(e^{jk\omega})$ (5.45)

<C> $x(\alpha t) \overset{FS}{\leftrightarrow} (\quad)a_k$ (Sec. 3.5.4)

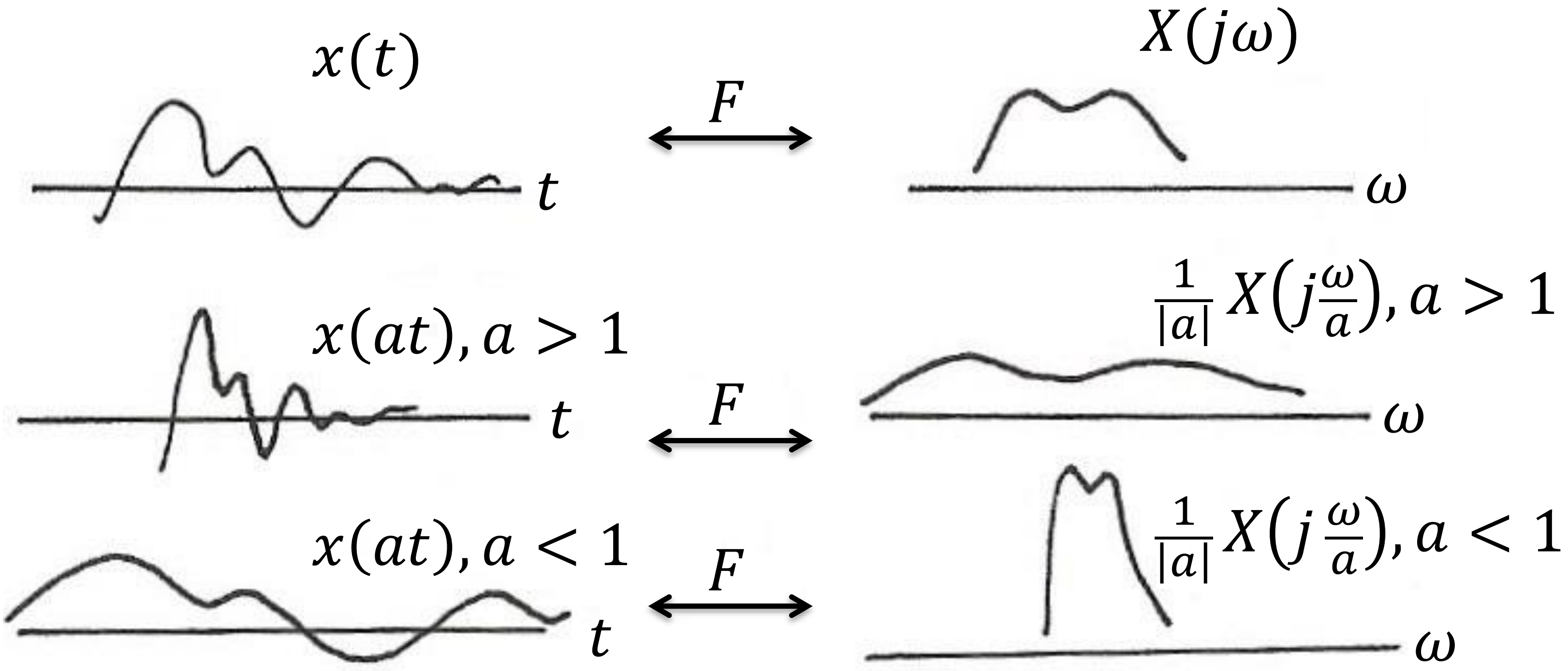
$(T' = T/\alpha, \omega_0' = \alpha\omega_0)$ (Table 3.1)

<B> $x_{(m)}[n] \overset{FS}{\leftrightarrow} \frac{1}{\enclose{circle}{m}} a_k$ (Table 3.2)

- Time/Frequency Scaling (p.38 of 4.0)

$$x(at) \xleftrightarrow{F} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

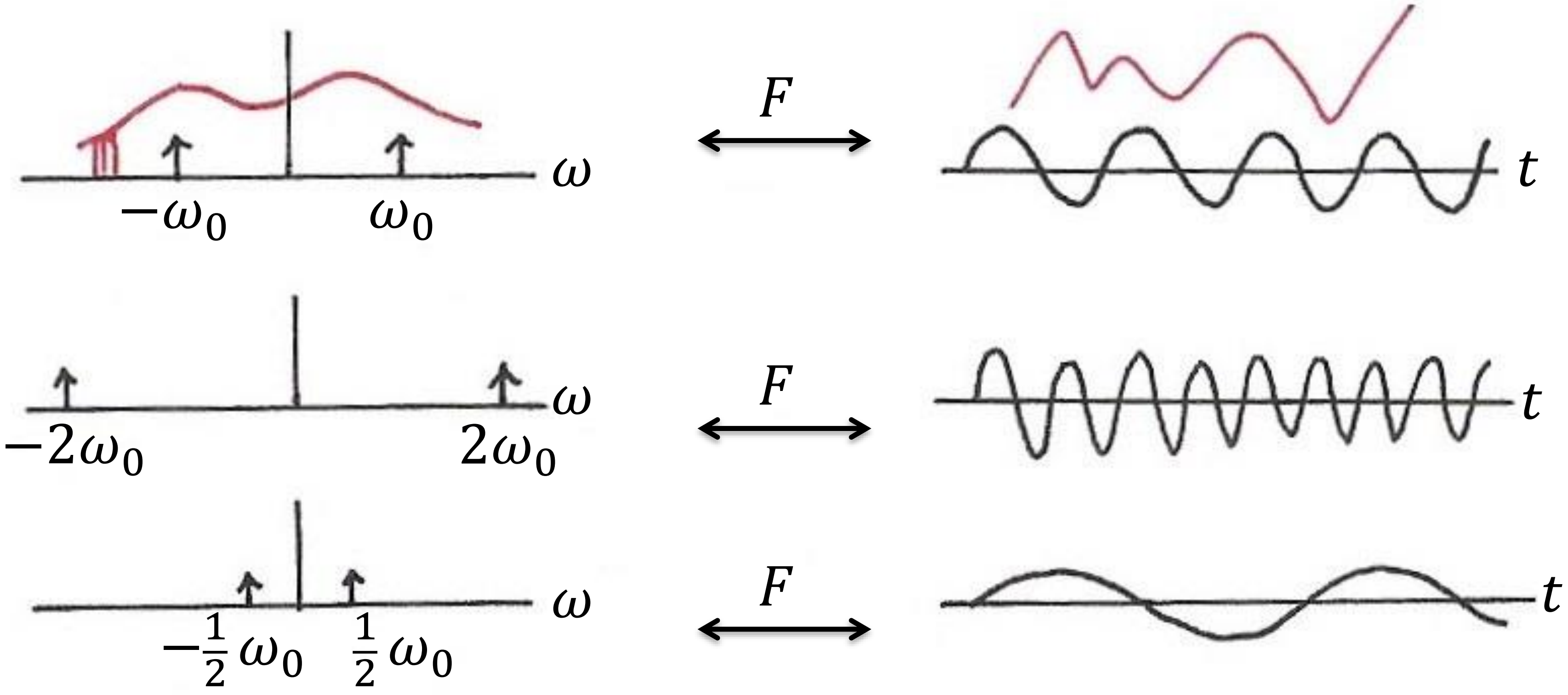
$$x(-t) \xleftrightarrow{F} X(-j\omega) \quad \text{(time reversal)}$$



