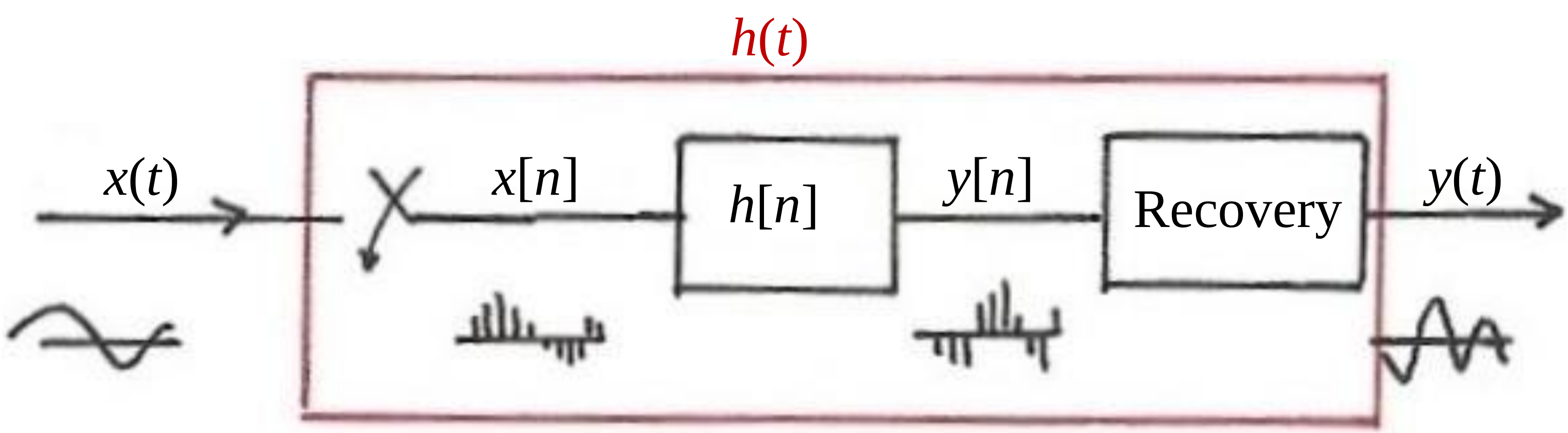
7.0 Sampling

7.1 The Sampling Theorem

A link between Continuous-time/Discrete-time Systems

$x[n]=x(nT)$, T : sampling period



Motivation: handling continuous-time signals/systems digitally using computing environment

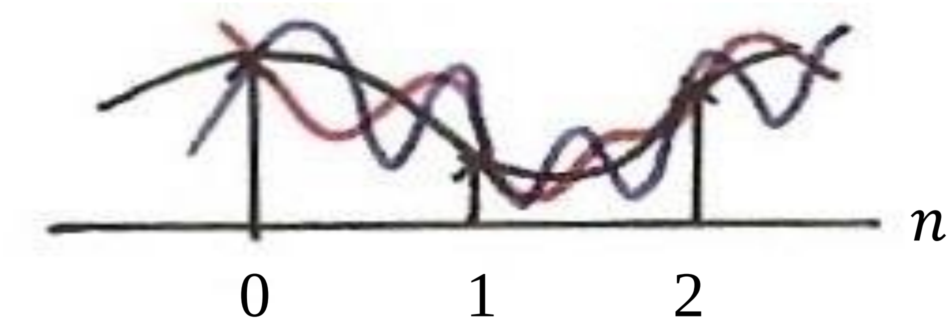
– accurate, programmable, flexible, reproducible, powerful
– compatible to digital networks and relevant technologies
– all signals look the same when digitized, except at different rates, thus can be supported by a single network

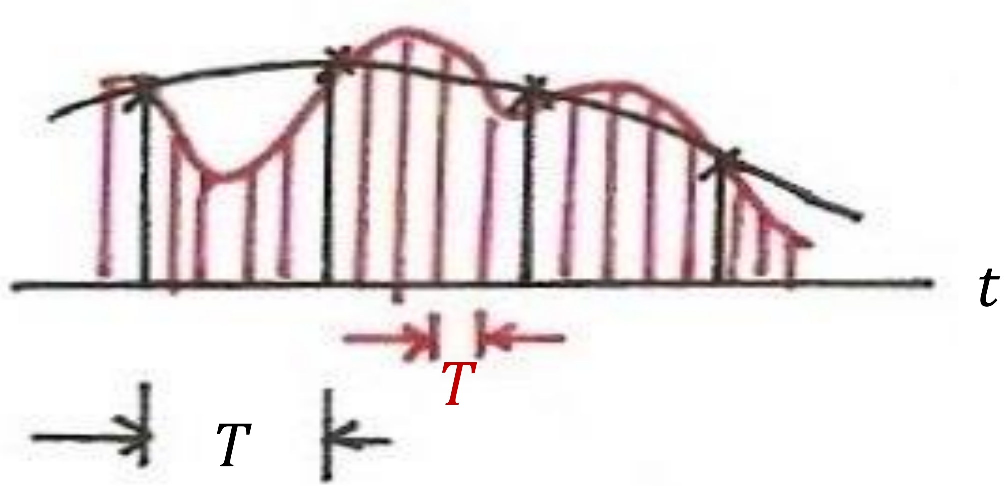
Question: under what kind of conditions can a continuous-time signal be uniquely specified by its discrete-time samples?

See Fig. 7.1, p.515 of text

– Sampling Theorem

Recovery from Samples ?





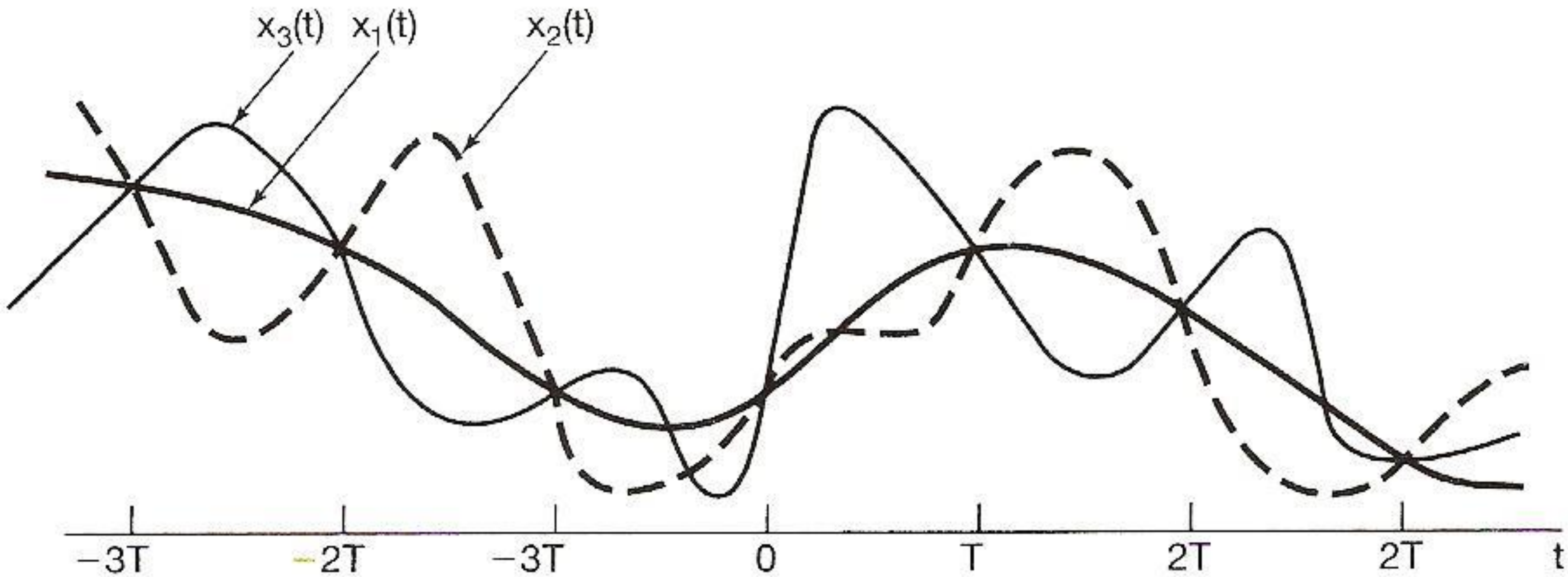
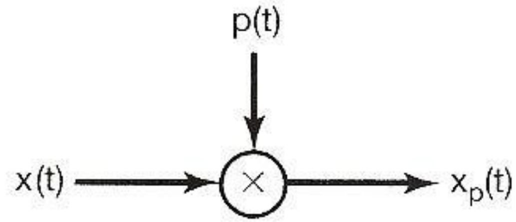


Figure 7.1 Three continuous-time signals with identical values at integer multiples of T.

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad T: \text{sampling period}$$

$$\omega_s = \frac{2\pi}{T} : \text{sampling frequency}$$

$$x_p(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$



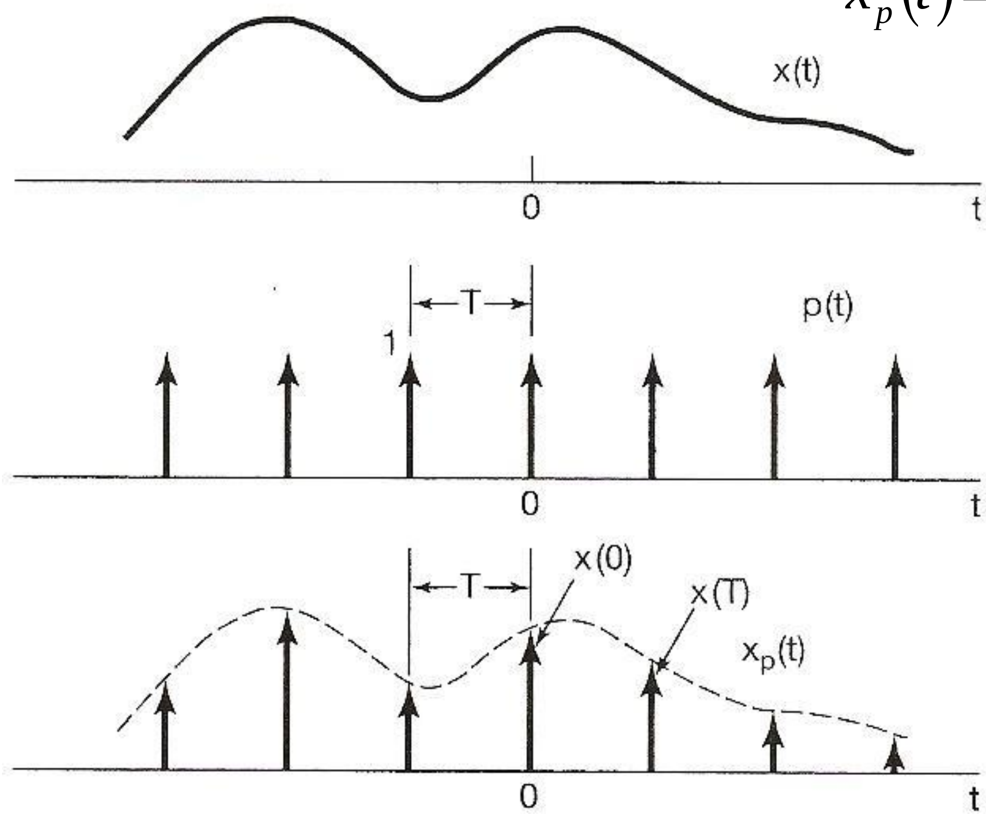


Figure 7.2 Impulse-train sampling.

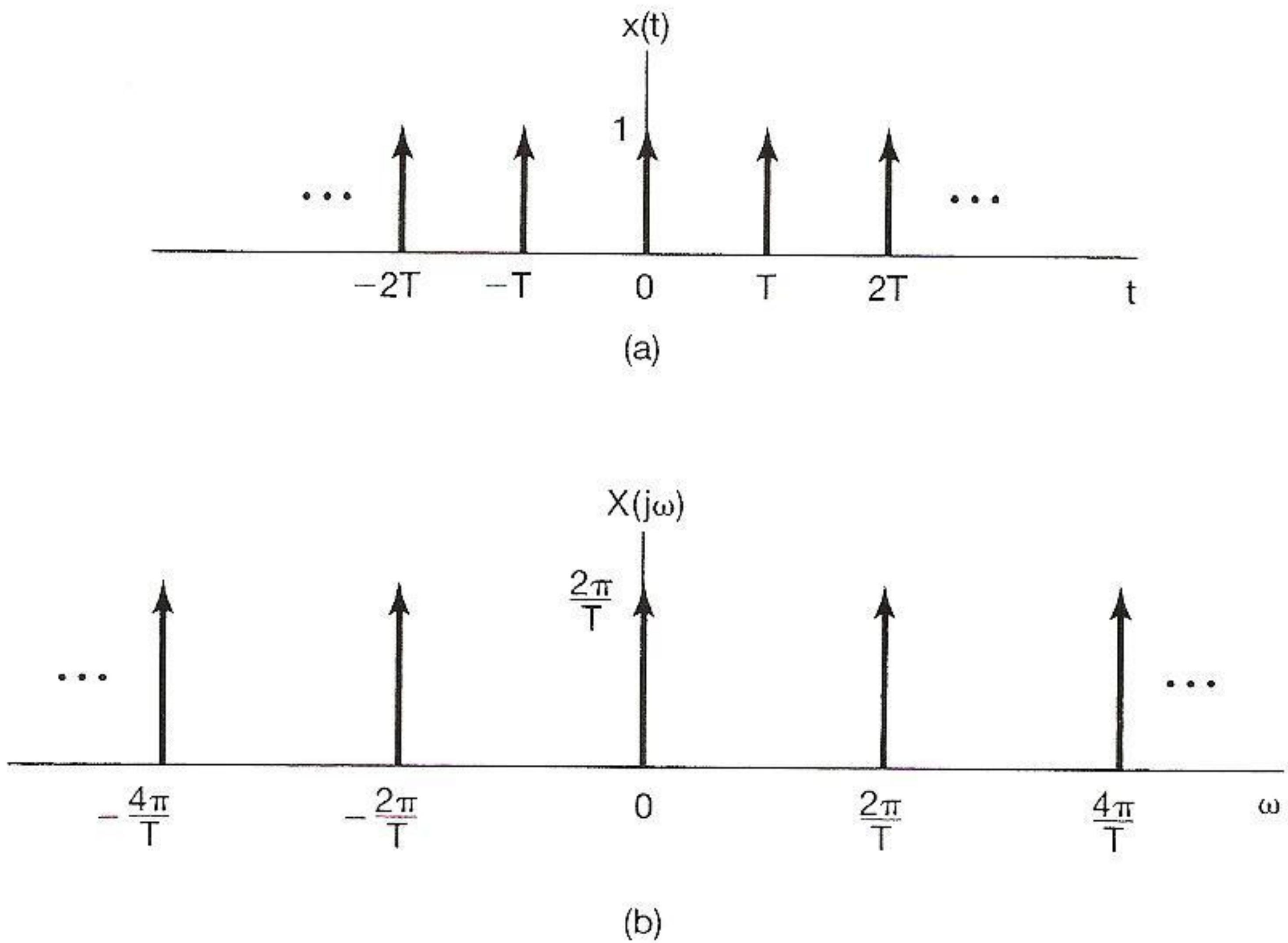


Figure 4.14 (a) Periodic impulse train; (b) its Fourier transform.

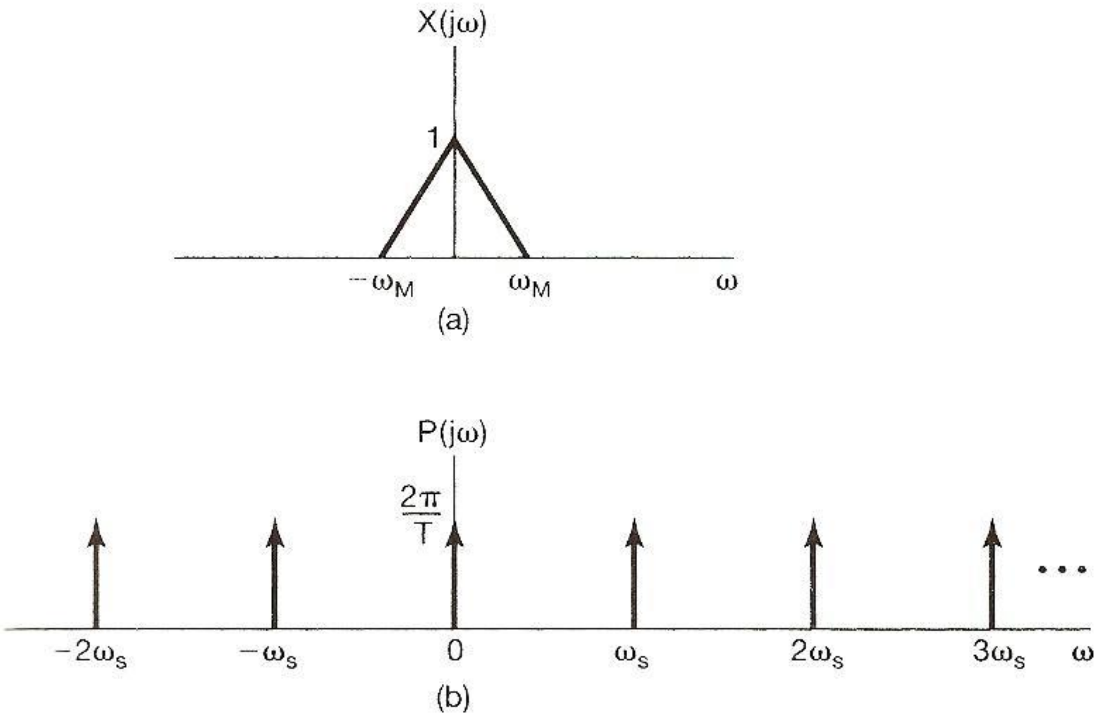


Figure 7.3 Effect in the frequency domain of sampling in the time domain: (a) spectrum of original signal; (b) spectrum of sampling function;

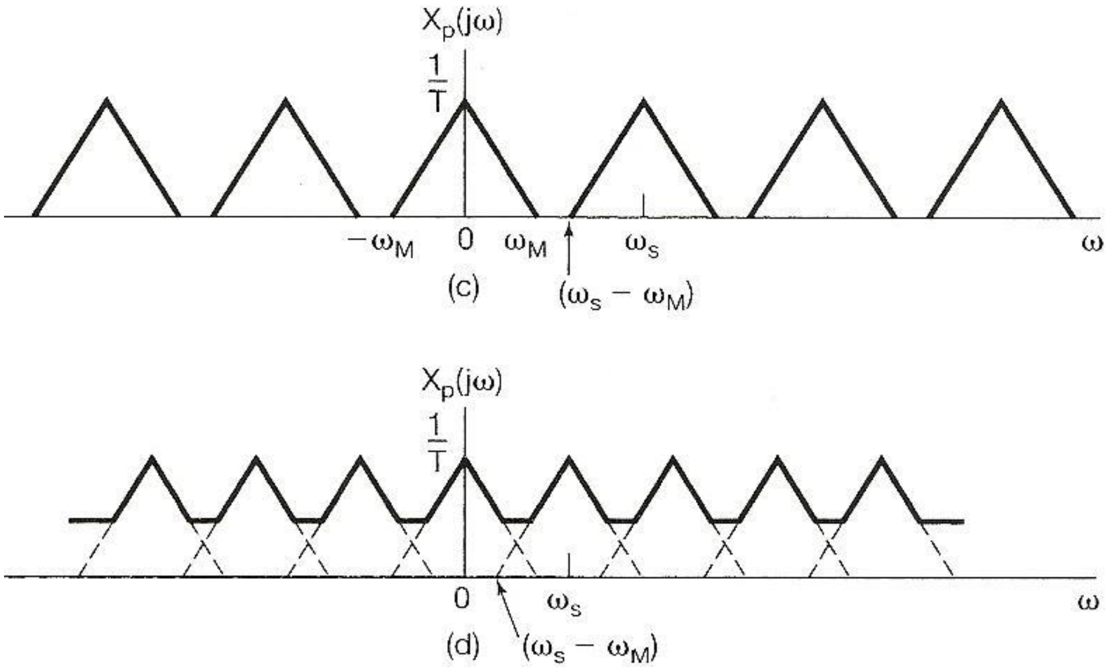


Figure 7.3 *Continued* (c) spectrum of sampled signal with $\omega_s > 2\omega_M$; (d) spectrum of sampled signal with $\omega_s < 2\omega_M$.