See Fig. 4.11, p.296 of text

Single Frequency (p.40 of 4.0)

$$\cos\omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t}+e^{-j\omega_0 t}\right)$$



- Parseval's Relation (p.37 of 4.0)

$$\int_{-\infty}^{\infty} | x(t) |^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} | X(j\omega) |^2 d\omega$$

total energy: energy per unit time integrated over the time

total energy: energy per unit frequency integrated over the frequency

$$\vec{A} = \sum_i a_i \, \hat{v}_i = \sum_k b_k \, \hat{u}_k$$

$$\| \vec{A} \|^2 = \sum_i |a_i|^2 = \sum_k |b_k|^2$$

Single Frequency

<A> $x(at) \overset{F}{\longleftrightarrow} \frac{1}{|a|}X(\frac{j\omega}{a})$ (4.34)

$$x(t) = \cos\omega_0 t \overset{F}{\longleftrightarrow} X(j\omega) = \pi\delta(\omega - \omega_0) + \cdots$$

$$x(at) = \cos(a\omega_0 t) \overset{F}{\longleftrightarrow} \enclose{circle}{\frac{1}{a}}\,\pi\,\enclose{circle}{\delta\left(\frac{\omega}{a} - \omega_0\right)} + \cdots$$

$$a \cdot \boxed{\int_{-\infty}^{\infty}\delta\left(\frac{\omega}{a}\right)\, d\left(\frac{\omega}{a}\right)} = a$$

$$= 1$$

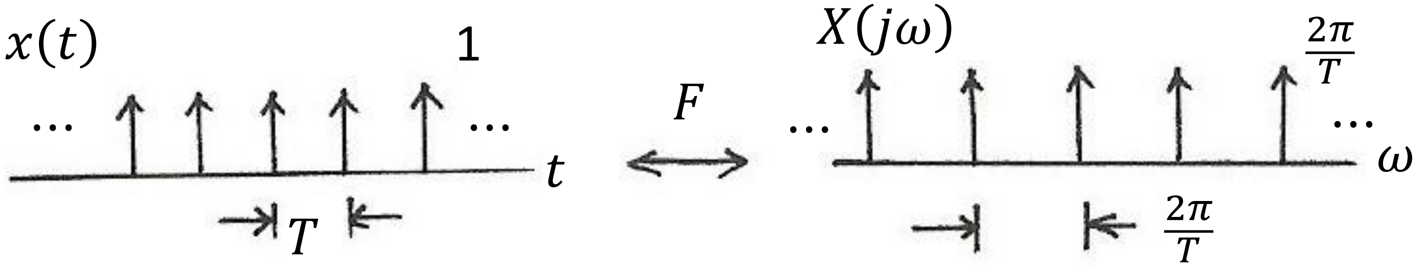
$$\Rightarrow$$

$$\delta\left(\frac{\omega}{a}\right) = a\delta(\omega)$$

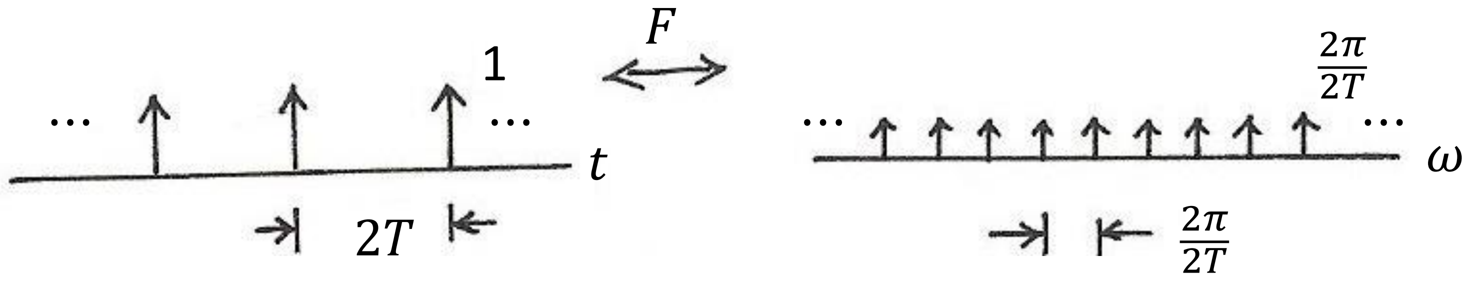
$$\delta\left(\frac{\omega}{a} - \omega_0\right) = \enclose{circle}{a}\,\delta(\omega - a\omega_0)$$

$$\Rightarrow\ x(at) = \cos(a\omega_0 t) \overset{F}{\longleftrightarrow} \frac{1}{a}X\left(\frac{j\omega}{a}\right) = \pi\delta(\omega - a\omega_0) + \cdots$$

Another Example



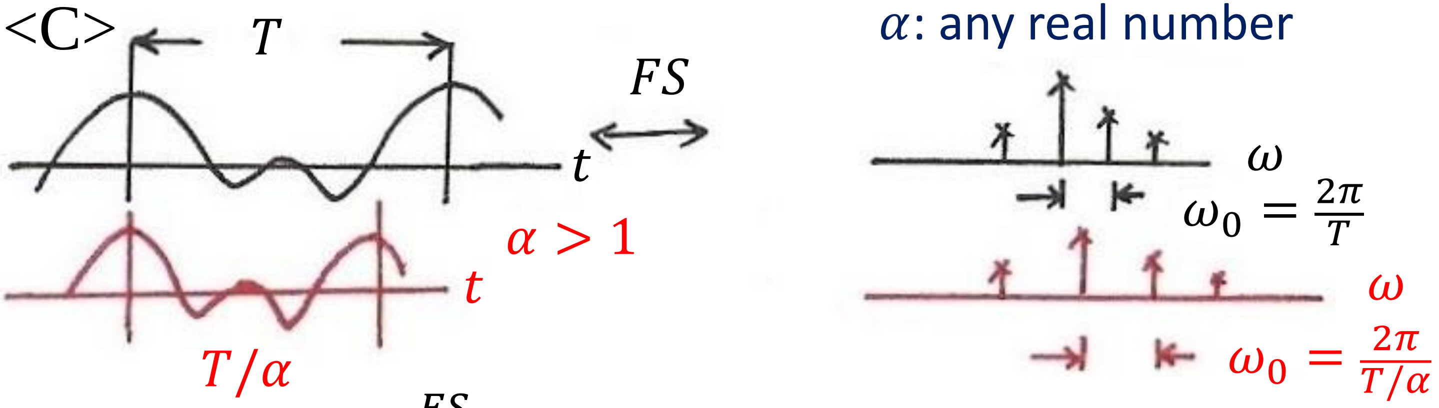
See Figure 4.14 (example 4.8), p.300 of text



Cases <C><D>

<C>

α: any real number

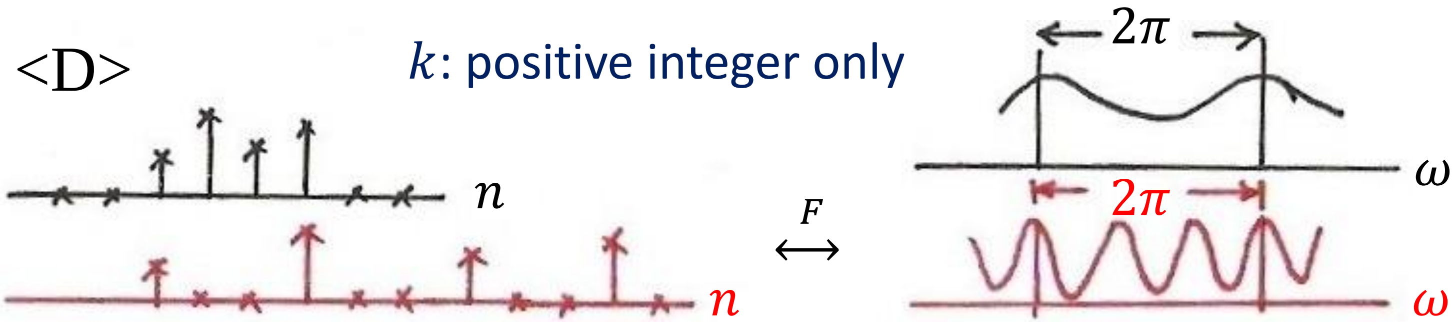


$$x(\alpha t) \overset{FS}{\longleftrightarrow} (\quad) a_k \ (T' = \mathrm{T}/\alpha\,, \omega_0' = \alpha\omega_0)$$

⇕ Duality

<D>

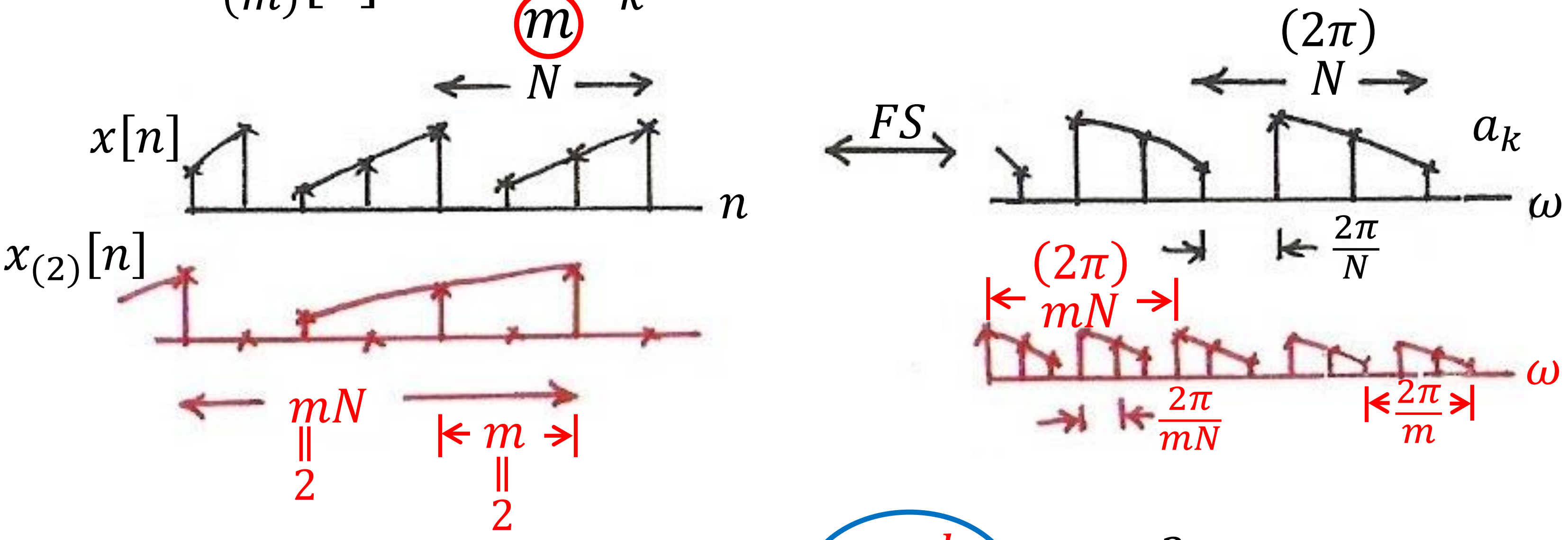
k: positive integer only



$$x_{(k)}[n] \overset{F}{\longleftrightarrow} (\quad) X(e^{jk\omega})$$

Cases <B>

$$x_{(m)}[n] \overset{FS}{\longleftrightarrow} \frac{1}{\enclose{circle}{m}} a_k$$