Impulse Train Sampling

$$X_p(j\omega) = \frac{1}{2\pi}\left[X(j\omega) * P(j\omega)\right]$$

$$P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s)$$

$$\omega_s = \frac{2\pi}{T} : \text{sampling frequency}$$

See Fig. 4.14, p.300 of text

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty}X(j(\omega - k\omega_s))$$

– periodic spectrum, superposition of scaled, shifted replicas of $X(j\omega)$

See Fig. 7.3, p.517 of text

Impulse Train Sampling

- Sampling Theorem (1/2)

  $$X(j\omega) = 0,\ |\,\omega\,| > \omega_M$$

  – $x(t)$ uniquely specified by its samples $x(nT)$, n=0, ±1, ± 2……

  $$\text{if } \omega_s = \frac{2\pi}{T} > 2\omega_M : \text{Nyquist rate}$$

  – precisely reconstructed by an ideal lowpass filter with Gain T and cutoff frequency $\omega_M < \omega_c < \omega_s - \omega_M$

  applied on the impulse train of sample values

  See Fig. 7.4, p.519 of text

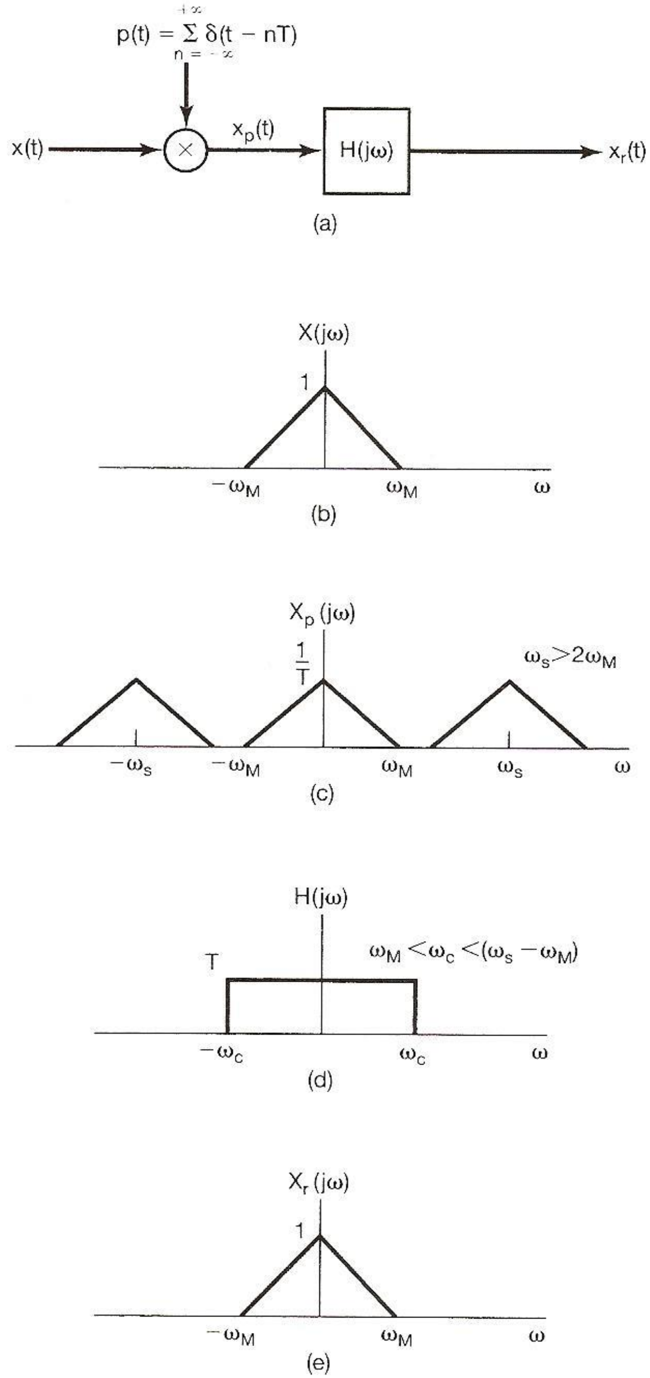


Figure 7.4 Exact recovery of a continuous-time signal from its samples using an ideal lowpass filter: (a) system for sampling and reconstruction; (b) representative spectrum for $x(t)$; (c) corresponding spectrum for $x_p(t)$; (d) ideal lowpass filter to recover $X(j\omega)$ from $X_p(j\omega)$; (e) spectrum of $x_r(t)$.

Impulse Train Sampling

- Sampling Theorem (2/2)

$$X(j\omega) = 0, \ |\omega| > \omega_M$$

– if $\omega_s \leq 2\ \omega_M$

spectrum overlapped, frequency components confused --- aliasing effect

can't be reconstructed by lowpass filtering

See Fig. 7.3, p.518 of text

Aliasing Effect

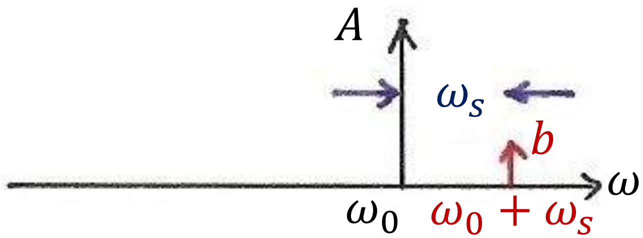
$x(t) = A\cos\omega_0 t$

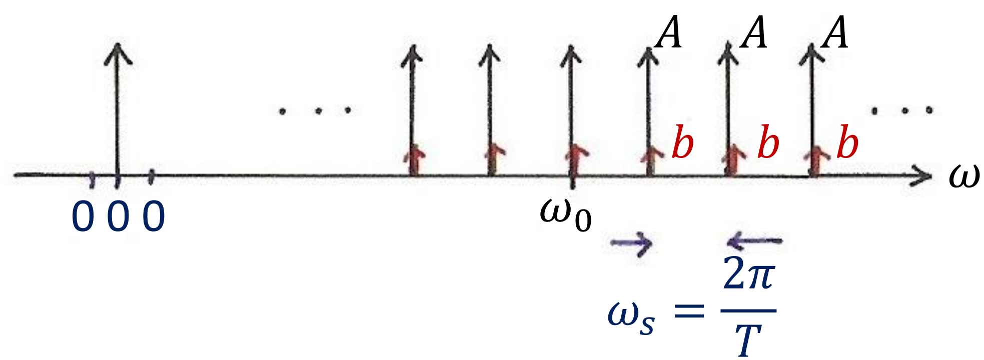
$y(t) = b\cos(\omega_0 + \omega_s)t,\ \ \omega_s = \frac{2\pi}{T}$

$x(nT) = A\cos\omega_0 nT$

$y(nT) = b\cos(\omega_0 + k\frac{2\pi}{T})nT$

$= b\cos\omega_0 nT$

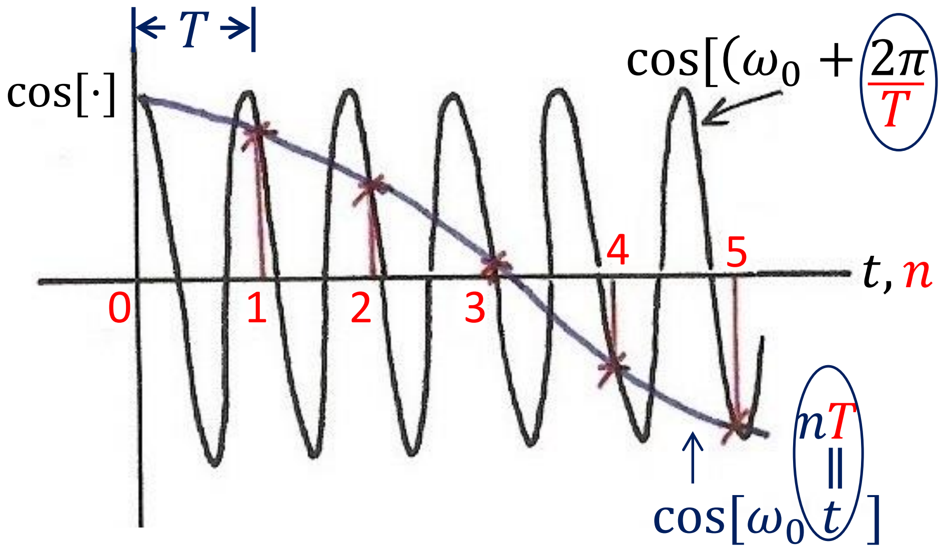




After sampling with $\omega_s = \frac{2\pi}{T}$, any two frequency components ω_1, ω_2 become indistinguishable, or sharing identical samples, or should be considered as identical frequency components if $|\omega_1 - \omega_2| = k\frac{2\pi}{T}$

$$\boxed{e^{j\left(\omega_0 + k\underset{\omega_s}{\underset{\parallel}{\frac{2\pi}{T}}}\right)nT} = e^{j\omega_0 nT} \quad (T = 1 \text{ for discrete-time signals })}$$

Continuous/Discrete Sinusoidals (p.36 of 1.0)



$T = 1$: discrete-time signals

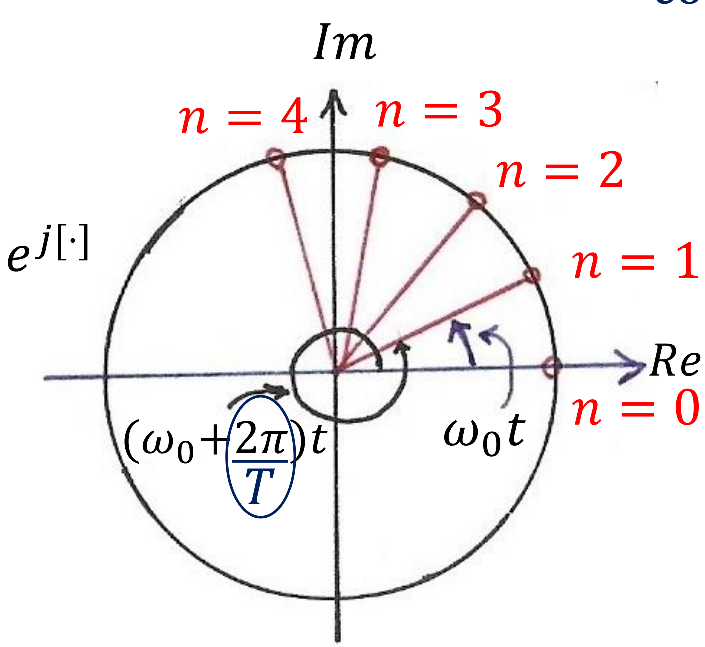
any T: sampling

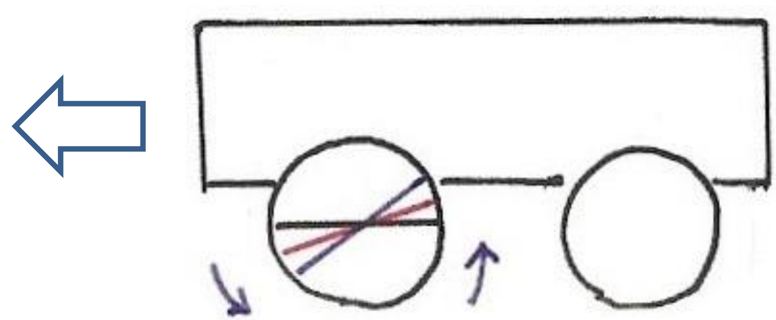
$$\cos \omega_0 t \neq \cos(\omega_0 + \frac{2\pi}{T}) t$$

$$\cos \omega_0 n T = \cos(\omega_0 + \frac{2\pi}{T}) n T$$

$$e^{j\omega_0 t} \neq e^{j(\omega_0 + \frac{2\pi}{T}) t}$$

$$e^{j\omega_0 nT} = e^{j(\omega_0 + \frac{2\pi}{T}) nT}$$

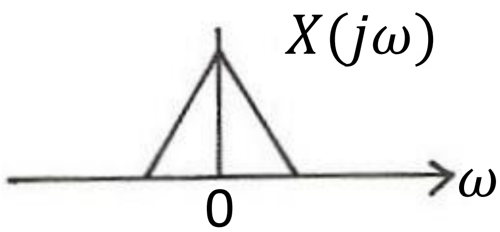




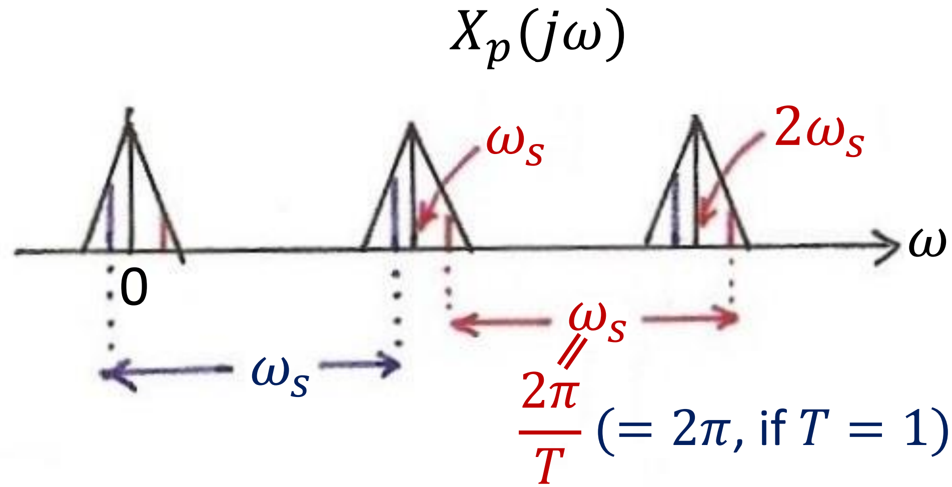
Sampling

sampling

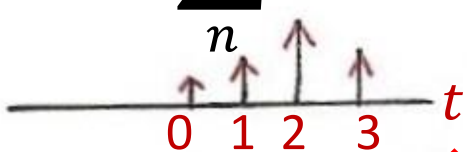
$$x(t) \overset{F}{\leftrightarrow} X(j\omega), \quad \xrightarrow{\text{sampling}} \quad x_p(t) \overset{F}{\leftrightarrow} X_p(j\omega)$$

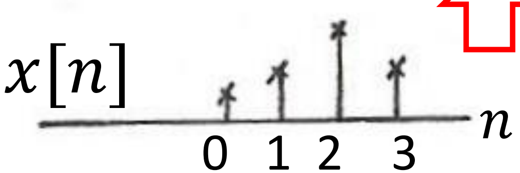


sampling



$$x(t) = \sum_n x[n]\,\delta(t-n)$$





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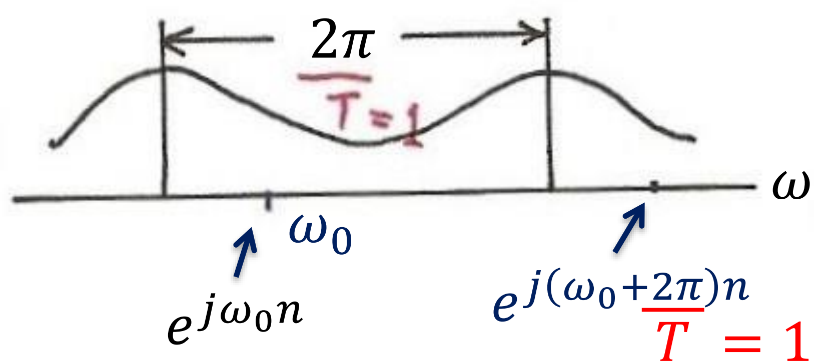
F(chap5)

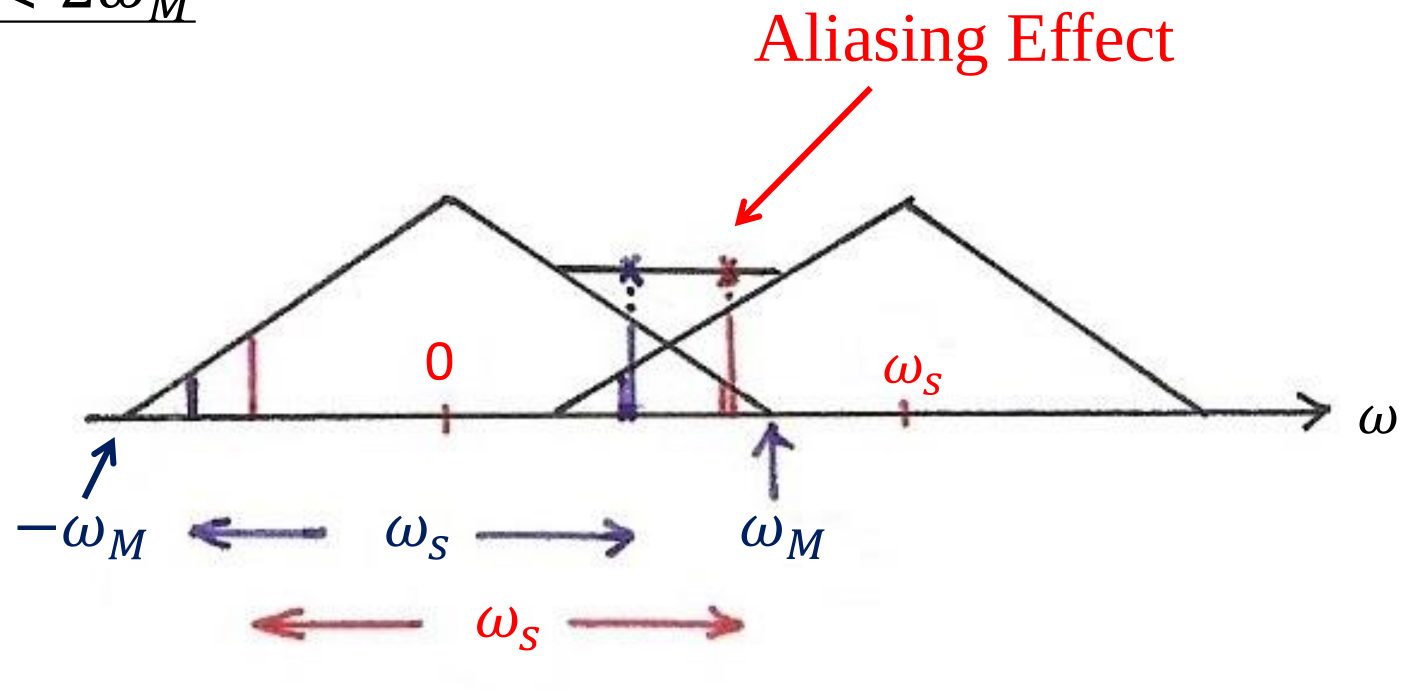


Aliasing Effect

$$z(t) = x(t) + y(t) = A\cos\omega_0 t + b\cos(\omega_0 + \omega_s)t$$

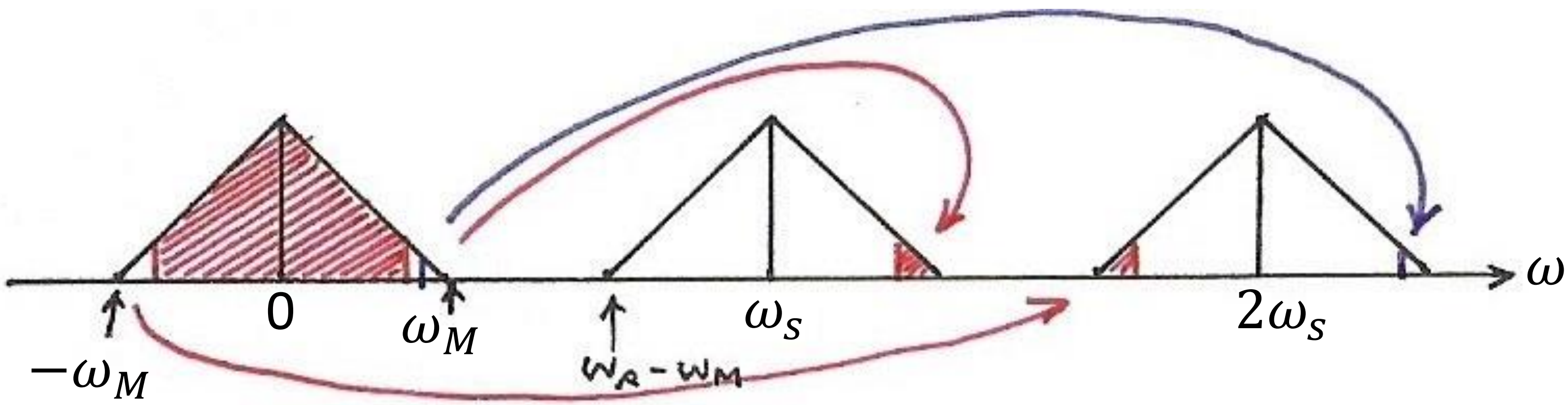
$$z(nT) = x(nT) + y(nT) = (A + b)\cos\omega_0 nT$$

$\underline{\omega_s < 2\omega_M}$

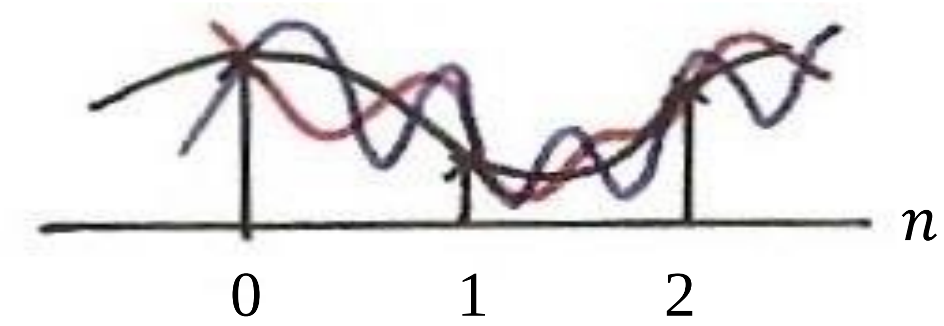


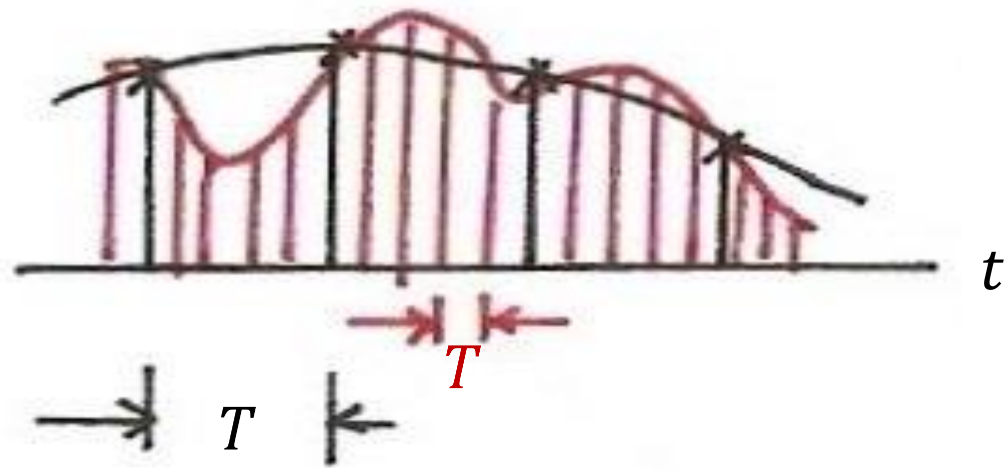
Sampling Thm

$\omega_s > 2\omega_M$

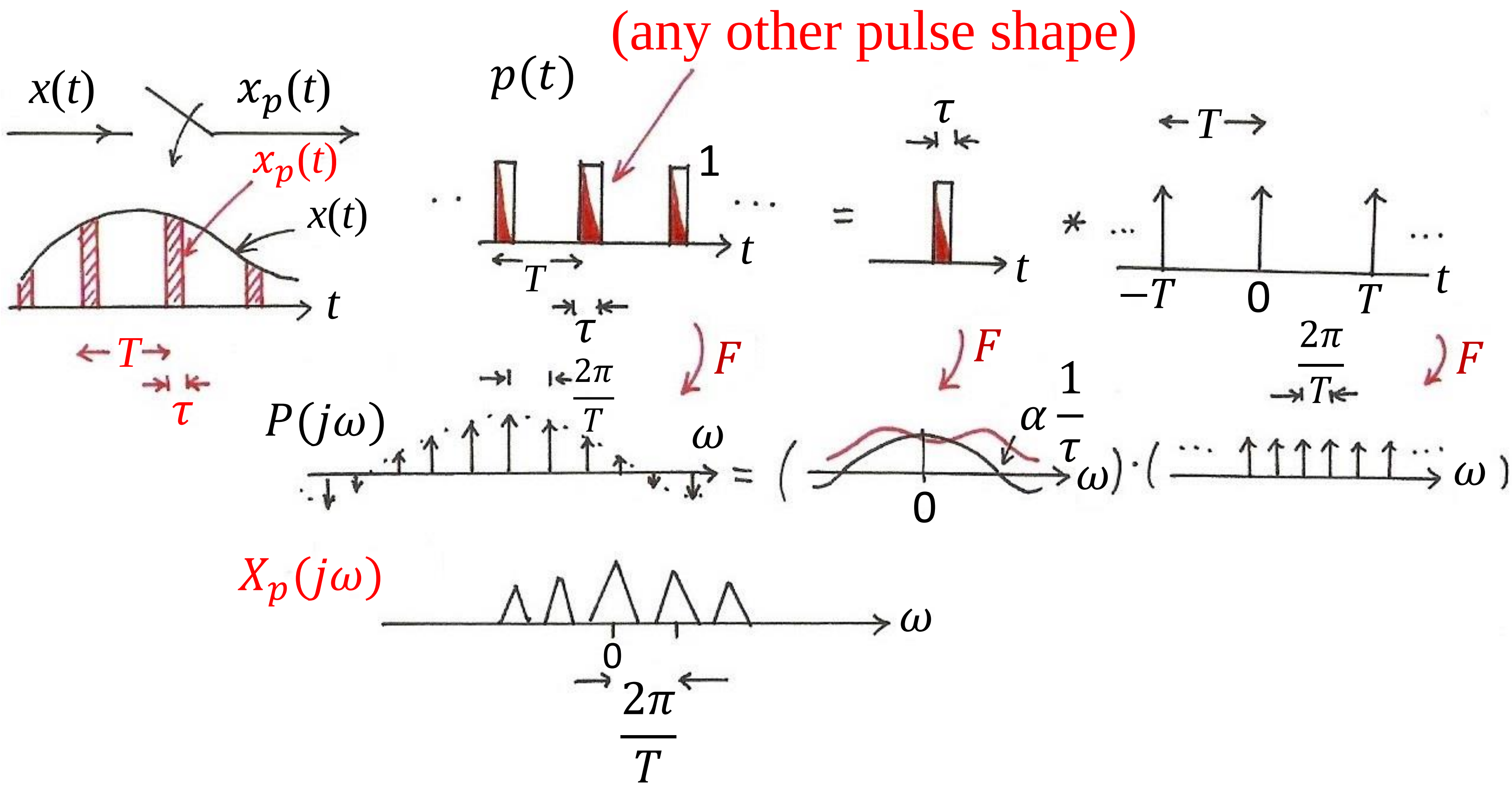


Recovery from Samples ? (p.3 of 7.0)



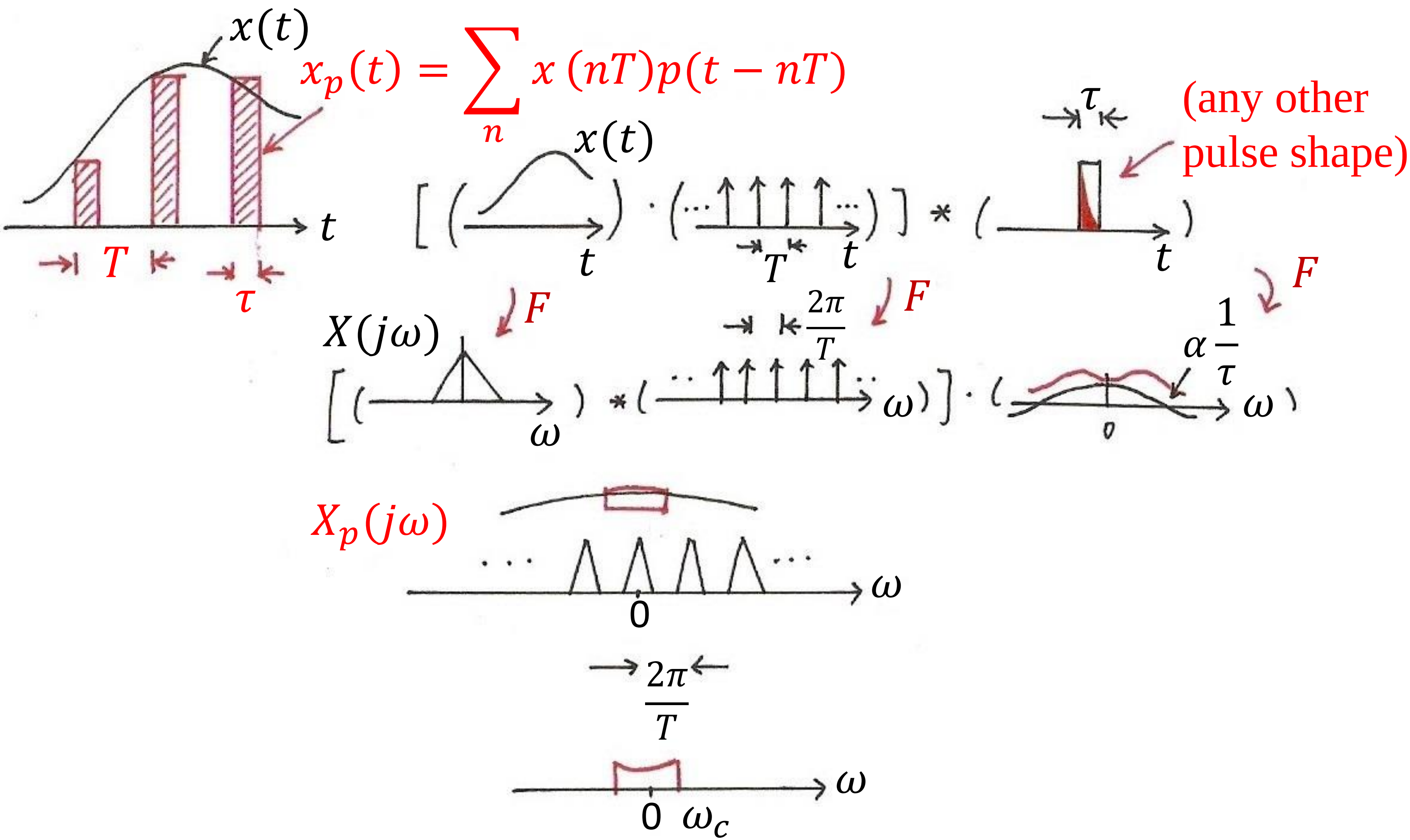


Practical Sampling



Practical Sampling

$$x_p(t) = \sum_n x(nT)p(t - nT)$$

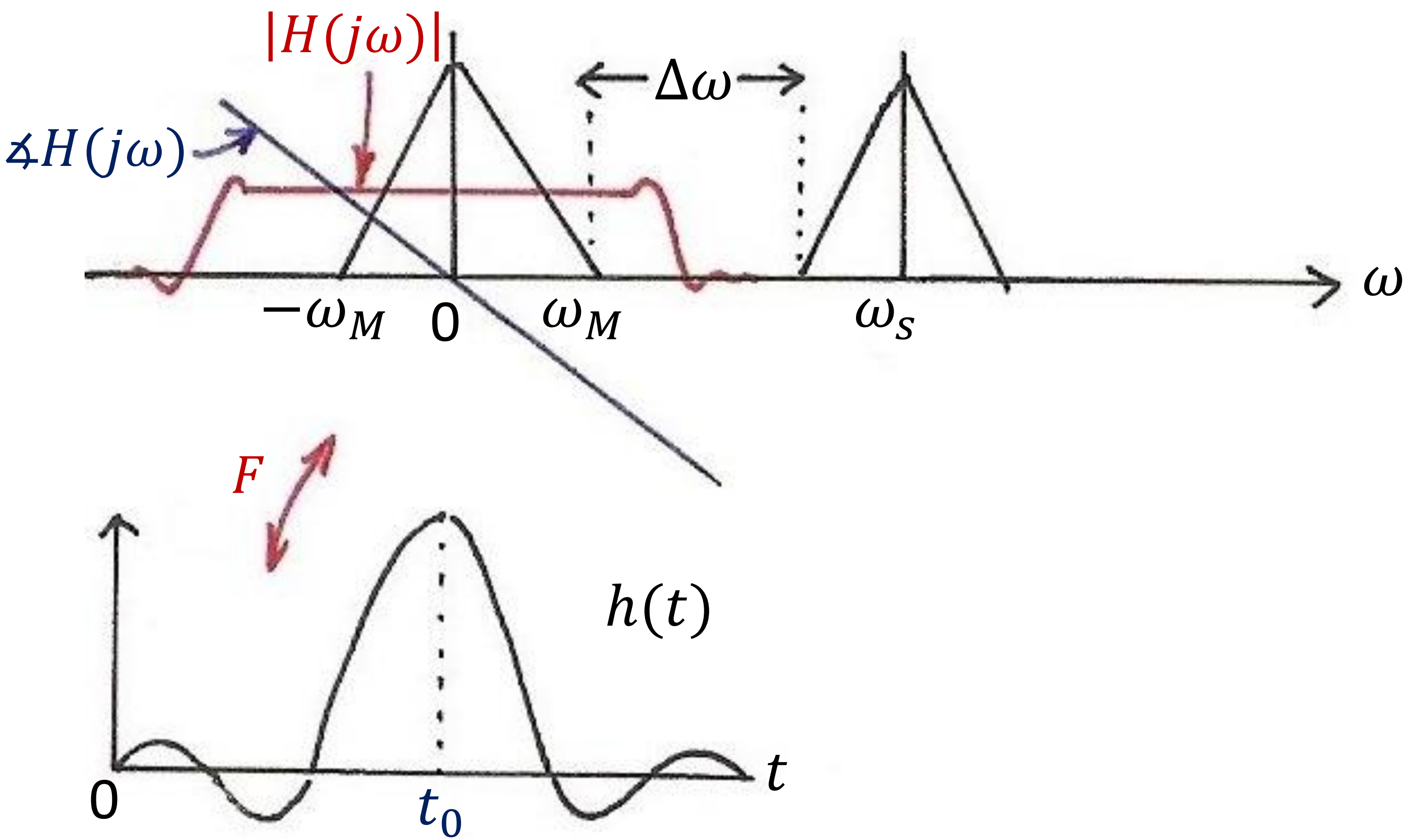


Impulse Train Sampling

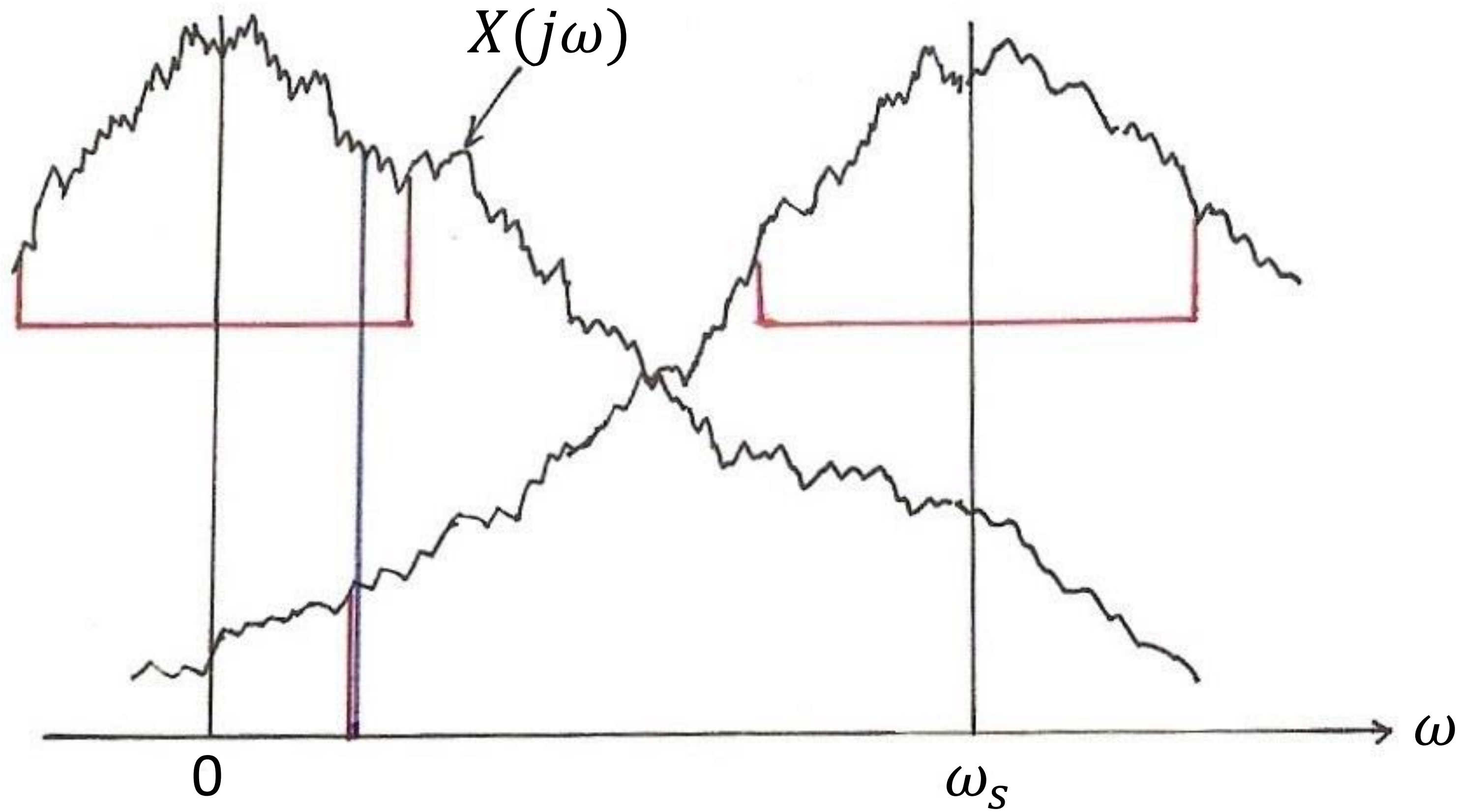
- Practical Issues
  - nonideal lowpass filters accurate enough for practical purposes determined by acceptable level of distortion

    oversampling $\omega_s = 2\ \omega_M + \Delta\ \omega$
  - sampled by pulse train with other pulse shapes
  - signals practically not bandlimited : pre-filtering