$$x_{(m)}[n] \overset{FS}{\longleftrightarrow} a'_k = \frac{1}{mN} \sum_{\substack{\cancel{n=<mN>} \\ l=<N>}} \enclose{circle}{x_{(m)}[\overset{lm}{\cancel{n}}]}\, e^{-jk\left(\frac{2\pi}{\cancel{m}N}\right)\cancel{n}} \; l\cancel{m}$$

$$= \frac{1}{m} \cdot \left[\frac{1}{N} \sum_{l=<N>} \enclose{circle}{x[l]}\, e^{-jk\left(\frac{2\pi}{N}\right)l}\right]$$

$$= \frac{1}{\enclose{circle}{m}} \cdot a_k$$

Examples

- Example 5.6, p.371 of text

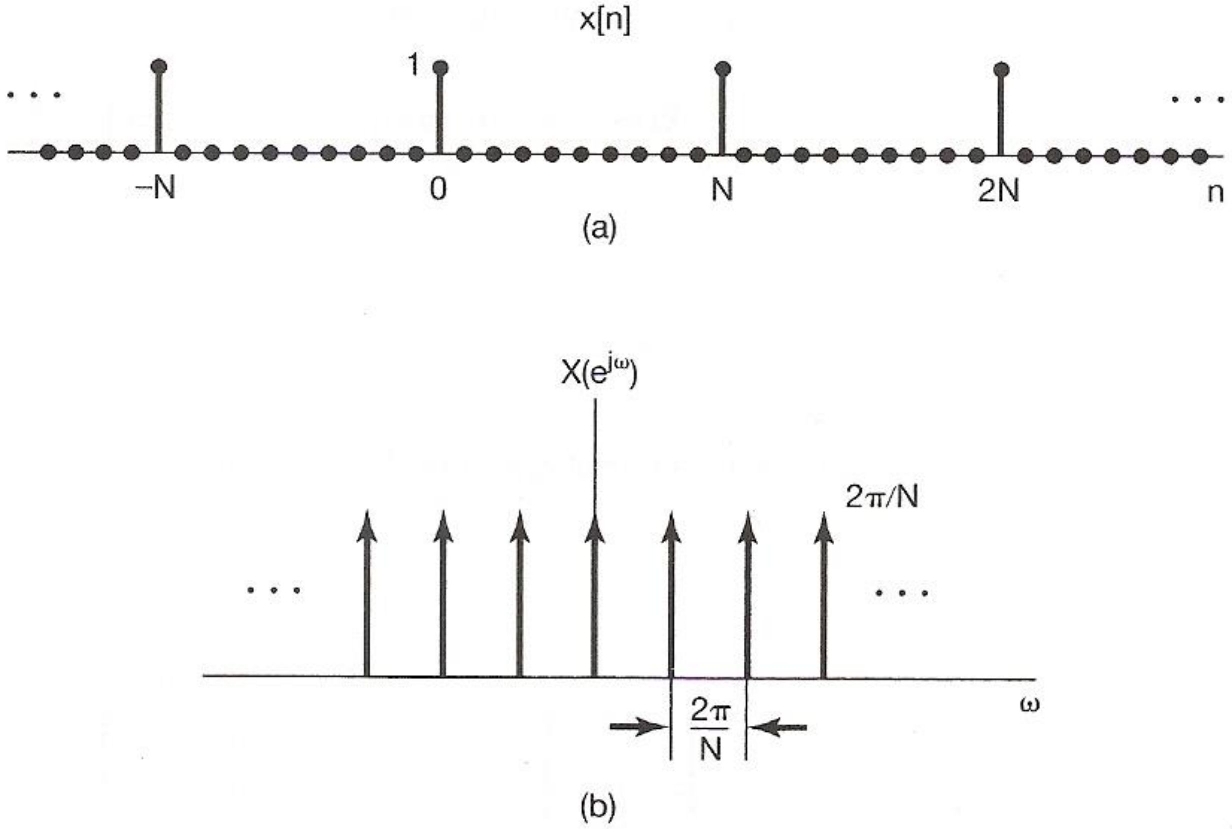


Figure 5.11 (a) Discrete-time periodic impulse train; (b) its Fourier transform.

Examples

- Example 4.8, p.299 of text (P.76 of 4.0)