Oversampling with Non-ideal Lowpass Filters



Signals not Bandlimited



Sampling with A Zero-order Hold

- Zero-order Hold:
  - holding the sampled value until the next sample taken
  - modeled by an impulse train sampler followed by a system with rectangular impulse response
- Reconstructed by a lowpass filter $H_r(j\omega)$

$$H_r(j\omega) = \frac{H(j\omega)}{H_0(j\omega)}$$

$$H_0(j\omega) = e^{-j\omega T/2}\left[\frac{2\sin(\omega T/2)}{\omega}\right]$$

$$H(j\omega) = \text{ideal lowpass filter in impulse train sampling}$$

See Fig. 7.6, 7.7, 7.8, p.521, 522 of text

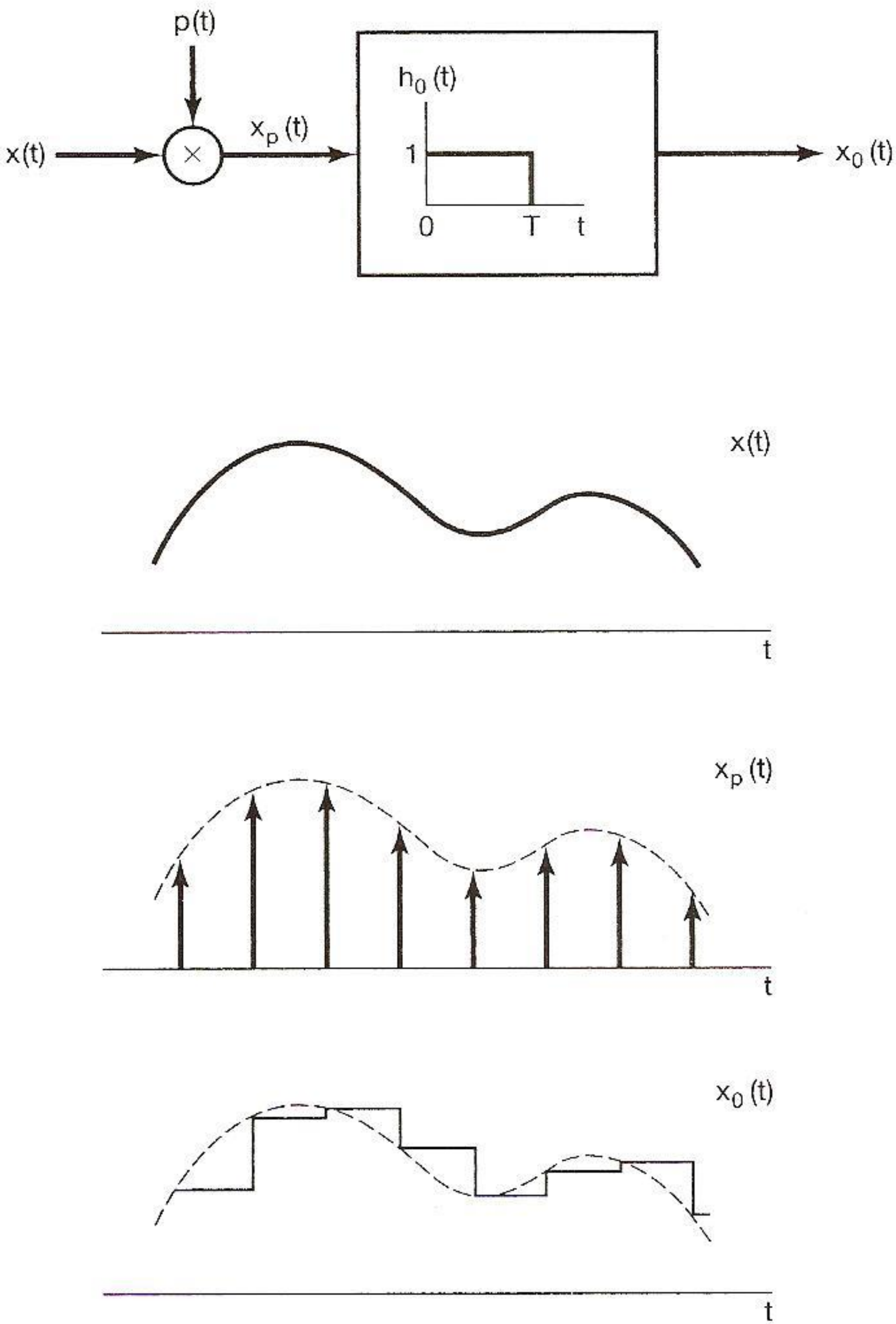


Figure 7.6 Zero-order hold as impulse-train sampling followed by an LTI system with a rectangular impulse response.

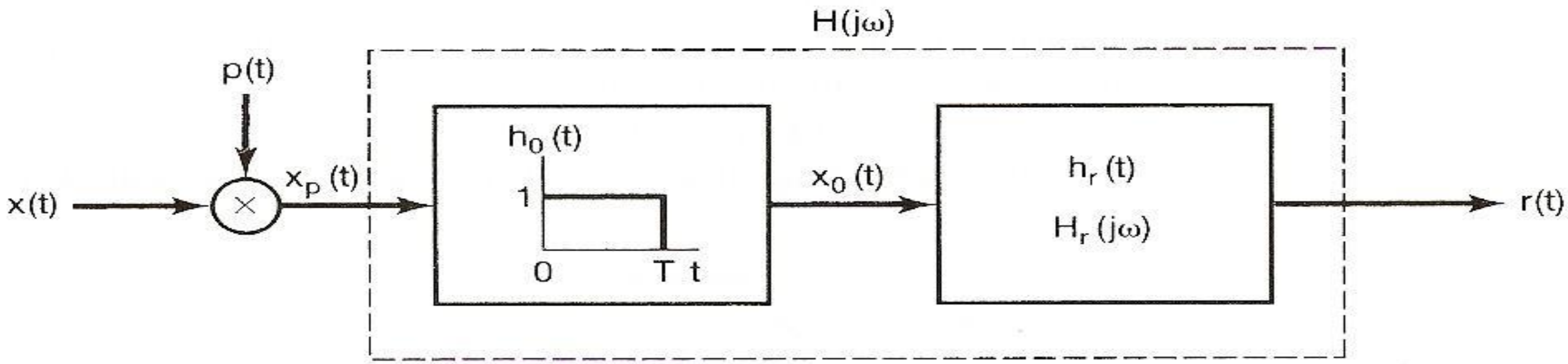


Figure 7.7 Cascade of the representation of a zero-order hold (Figure 7.6) with a reconstruction filter.

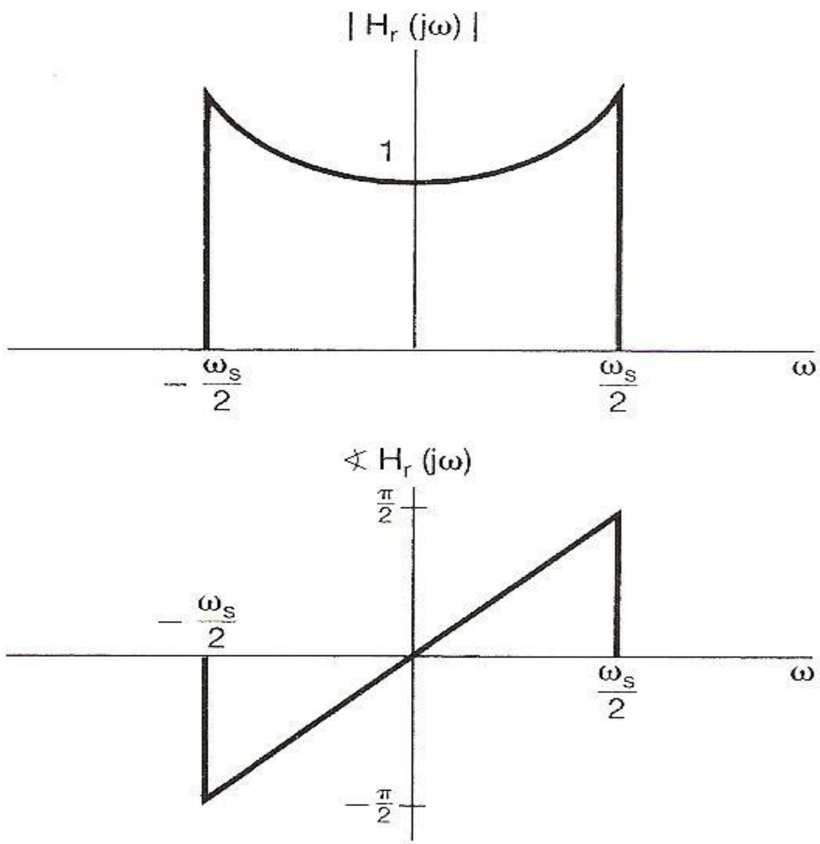


Figure 7.8 Magnitude and phase for the reconstruction filter for a zero-order hold.

Interpolation

- Impulse train sampling/ideal lowpass filtering

$$x(t) = x_p(t) * h(t) = \sum_{n=-\infty}^{\infty} x(nT)\, h(t - nT)$$

$$h(t) = \frac{\omega_c T \sin(\omega_c t)}{\pi \omega_c t}$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT) \frac{\omega_c T}{\pi} \frac{\sin(\omega_c (t - nT))}{\omega_c (t - nT)}$$

See Fig. 7.10, p.524 of text

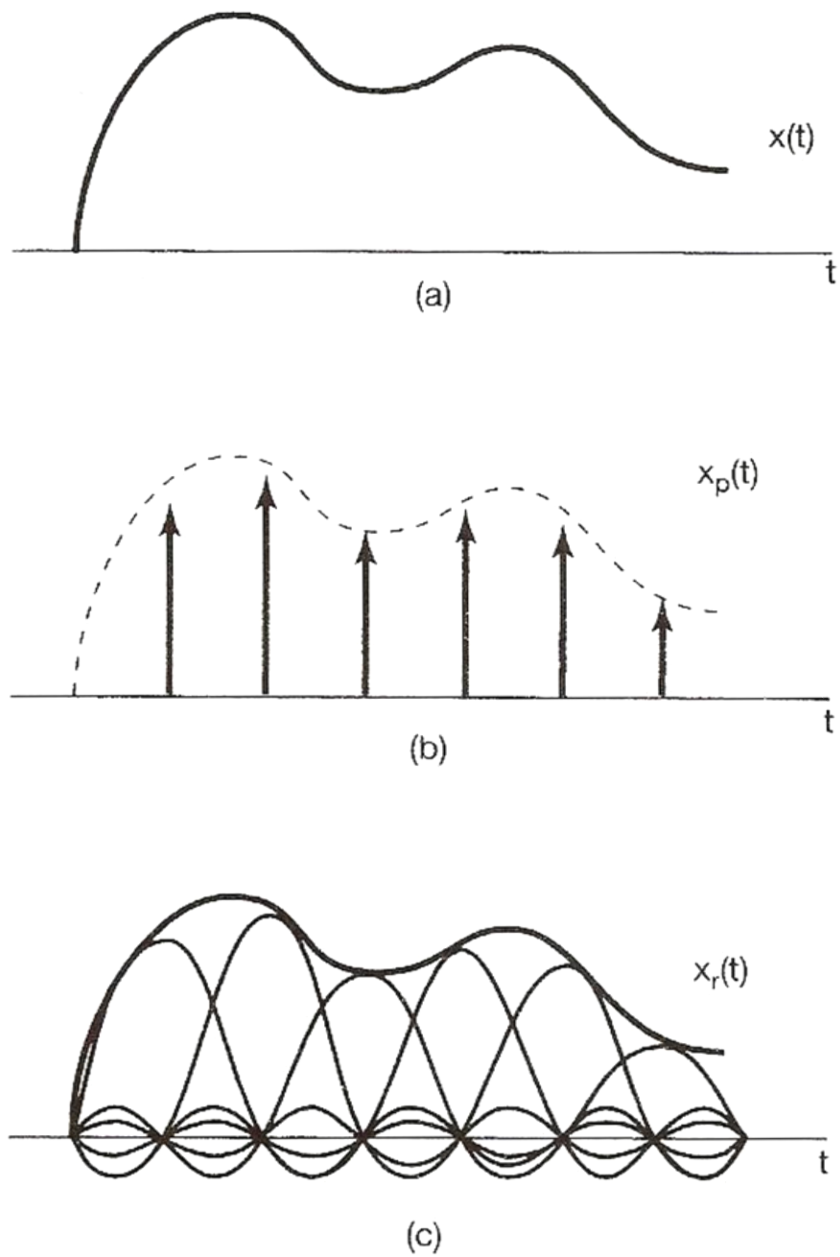
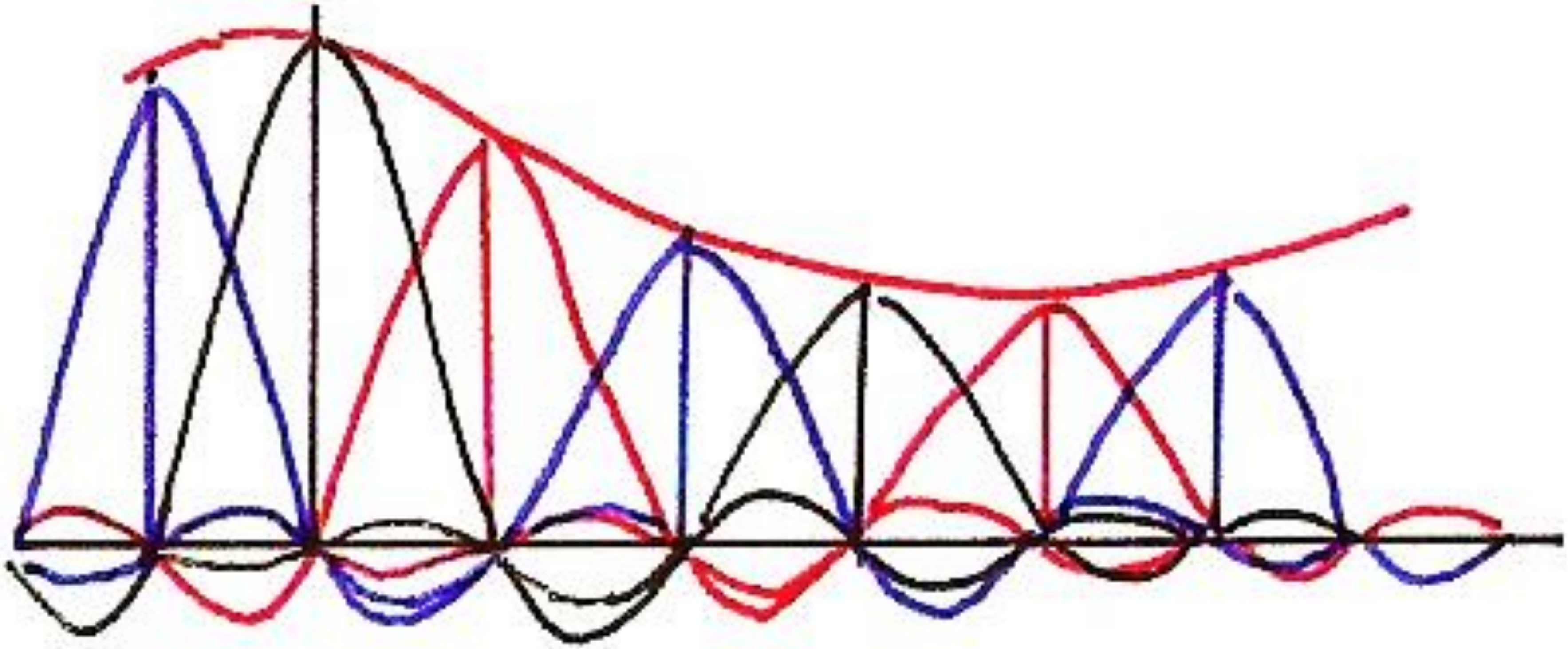


Figure 7.10 Ideal band-limited interpolation using the sinc function: (a) band-limited signal $x(t)$; (b) impulse train of samples of $x(t)$; (c) ideal band-limited interpolation in which the impulse train is replaced by a superposition of sinc functions [eq. (7.11)].