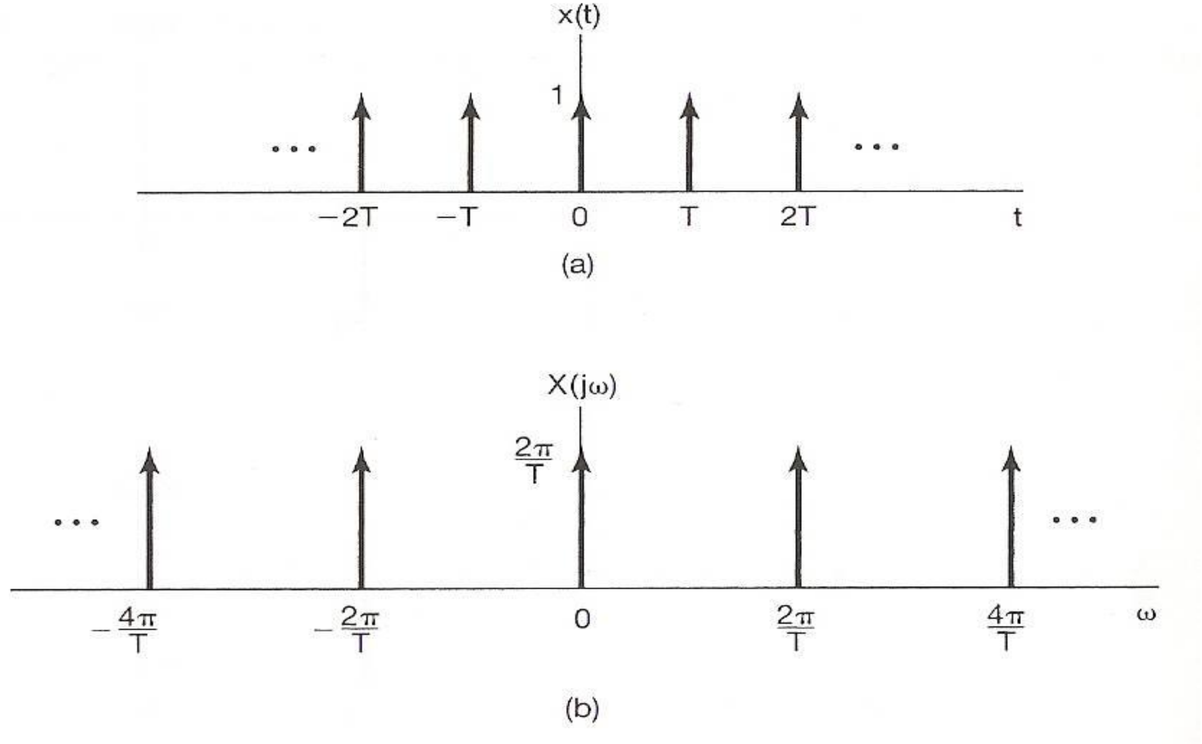


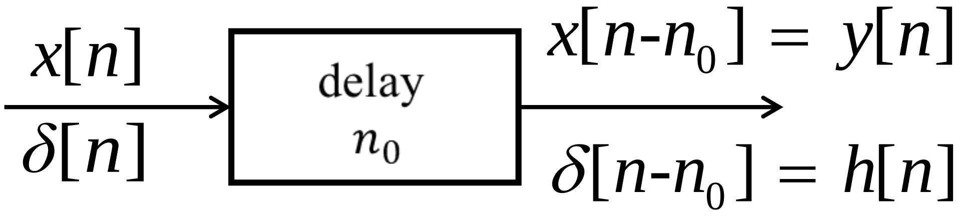
Figure 4.14 (a) Periodic impulse train; (b) its Fourier transform.

Discrete $(\Delta t = T)$ → Periodic $(W = \frac{2\pi}{T})$

Periodic $(T = T)$ → Discrete $(\omega_0 = \frac{2\pi}{T})$

Examples

- Example 5.11, p.383 of text



$x[n]$

$\delta[n]$

delay

n_0

$x[n\text{-}n_0] = y[n]$

$\delta[n\text{-}n_0] = h[n]$

$$H\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}\delta[n\text{-}n_0]\,e^{-j\omega n}=e^{-j\omega n_0}$$

$$Y\left(e^{j\omega}\right)=e^{-j\omega n_0}X\left(e^{j\omega}\right)$$

time shift property

Examples

- Example 5.14, p.387 of text

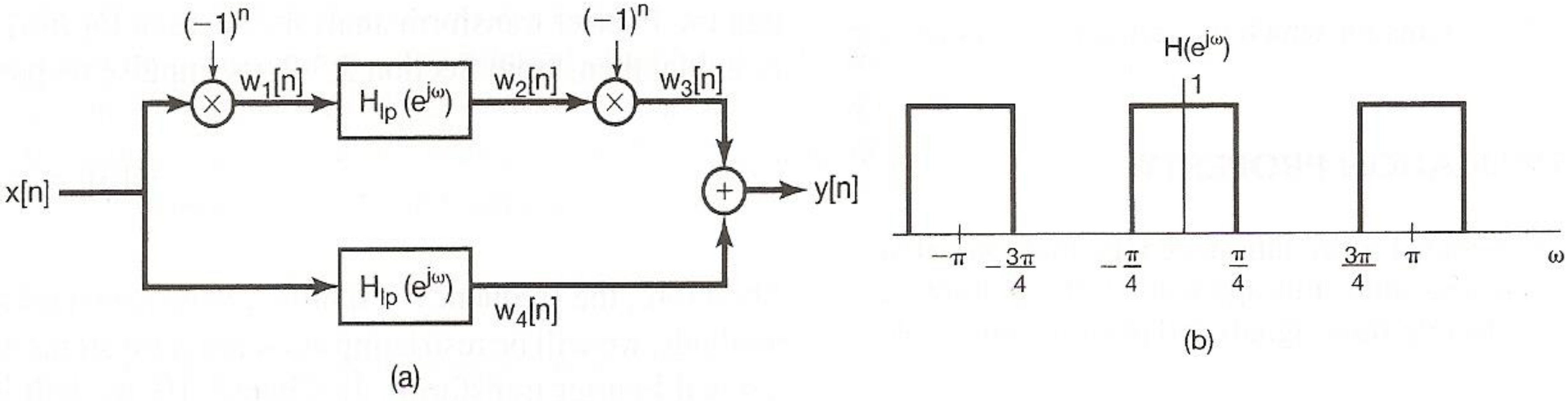


Figure 5.18 (a) System interconnection for Example 5.14; (b) the overall frequency response for this system.

$$w_1[n] = (-1)^n \mathrm{x}[n] = e^{j\pi n}\mathrm{x}[n]$$

$$W_1(e^{j\omega}) = X(e^{j(\omega-\pi)})$$

$$H(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)}) + H_{lp}(e^{j\omega})$$

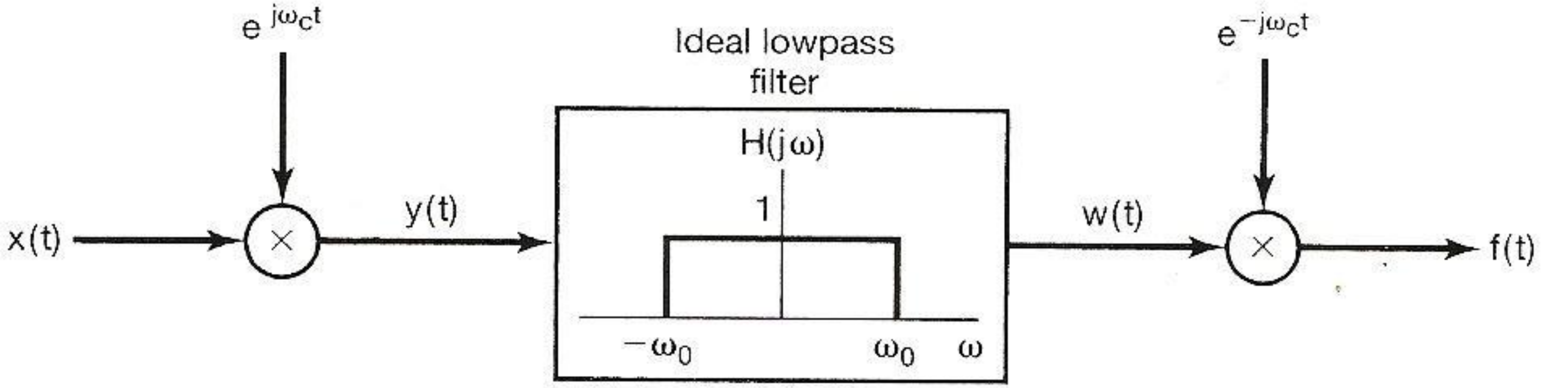


Figure 4.26 Implementation of a bandpass filter using amplitude modulation with a complex exponential carrier.

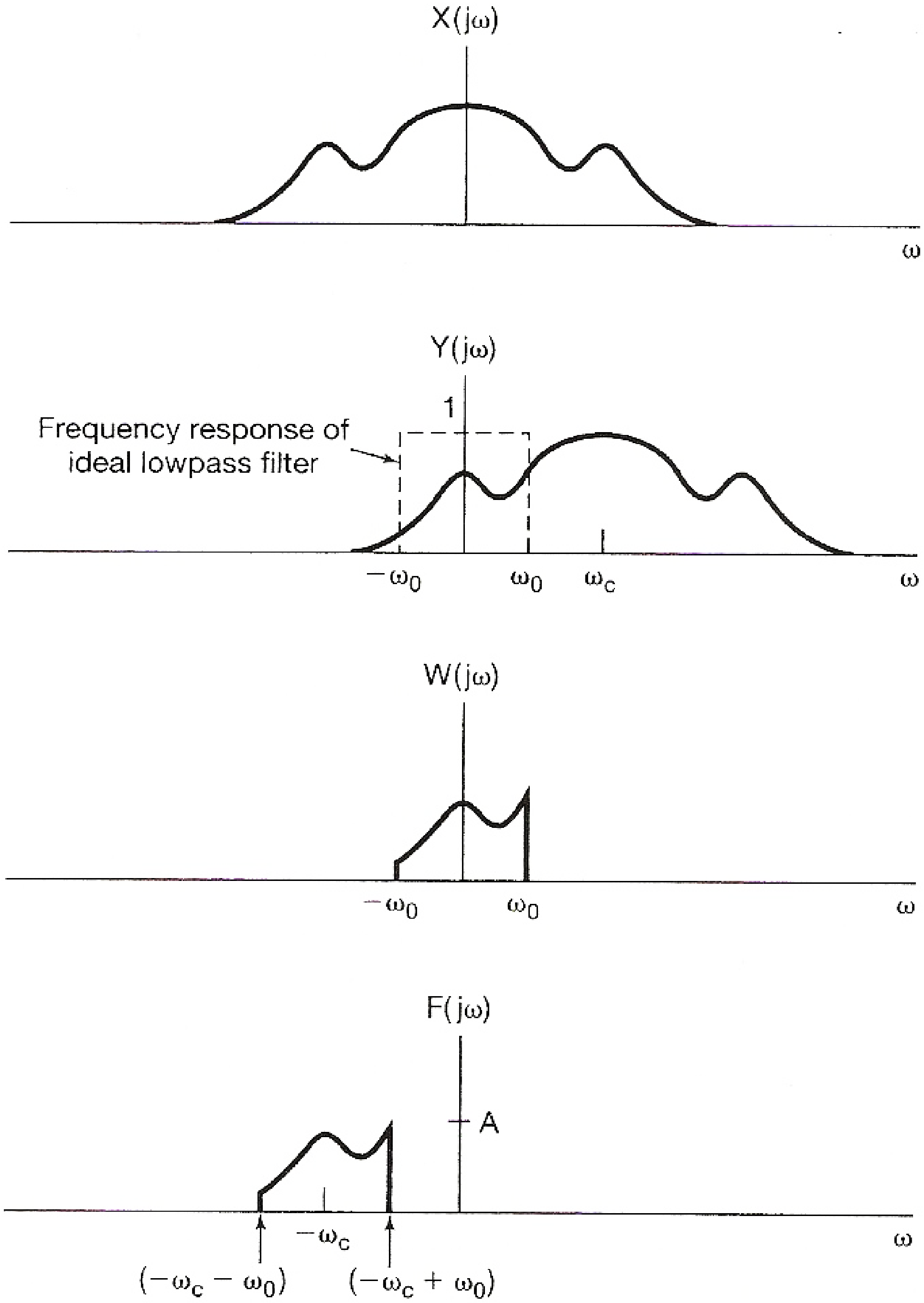


Figure 4.27 Spectra of the signals in the system of Figure 4.26.

Examples

- Example 5.17, p.395 of text

$$g(t) = \begin{cases} 1, & |t| \leq T_1 \\ 0, & T_1 \leq |t| \leq \pi, \text{ in } [-\pi, \pi] \end{cases}$$

$$\text{periodic with } T = 2\pi,\ \omega_0 = \frac{2\pi}{T} = 1$$

$$a_k = \frac{\sin(kT_1)}{k\pi},\ \text{(example 3.5 of text)}$$

$$x[n] = \frac{\sin(\pi n/2)}{\pi n},\ \text{by duality}$$

$$X(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \pi/2 \\ 0, & \pi/2 < \omega \leq \pi \end{cases}$$

Examples

- Example 3.5, p.193 of text (P. 58 of 3.0)

(a)

(b)

(c)