Ideal Interpolation



Interpolation

- Zero-order hold can be viewed as a “coarse” interpolation

  See Fig. 7.11, p.524 of text

- Sometimes additional lowpass filtering naturally applied

  e.g. viewed at a distance by human eyes, mosaic smoothed naturally

  See Fig. 7.12, p.525 of text

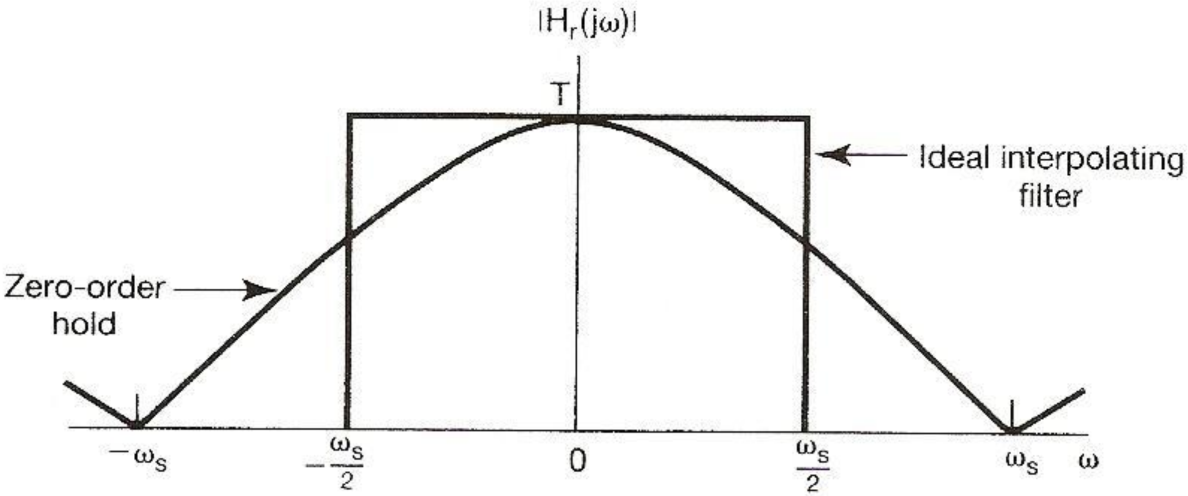
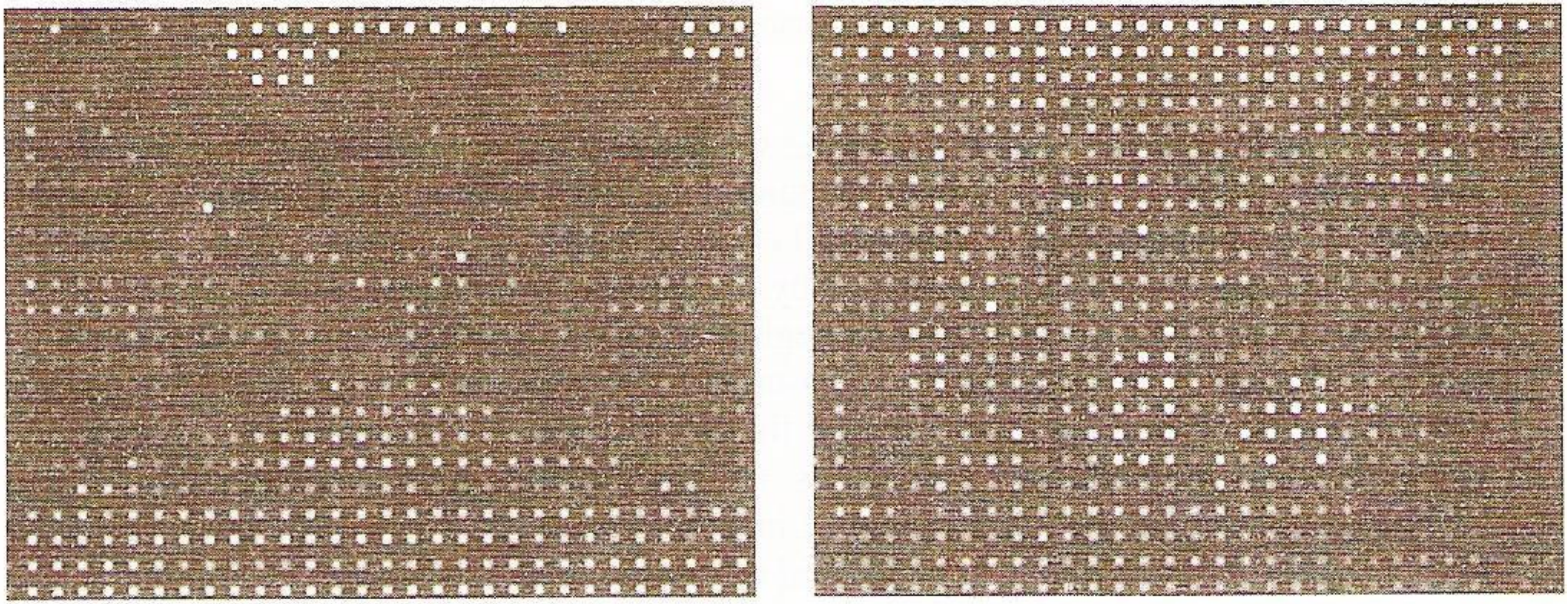
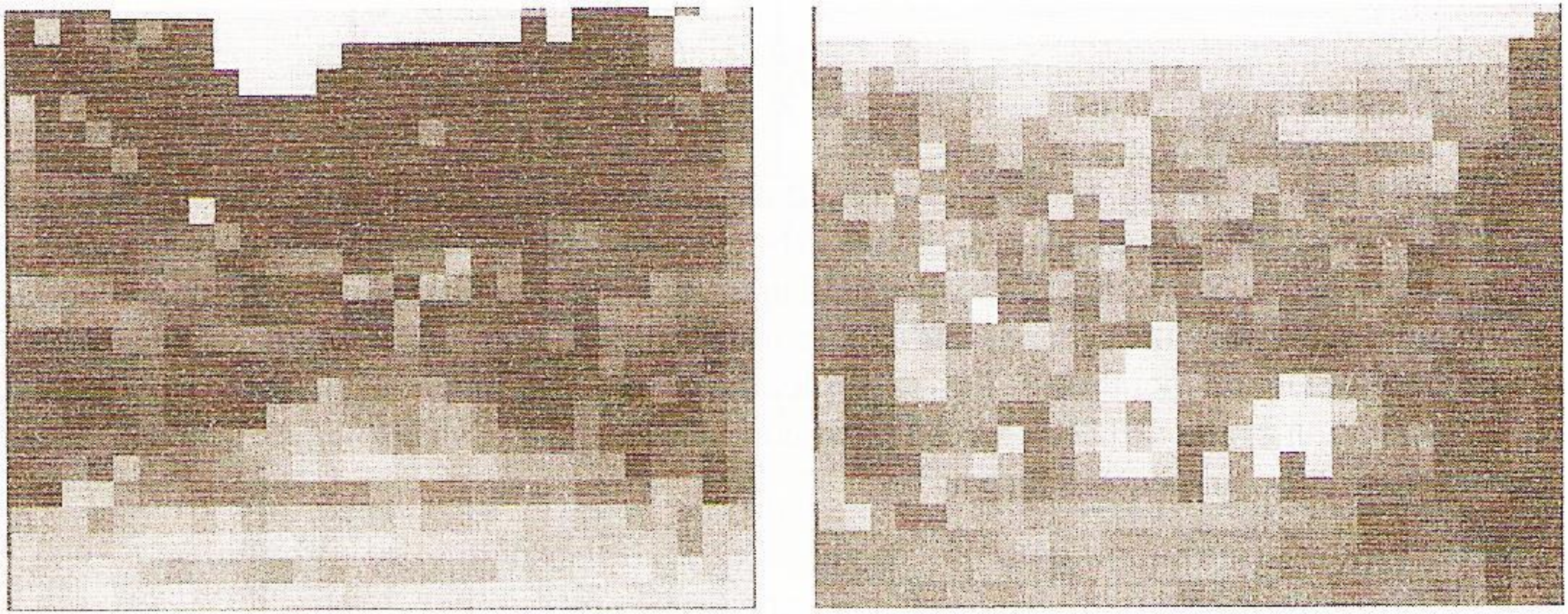
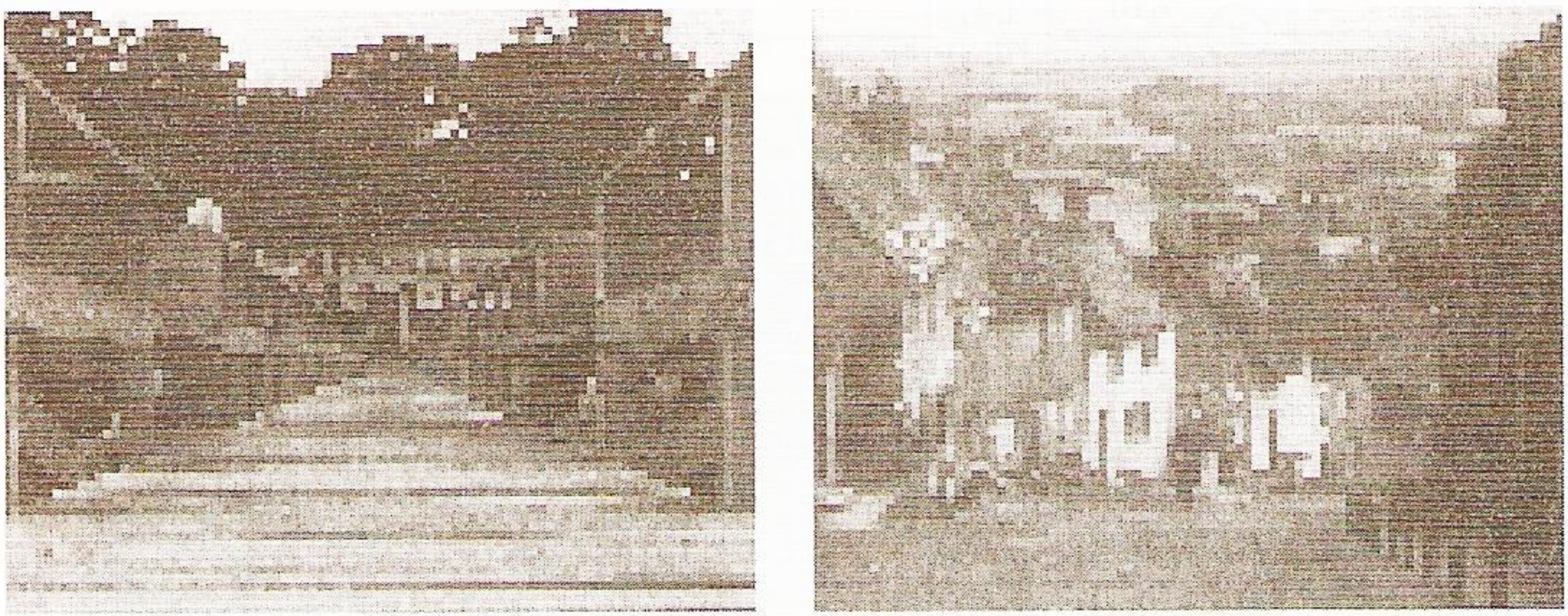


Figure 7.11 Transfer function for the zero-order hold and for the ideal interpolating filter.


(a)



(b)



(c)

Figure 7.12 (a) The original pictures of Figures 6.2(a) and (g) with impulse sampling; (b) zero-order hold applied to the pictures in (a). The visual system naturally introduces lowpass filtering with a cutoff frequency that decreases with distance. Thus, when viewed at a distance, the discontinuities in the mosaic in Figure 7.12(b) are smoothed; (c) result of applying a zero-order hold after impulse sampling with one-fourth the horizontal and vertical spacing used in (a) and (b).

Interpolation

- Higher order holds
  - zero-order : output discontinuous
  - first-order : output continuous, discontinuous derivatives

$$H(j\omega) = \frac{1}{T}\left[\frac{\sin(\omega T/2)}{\omega/2}\right]^2$$

See Fig. 7.13, p.526, 527 of text

  - second-order : continuous up to first derivative discontinuous second derivative

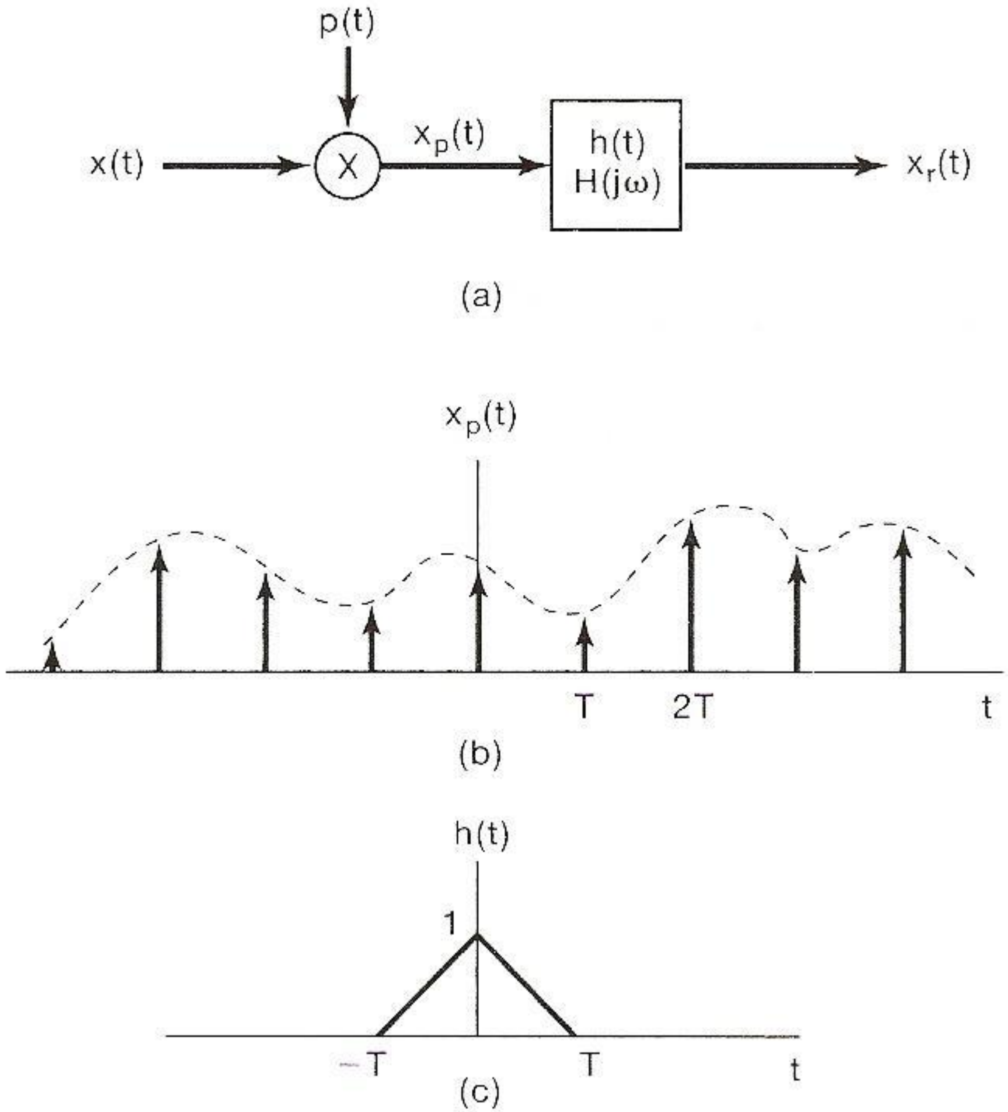


Figure 7.13 Linear interpolation (first-order hold) as impulse-train sampling followed by convolution with a triangular impulse response: (a) system for sampling and reconstruction; (b) impulse train of samples; (c) impulse response representing a first-order hold;

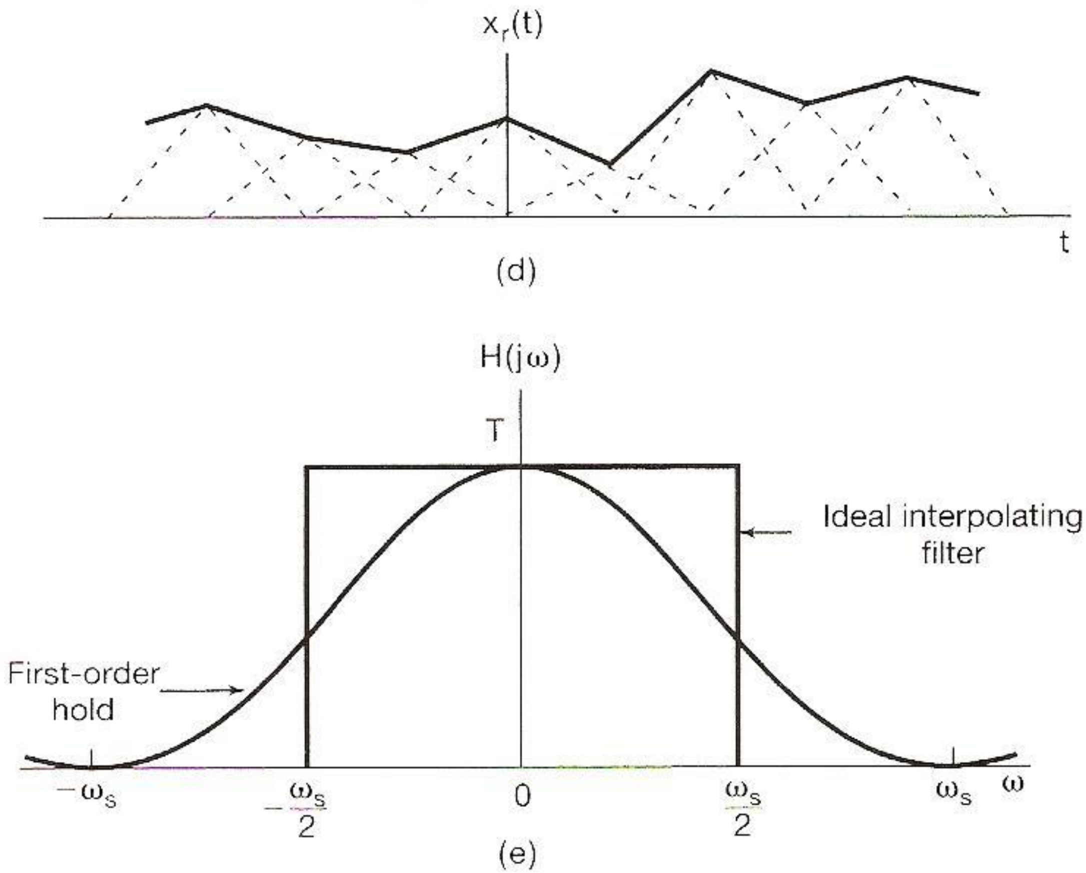


Figure 7.13 *Continued* (d) first-order hold applied to the sampled signal; (e) comparison of transfer function of ideal interpolating filter and first-order hold.

Aliasing

- Consider a signal $x(t)$=cos $\omega_0 t$
  - sampled at sampling frequency $\omega_s = \dfrac{2\pi}{T}$

    reconstructed by an ideal lowpass filter

    with $\omega_c = \dfrac{\omega_s}{2}$

    $x_r(t)$: reconstructed signal

    fixed ω_s, varying ω_0

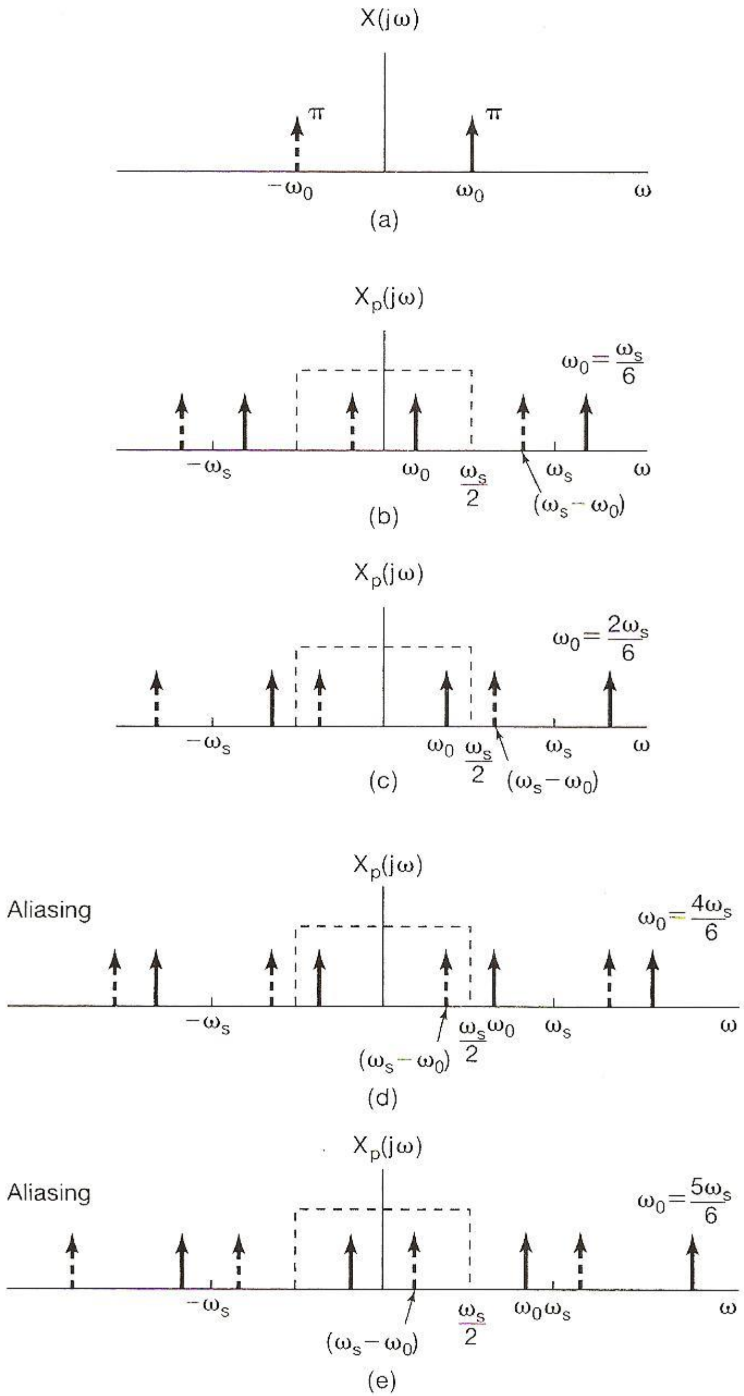


Figure 7.15 Effect in the frequency domain of oversampling and undersampling: (a) spectrum of original sinusoidal signal; (b), (c) spectrum of sampled signal with $\omega_s > 2\omega_0$; (d), (e) spectrum of sampled signal with $\omega_s < 2\omega_0$. As we increase ω_0 in moving from (b) through (d), the impulses drawn with solid lines move to the right, while the impulses drawn with dashed lines move to the left. In (d) and (e), these impulses have moved sufficiently that there is a change in the ones falling within the passband of the ideal lowpass filter.

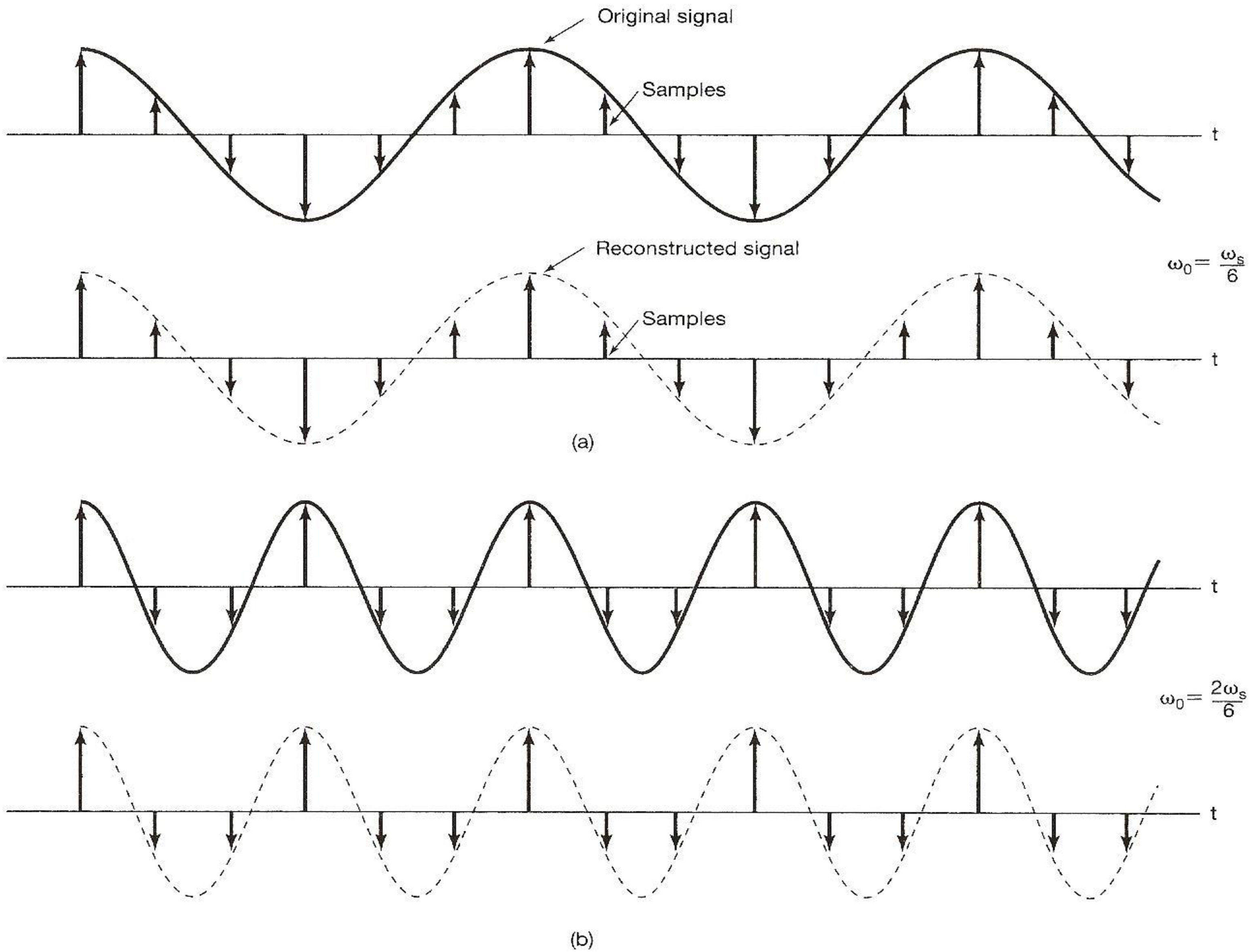


Figure 7.16 Effect of aliasing on a sinusoidal signal. For each of four values of ω_0, the original sinusoidal signal (solid curve), its samples, and the reconstructed signal (dashed curve) are illustrated: (a) $\omega_0 = \omega_s/6$; (b) $\omega_0 = 2\omega_s/6$; (c) $\omega_0 = 4\omega_s/6$; (d) $\omega_0 = 5\omega_s/6$. In (a) and (b) no aliasing occurs, whereas in (c) and (d) there is aliasing.

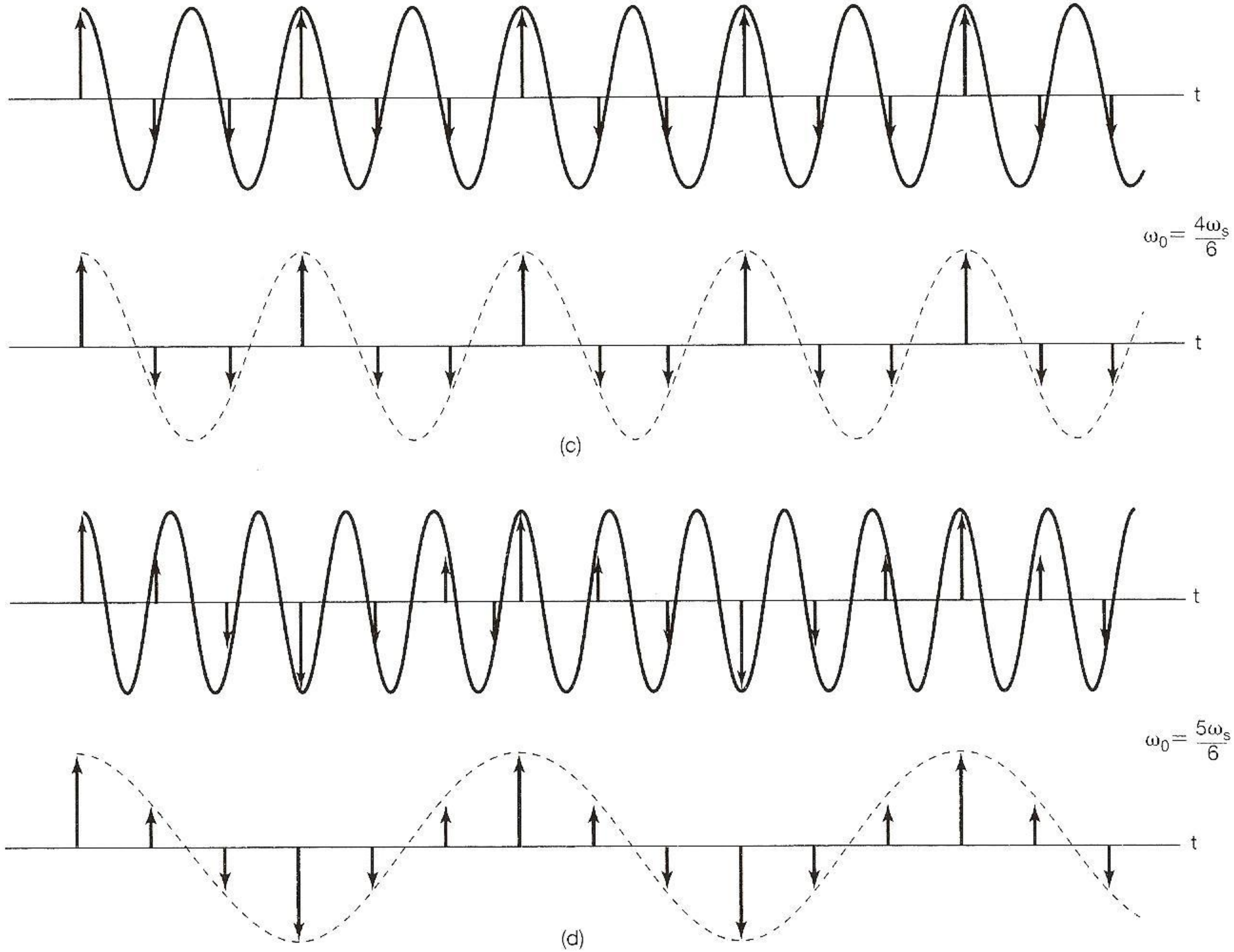


Figure 7.16 *Continued*

Aliasing

- Consider a signal $x(t)=\cos \omega_0 t$
  - (a) (b) $\omega_0 < \dfrac{\omega_s}{2}$ $\quad x_r(t) = \cos \omega_0 t = x(t)$

    (c) (d) $\dfrac{\omega_s}{2} < \omega_0 < \omega_s$ $\quad x_r(t) = \cos(\omega_s - \omega_0)t \neq x(t)$

    when aliasing occurs, the original frequency ω_0 takes on the identity of a lower frequency, $\omega_s - \omega_0$

  - w_0 confused with not only $\omega_s + \omega_0$, but $\omega_s - \omega_0$

    See Fig. 7.15, 7.16, p.529-531 of text

Aliasing

- Consider a signal $x(t)$=cos $\omega_0 t$
  - many $x_r(t)$ exist such that

$$x_r(nT) = x(nT), \quad n = 0, \pm 1, \pm 2, \ldots$$

    the problem is how to get the right one

  - if $x(t)$ = cos($\omega_0 t + \phi$)

    the impulses have extra phases $e^{j\phi}$, $e^{-j\phi}$

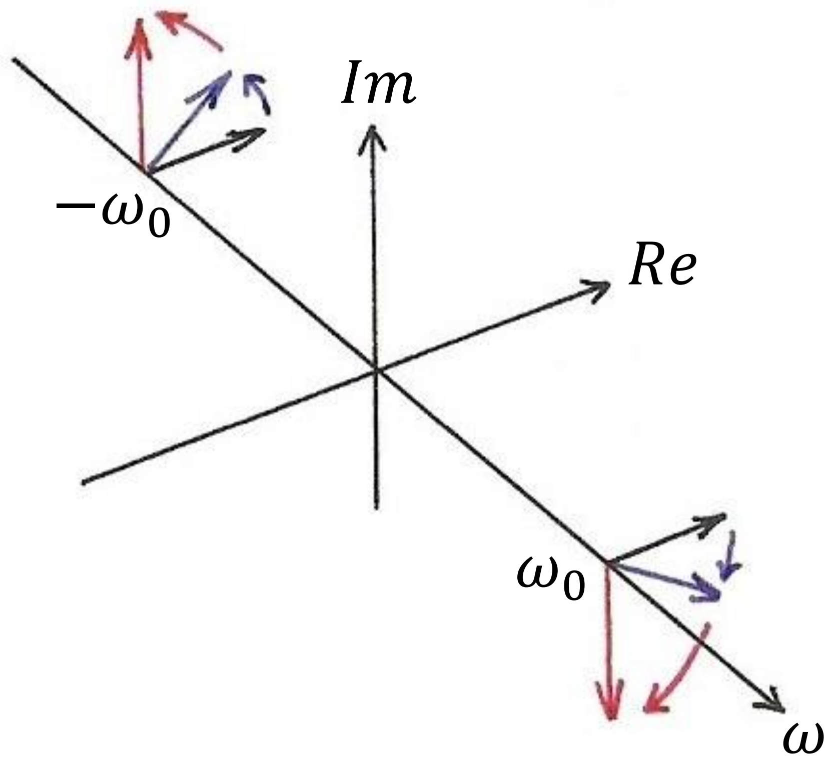
  - $\cos x = \frac{1}{2}\left(e^{jx} + e^{-jx}\right)$

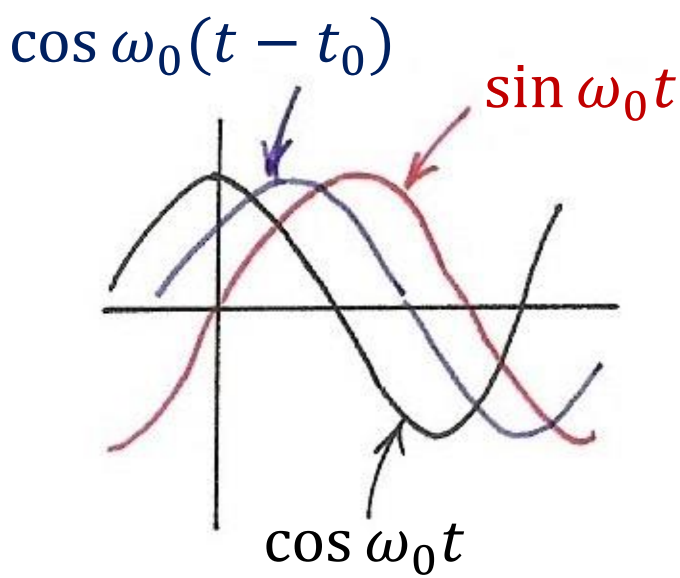
$$\cos(\omega_0 t + \phi) = \frac{1}{2}\left(e^{j(\omega_0 t + \phi)} + e^{-j(\omega_0 t + \phi)}\right)$$

Sinusoidals (p.25 of 4.0)

$$\cos\omega_0 t \overset{F}{\leftrightarrow} \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)],\ \ \tfrac{1}{2}[e^{j\omega_0 t}+e^{-j\omega_0 t}]$$

$$\sin\omega_0 t \overset{F}{\leftrightarrow} \frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)],\ \tfrac{1}{2j}[e^{j\omega_0 t}-e^{-j\omega_0 t}]$$





$$x(t-t_0) \leftrightarrow e^{-j\omega_0 t}\cdot X(j\omega)$$

Aliasing

- Consider a signal $x(t)$=cos $(\omega_0 t + \phi)$
  - (a) (b) $\omega_0 < \dfrac{\omega_s}{2}$

$$x_r(t) = \cos(\omega_0 t + \phi) = x(t)$$

(c) (d) $\dfrac{\omega_s}{2} < \omega_0 < \omega_s$

$$x_r(t) = \cos[(\omega_s - \omega_0)t - \phi] \neq x(t)$$

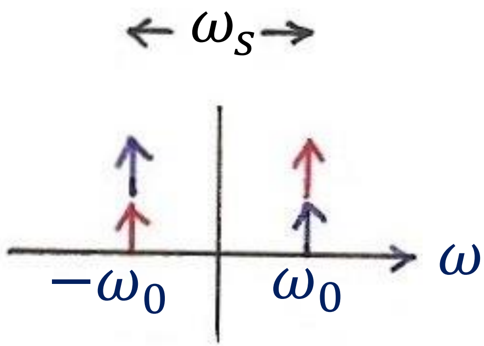
phase also changed

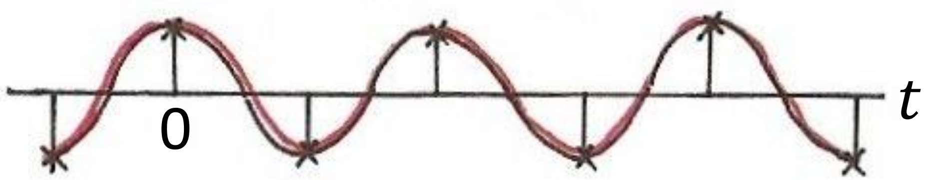
Example 7.1 of Text

$$x(t) = \cos(\omega_0 t + \phi), \;\; \omega_s = 2\omega_0$$

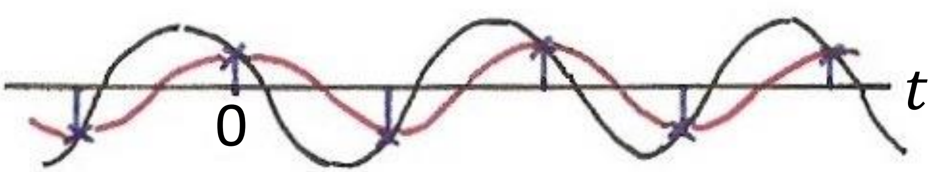
$$x_r(t) = (\cos\phi)\cos(\omega_0 t), \;\; {\color{red} t = nT}$$

Sampling is “time-varying”





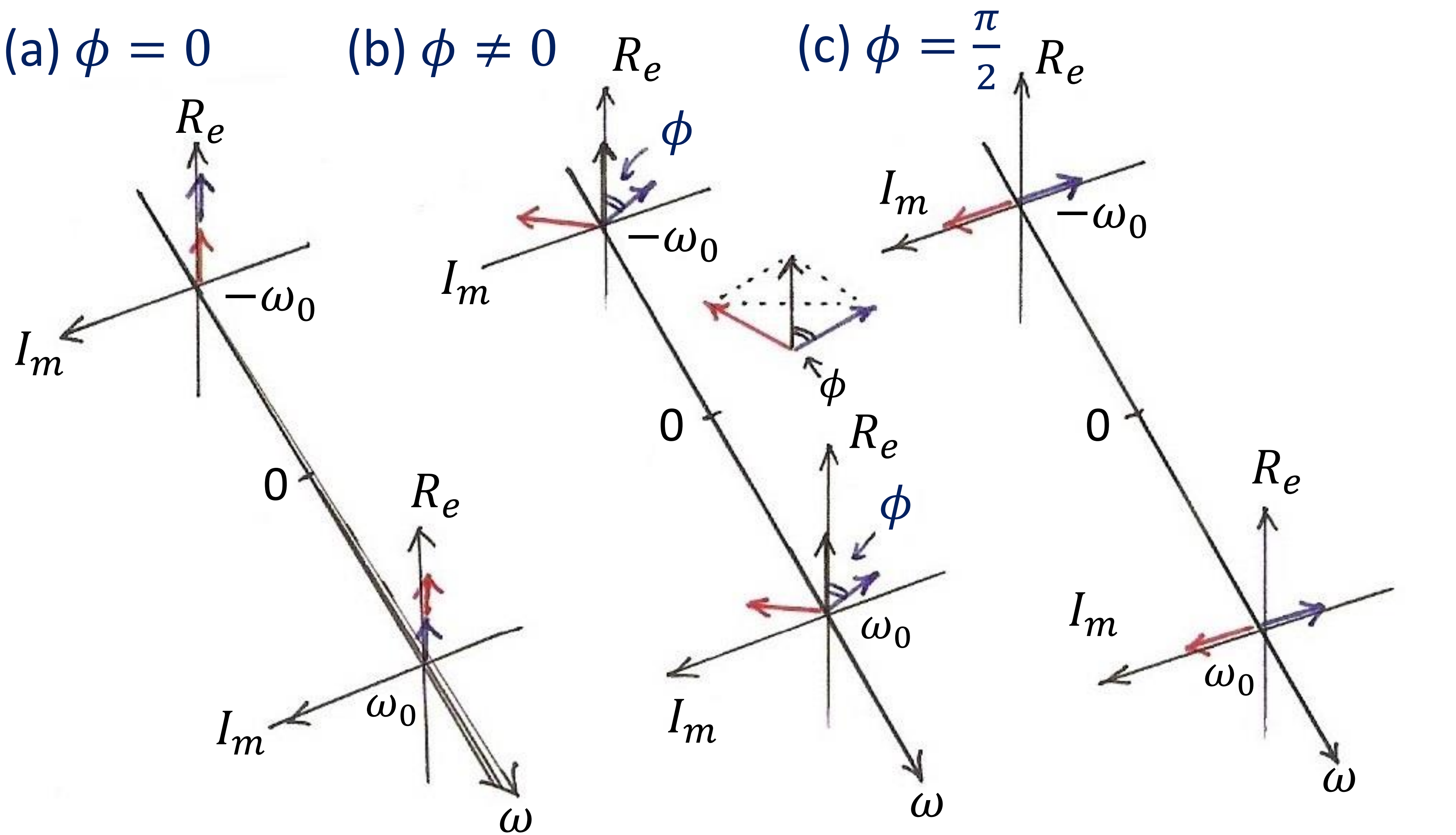
(a) $\phi = 0$



(b) $\phi \neq 0$

(c) $\phi = \frac{\pi}{2}$

Example 7.1 of Text



Example 7.1 of Text

$$x(t)=\cos\left(\frac{\omega_s}{2}t+\phi\right)=\frac{1}{2}\left[e^{j\left(\frac{\omega_s}{2}\overset{\overset{nT}{/\!/}}{t}+\phi\right)}+e^{-j\left(\frac{\omega_s}{2}\overset{\overset{nT}{/\!/}}{t}+\phi\right)}\right]$$