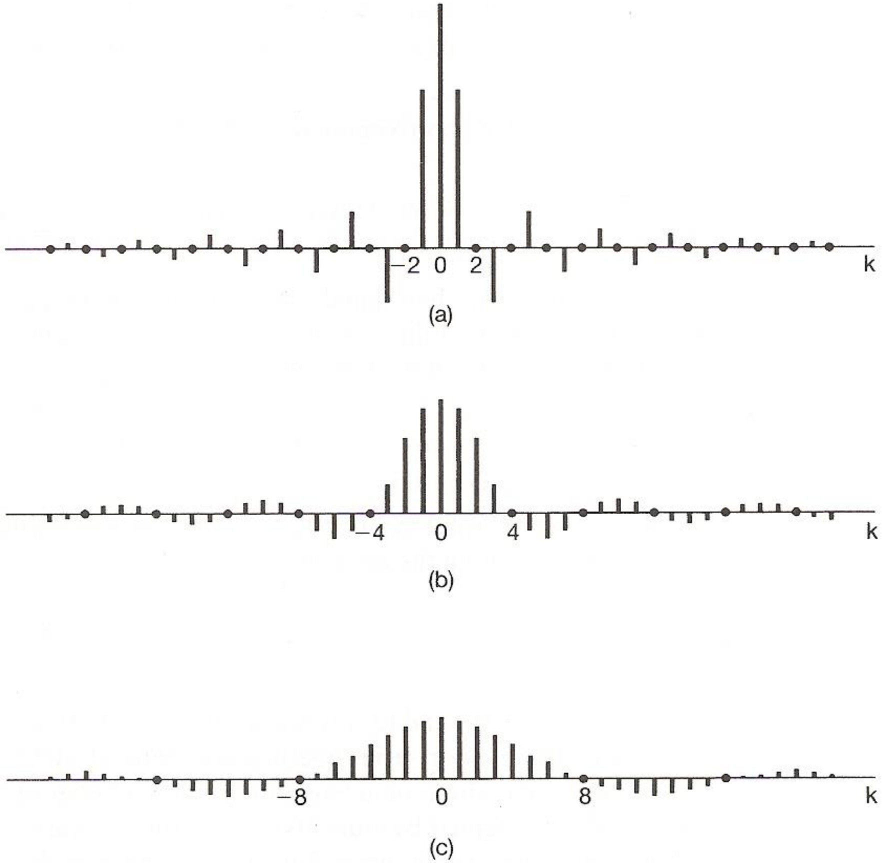
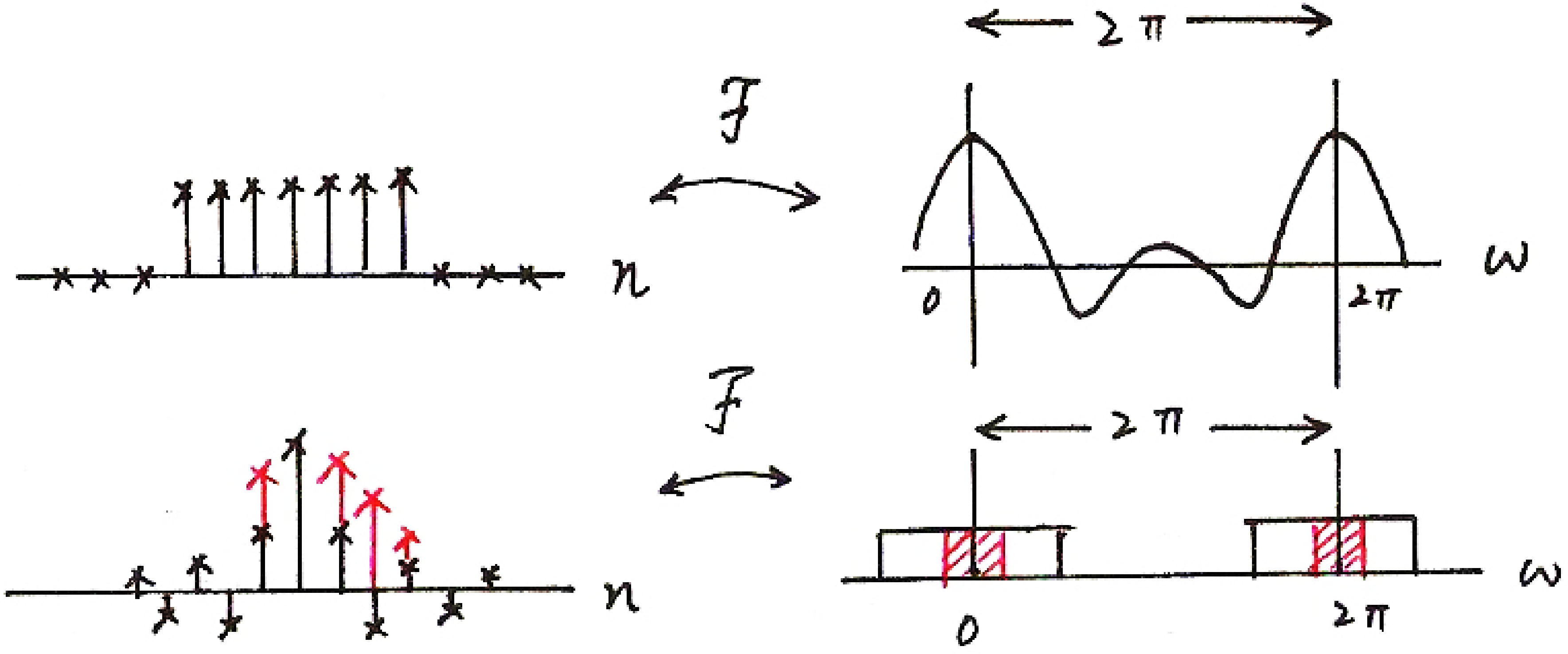


Figure 3.7 Plots of the scaled Fourier series coefficients Ta_k for the periodic square wave with T_1 fixed and for several values of T: (*a*) $T = 4T_1$; (*b*) $T = 8T_1$; (*c*) $T = 16T_1$. The coefficients are regularly spaced samples of the envelope $(2 \sin \omega T_1)/\omega$, where the spacing between samples, $2\pi/T$, decreases as T increases.

Rectangular/Sinc (p.21 of 5.0)



Problem 5.36(c), p.411 of text

$$y[n] + y[n-1] + \tfrac{1}{4} y[n-2] = x[n-1] - \tfrac{1}{2} x[n-2]$$

$$H(e^{j\omega}) = \frac{e^{-j\omega} - \tfrac{1}{2} e^{-j2\omega}}{1 + e^{-j\omega} + \tfrac{1}{4} e^{-j2\omega}}$$

Inverse system

$$G(e^{j\omega}) = \frac{1}{H(e^{j\omega})} = \frac{e^{-j\omega}\left(e^{j\omega} + 1 + \tfrac{1}{4} e^{-j\omega}\right)}{e^{-j\omega}\left(1 - \tfrac{1}{2} e^{-j\omega}\right)}$$

$$g[n] = \left(\tfrac{1}{2}\right)^{n+1} u[n+1] + \left(\tfrac{1}{2}\right)^{n} u[n] + \tfrac{1}{4}\left(\tfrac{1}{2}\right)^{n-1} u[n-1]$$

Not causal

$$g'[n] = \left(\tfrac{1}{2}\right)^{n} u[n] + \left(\tfrac{1}{2}\right)^{n-1} u[n-1] + \tfrac{1}{4}\left(\tfrac{1}{2}\right)^{n-2} u[n-2]$$

Inverse with delay : output $x[n-1]$

Problem 5.43, p.413 of text

$$g[n] = x[2n], \quad G(e^{j\omega}) = ?$$

$$v[n] = \tfrac{1}{2}\left\{\left(-1\right)^n x[n] + x[n]\right\}$$

odd index samples are zero

$$x[2n] = v[2n] = g[n]$$

$$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} v[2n] e^{-j\omega n}, \quad m = 2n$$

$$= \sum_{m=-\infty}^{\infty} v[m] e^{-j\omega \frac{m}{2}} = V(e^{j\omega/2})$$

$$= \tfrac{1}{2}\left[X(e^{j(\frac{\omega}{2}-\pi)}) + X(e^{j\frac{\omega}{2}})\right]$$

Problem 5.46, p.415 of text

$$\alpha^{\mathrm{n}}\mathrm{u}[n] \xleftrightarrow{\mathrm{F}} \frac{1}{1-\alpha e^{-j\omega}}$$

$$(n+1)\alpha^{\mathrm{n}}\mathrm{u}[n] \xleftrightarrow{\mathrm{F}} \frac{1}{(1-\alpha e^{-j\omega})^2}$$

$$nx[n] \xleftrightarrow{\mathrm{F}} j\frac{dX(e^{j\omega})}{d\omega}, \text{ example } 5.13, \mathrm{P}.385 \text{ of text}$$

$$\frac{(n+r-1)!}{n!(r-1)!}\alpha^{\mathrm{n}}\mathrm{u}[n] \xleftrightarrow{\mathrm{F}} \frac{1}{(1-\alpha e^{-j\omega})^r}$$

true for $r = 1,\ 2$

when $r = k$ is true

show $r = k+1$ is also true

Problem 5.56, p.422 of text

$x[m,n]$ is a two-dimensional signal

$$X\left(e^{j\omega_1}, e^{j\omega_2}\right) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[m,n] e^{-j(\omega_1 m + \omega_2 n)}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} x[m,n] e^{-j\omega_1 m}\right] e^{-j\omega_2 n}$$

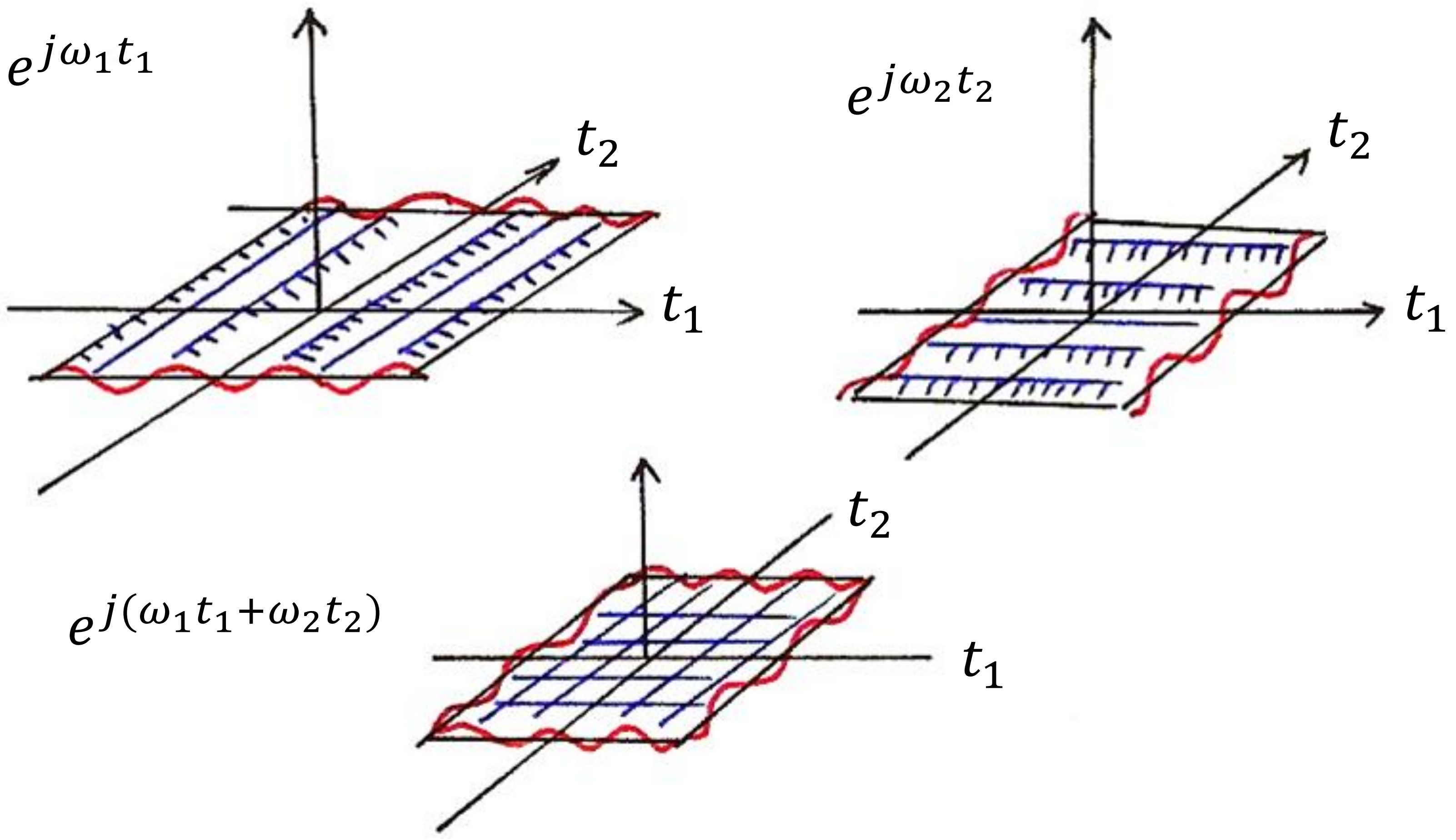
$$= \sum_{n=-\infty}^{\infty} X_n\left(e^{j\omega_1}\right) e^{-j\omega_2 n}$$

$$x[m,n] = \left(\frac{1}{2\pi}\right)^2 \int_{2\pi} \int_{2\pi} X\left(e^{j\omega_1}, e^{j\omega_2}\right) e^{j\omega_1 m}\, e^{j\omega_2 n}\, d\omega_1 d\omega_2$$

$$= \frac{1}{2\pi} \int_{2\pi} \left[\frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega_1}, e^{j\omega_2}\right) e^{j\omega_1 m}\, d\omega_1\right] e^{j\omega_2 n}\, d\omega_2$$

Problem 3.70, p.281 of text

- 2-dimensional signals (P. 65 of 3.0)



Problem 3.70, p.281 of text

- 2-dimensional signals (P. 64 of 3.0)