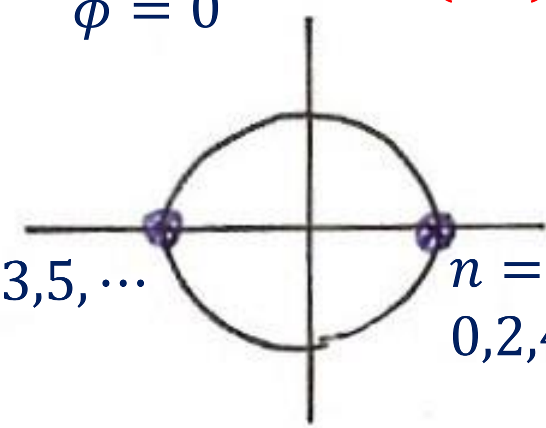
$$t=nT,\quad \frac{\omega_s}{2}\cdot nT=\left(\frac{\cancel{\omega_s}}{\cancel{2}}\right)\cdot n\left(\frac{\cancel{2}\pi}{\cancel{\omega_s}}\right)=n\pi$$

$$x(nT)=\frac{1}{2}\left[e^{jn\pi}\cdot e^{j\phi}+e^{-jn\pi}\cdot e^{-j\phi}\right]$$

$$e^{jn\pi}=e^{-jn\pi}=\pm 1=(e^{j\pi})^n=(e^{-j\pi})^n=(-1)^n$$

$$x(nT)=\frac{1}{2}e^{jn\pi}\left(e^{j\phi}+e^{-j\phi}\right)=\underset{\underset{(-1)^n}{\parallel}}{\underline{\left(e^{jn\pi}\right)}}\cdot(\cos\phi)$$

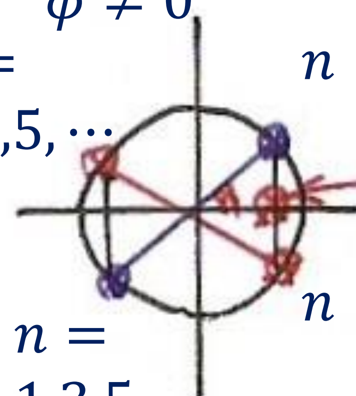
$\phi=0$



$n=1,3,5,\cdots$

$n=0,2,4,\cdots$

$\phi\neq 0$



$n=1,3,5,\cdots$

$n=0,2,4,\cdots$

$\cos\phi$

$n=0,2,4,\cdots$

$n=1,3,5,\cdots$

Examples

- Example 7.1, p.532 of text

  (Problem 7.39, p.571 of text)

$$x(t) = \cos\left(\tfrac{\omega_s}{2} t + \phi\right)$$

$$= (\cos\phi)\cos\left(\tfrac{\omega_s}{2} t\right) - (\sin\phi)\sin\left(\tfrac{\omega_s}{2} t\right)$$

$$\mathrm{t = nT,}\quad \tfrac{\omega_s}{2}(nT) = \tfrac{\omega_s}{2}\cdot n \cdot \tfrac{2\pi}{\omega_s} = n\pi$$

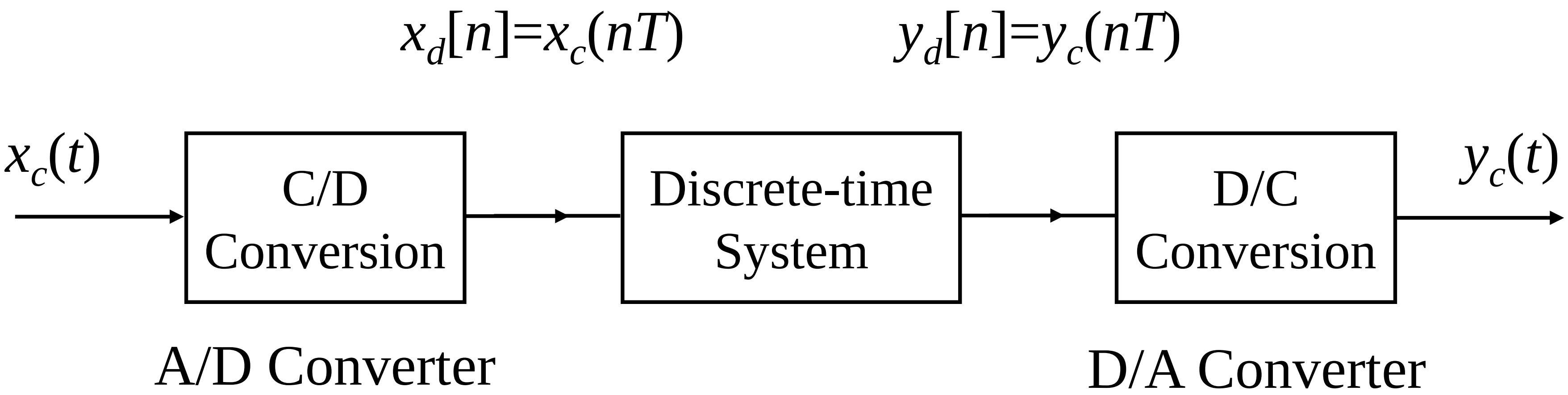
$$x(nT) = (\cos\phi)\cos(n\pi) = (-1)^n(\cos\phi)$$

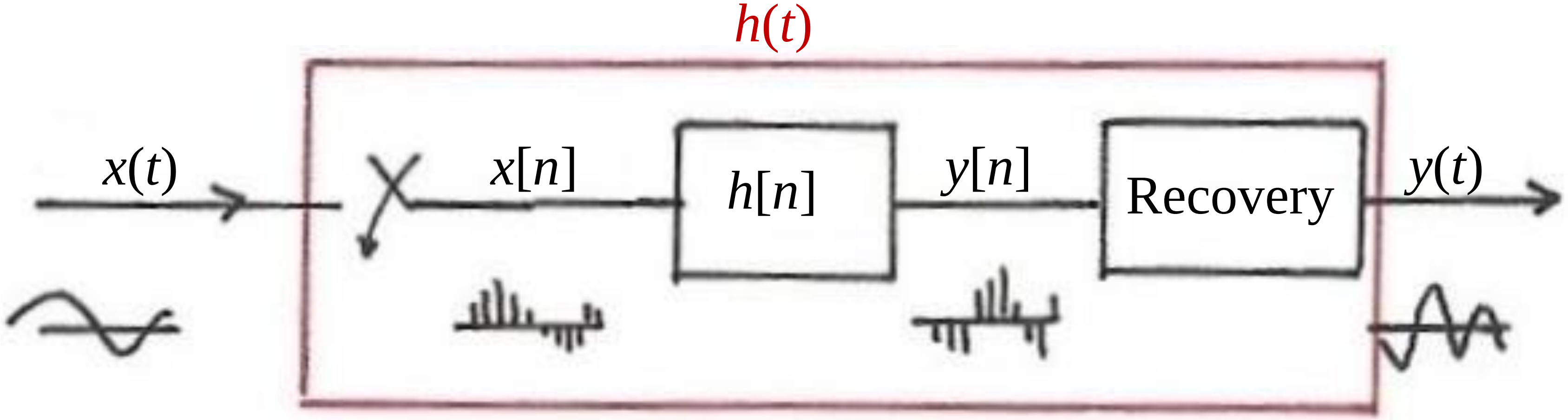
sampled and low-pass filtered

$$x_r(t) = (\cos\phi)\cos\left(\tfrac{\omega_s}{2} t\right)$$

7.2 Discrete-time Processing of Continuous-time Signals

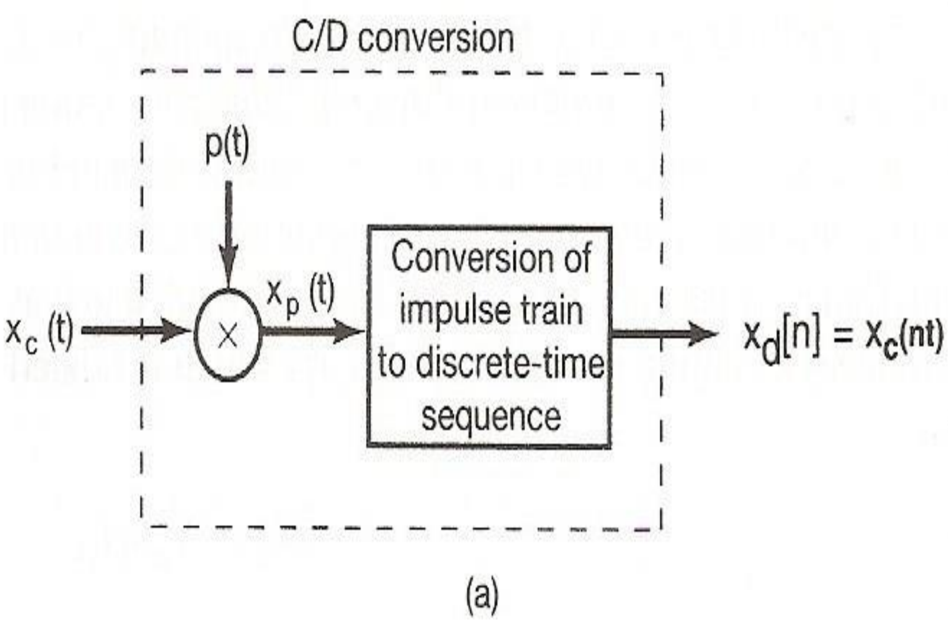
- Processing continuous-time signals digitally





Formal Formulation/Analysis

- C/D Conversion
  - (1) impulse train sampling with sampling period *T*
  - (2) mapping the impulse train to a sequence with unity spacing
    - – normalization (or scaling) in time



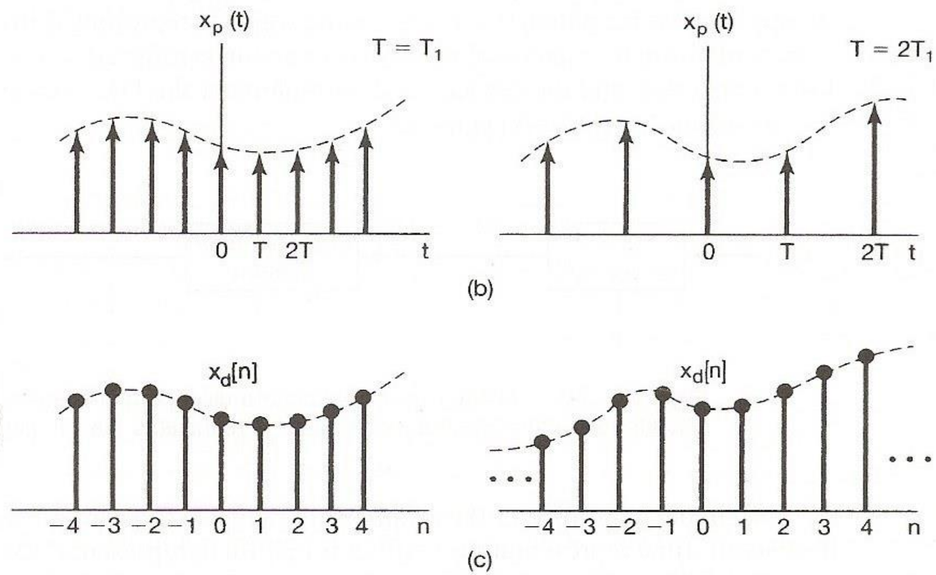


Figure 7.21 Sampling with a periodic impulse train followed by conversion to a discrete-time sequence: (a) overall system; (b) $x_p(t)$ for two sampling rates. The dashed envelope represents $x_c(t)$; (c) the output sequence for the two different sampling rates.

Formal Formulation/Analysis

- Frequency Domain Representation

  ω for continuous-time, Ω for discrete-time, only in this section

$$x_c(t),\ y_c(t) \xleftrightarrow{F} X_c(j\omega),\ Y_c(j\omega)$$

$$x_d[n],\ y_d[n] \xleftrightarrow{F} X_d\left(e^{j\Omega}\right),\ Y_d\left(e^{j\Omega}\right)$$

Formal Formulation/Analysis

- Frequency Domain Relationships
  - continuous-time

$$x_p(t) = \sum_{k=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

$$X_p(j\omega) = \sum_{k=-\infty}^{\infty} x_c(nT)e^{-j\omega nT} \qquad (4.9)$$

  - discrete-time

$$x_d[n] = x_c(nT)$$

$$X_d\left(e^{j\Omega}\right) = \sum_{k=-\infty}^{\infty} x_c(nT)e^{-j\Omega n} \qquad (5.9)$$

Formal Formulation/Analysis

- Frequency Domain Relationships
  - relationship

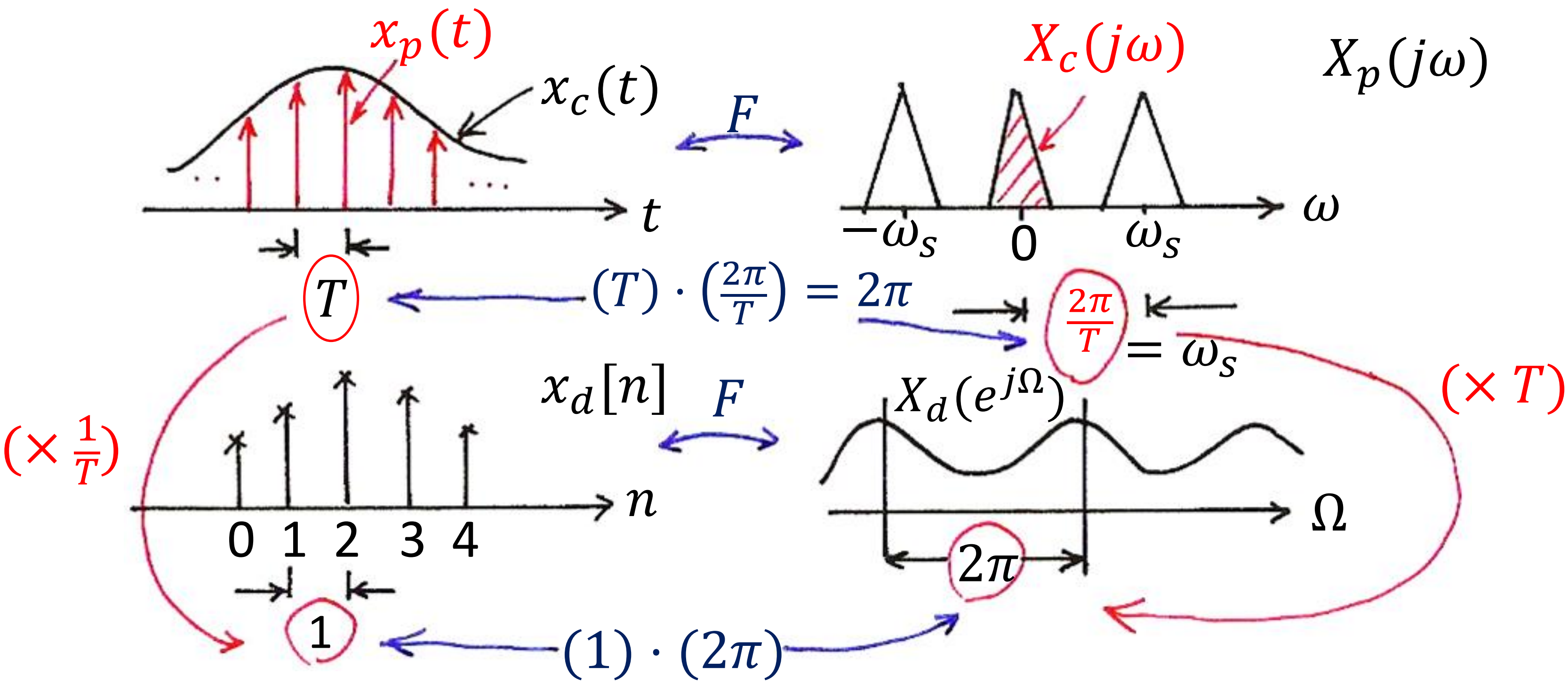
$$X_d\left(e^{j\Omega}\right) = X_p\left(j\Omega/T\right),\ \omega = \frac{\Omega}{T}$$

$$X_p\left(j\omega\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\omega - k\omega_s\right)\right) \qquad (7.6)$$

$$X_d\left(e^{j\Omega}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - 2\pi k\right)/T\right)$$

See Fig. 7.22, p.537 of text

C/D Conversion



– $X_d(e^{j\Omega})$ is a frequency-scaled (by T) version of $X_p(j\omega)$

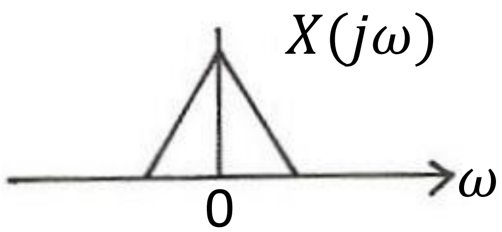
$x_d[n]$ is a time-scaled (by $1/T$) version of $x_p(t)$

– $X_d(e^{j\Omega})$ periodic with period 2π

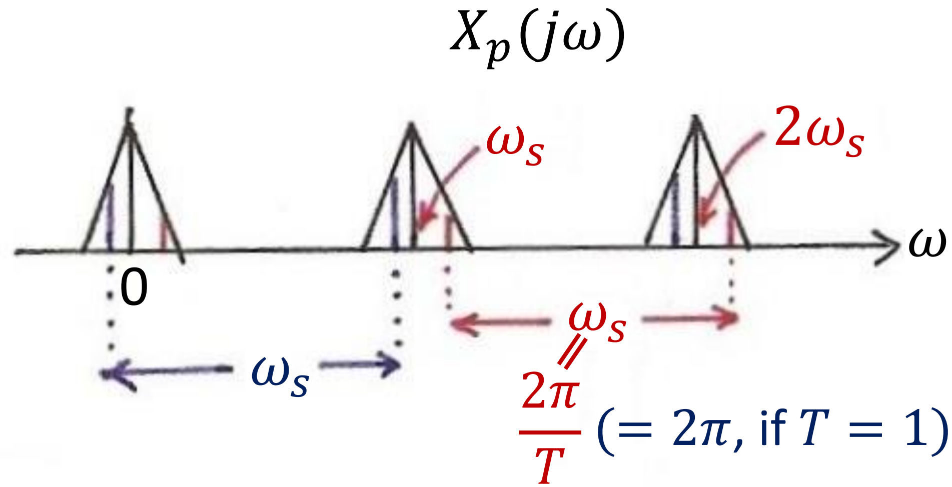
$X_p(j\omega)$ periodic with period $2\pi/T=\omega_s$

Sampling (p.15 of 7.0)

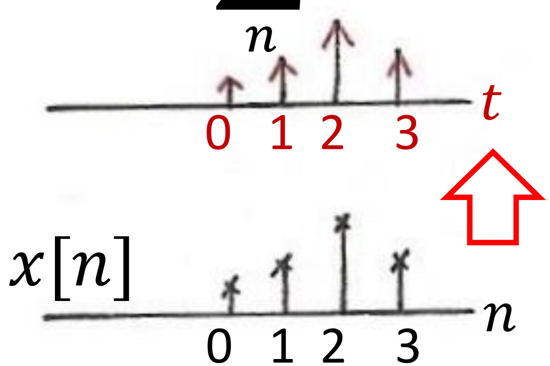
$$x(t) \overset{F}{\leftrightarrow} X(j\omega), \quad \xrightarrow{\text{sampling}} \quad x_p(t) \overset{F}{\leftrightarrow} X_p(j\omega)$$



sampling →



$$x(t) = \sum_n x[n]\,\delta(t-n)$$



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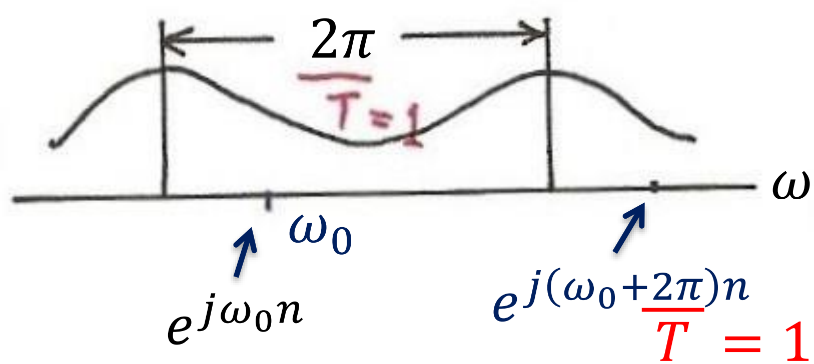


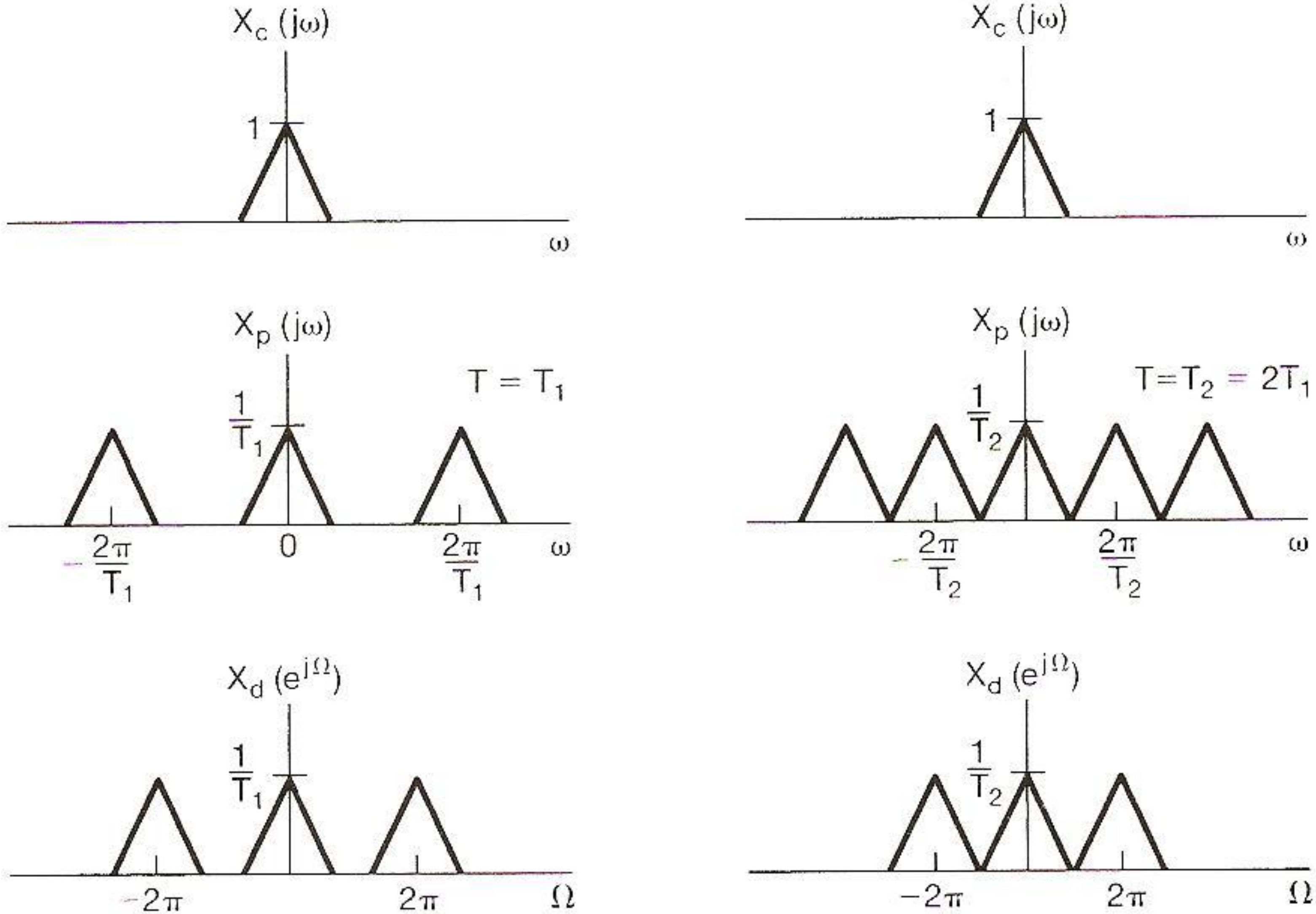


Figure 7.22 Relationship between $X_c(j\omega)$, $X_p(j\omega)$, and $X_d(e^{j\Omega})$ for two different sampling rates.

Formal Formulation/Analysis

- D/C Conversion

  (1) mapping a sequence to an impulse train

  (2) lowpass filtering

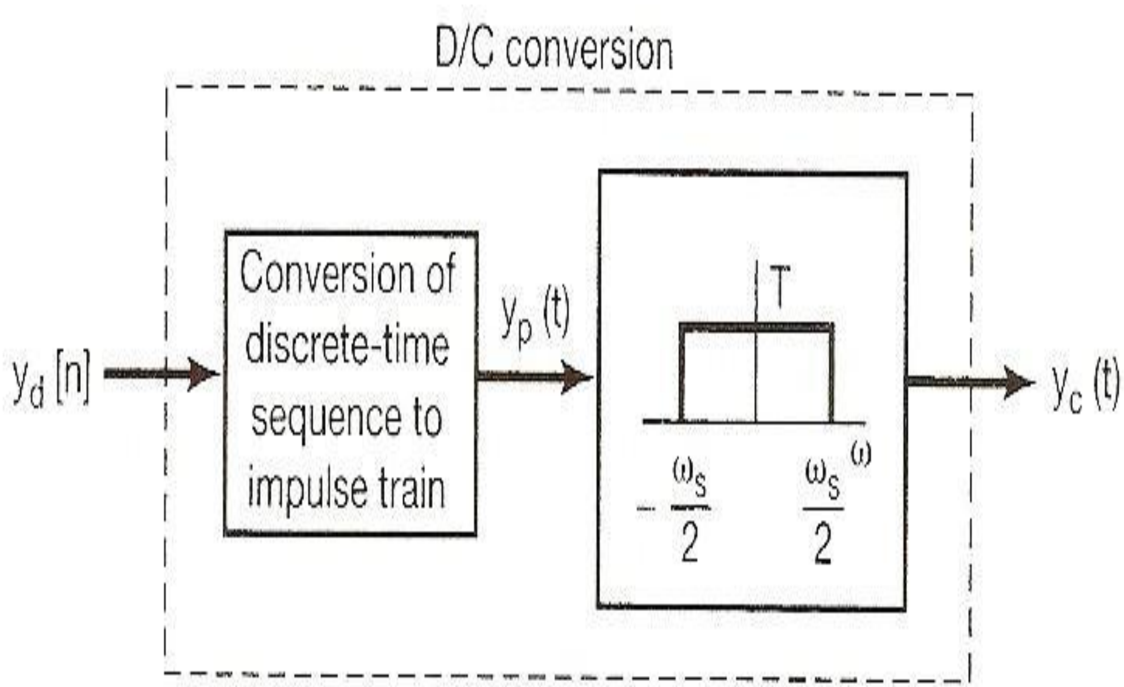


Figure 7.23 Conversion of a discrete-time sequence to a continuous-time signal.

Formal Formulation/Analysis

- Complete System

See Fig. 7.24, 7.25, 7.26, p.538, 539, 540 of text

$$Y_c(j\omega) = X_c(j\omega)H_d\left(e^{j\omega T}\right) \qquad (7.24)$$

equivalent to a continuous-time system

$$H_c(j\omega) = \begin{cases} H_d\left(e^{j\omega T}\right), & |\omega| < \omega_s/2 \\ 0, & |\omega| \geq \omega_s/2 \end{cases} \qquad (7.25)$$

if the sampling theorem is satisfied

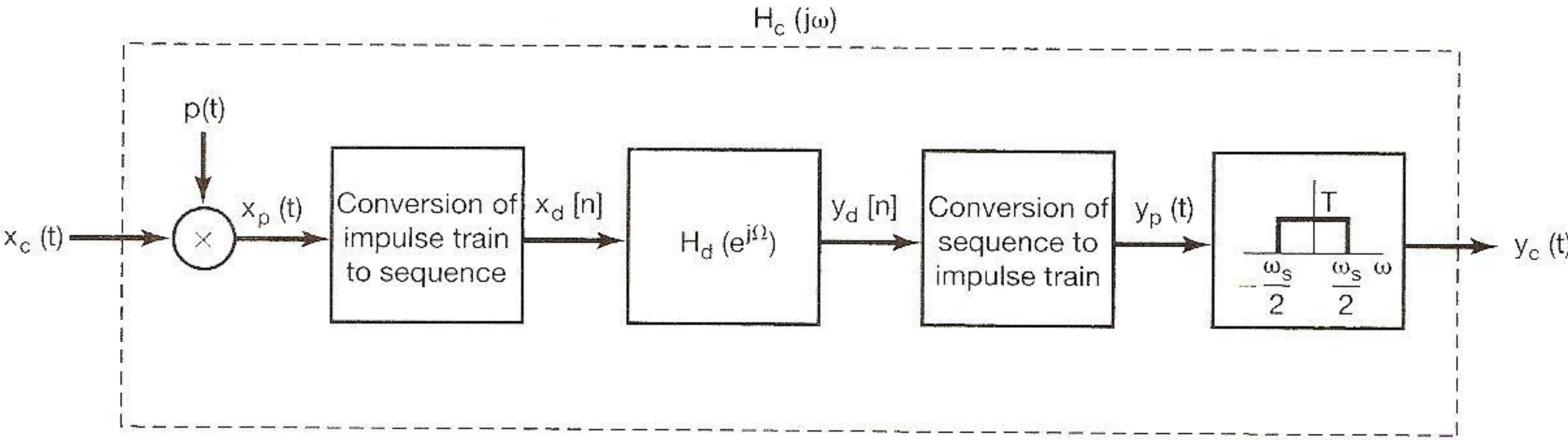
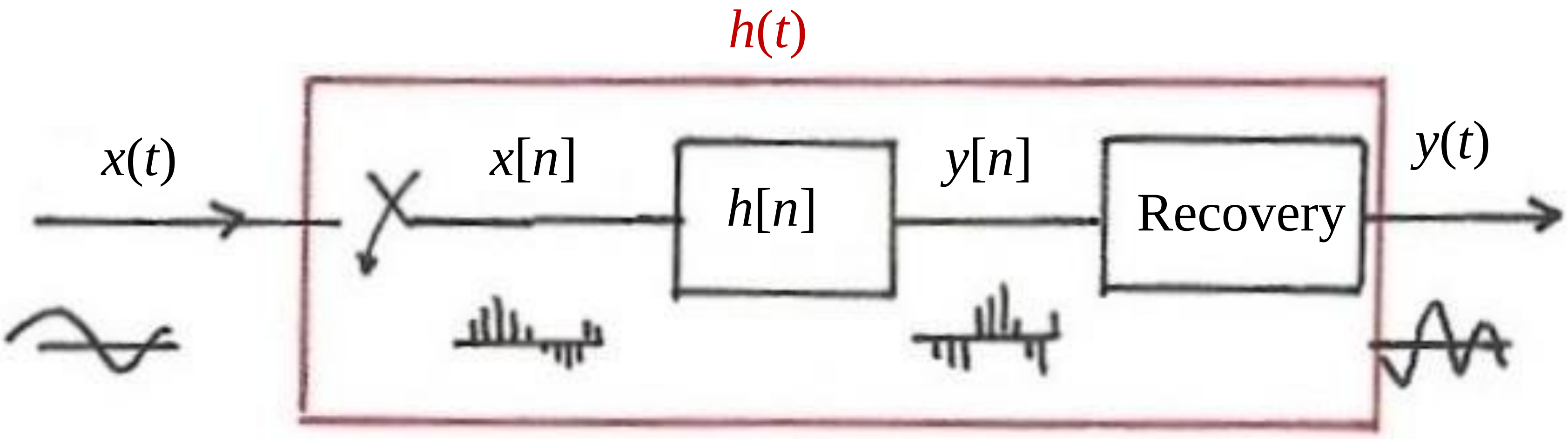


Figure 7.24 Overall system for filtering a continuous-time signal using a discrete-time filter.



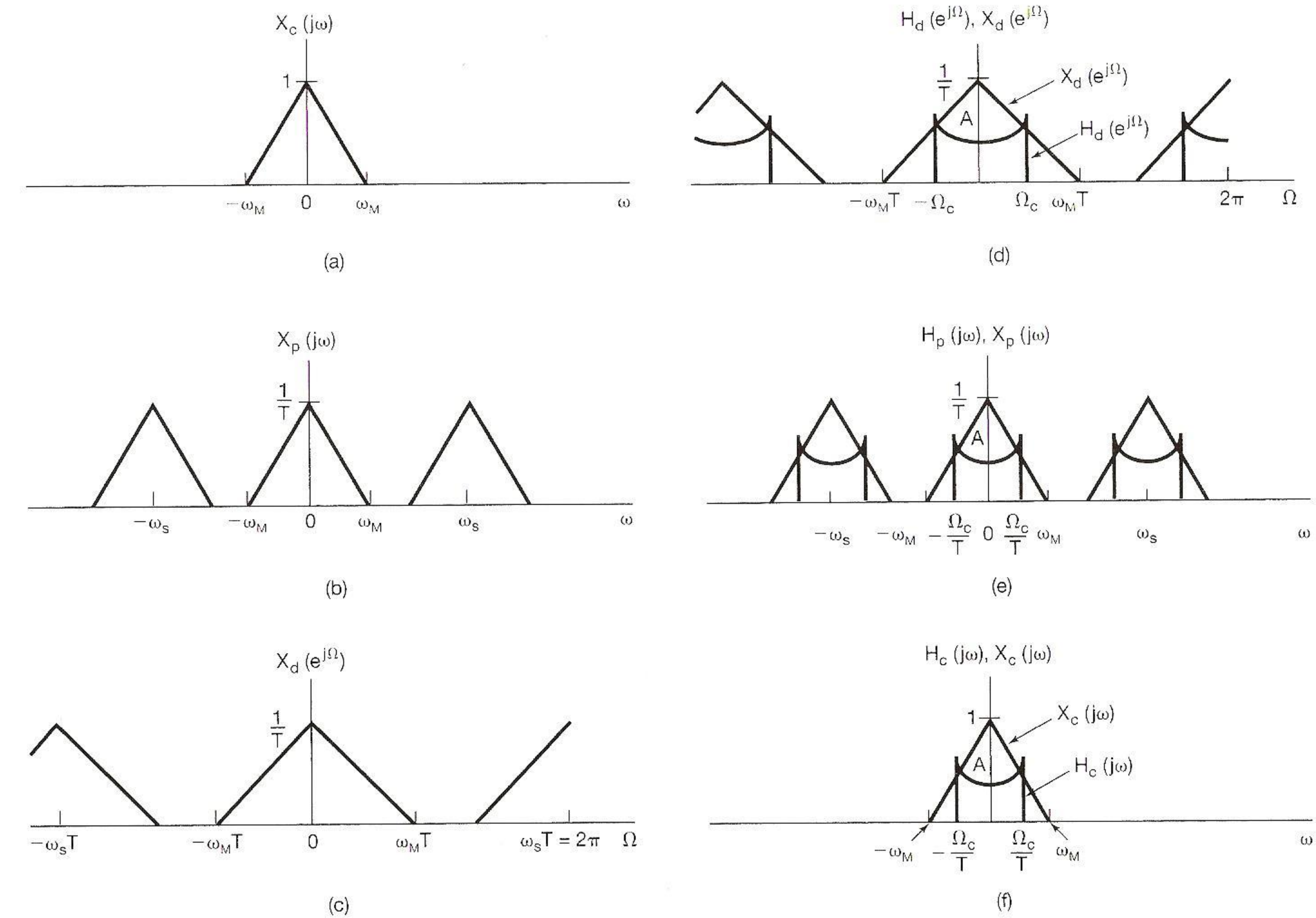


Figure 7.25 Frequency-domain illustration of the system of Figure 7.24: (a) continuous-time spectrum $X_c(j\omega)$; (b) spectrum after impulse-train sampling; (c) spectrum of discrete-time sequence $x_d[n]$; (d) $H_d(e^{j\Omega})$ and $X_d(e^{j\Omega})$ that are multiplied to form $Y_d(e^{j\Omega})$; (e) spectra that are multiplied to form $Y_p(j\omega)$; (f) spectra that are multiplied to form $Y_c(j\omega)$.

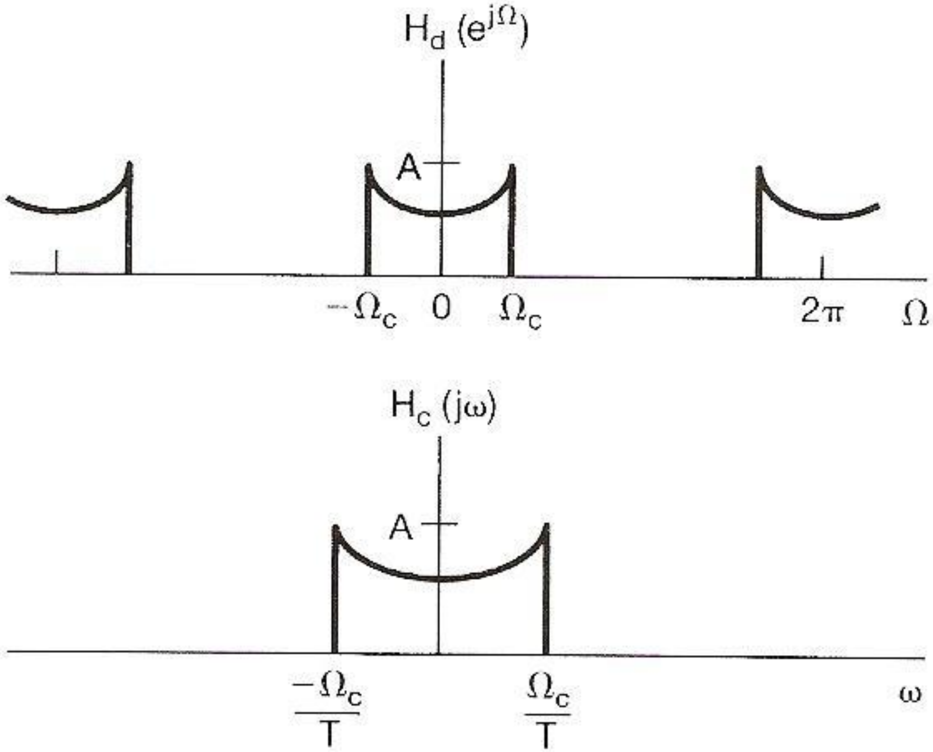


Figure 7.26 Discrete-time frequency response and the equivalent continuous-time frequency response for the system of Figure 7.24.

Discrete-time Processing of Continuous-time Signals

- Note
  - the complete system is linear and time-invariant if the sampling theorem is satisfied
  - sampling process itself is NOT time-invariant

Examples

- Digital Differentiator
  - band-limited differentiator

$$H_c(j\omega) = j\omega, \ |\omega| < \omega_c = \omega_s/2$$
$$0, \ |\omega| \geq \omega_c$$

  - discrete-time equivalent

$$H_d\left(e^{j\Omega}\right) = j(\Omega/T), \ |\Omega| < \pi$$

See Fig. 7.27, 7.28, p.541, 542 of text

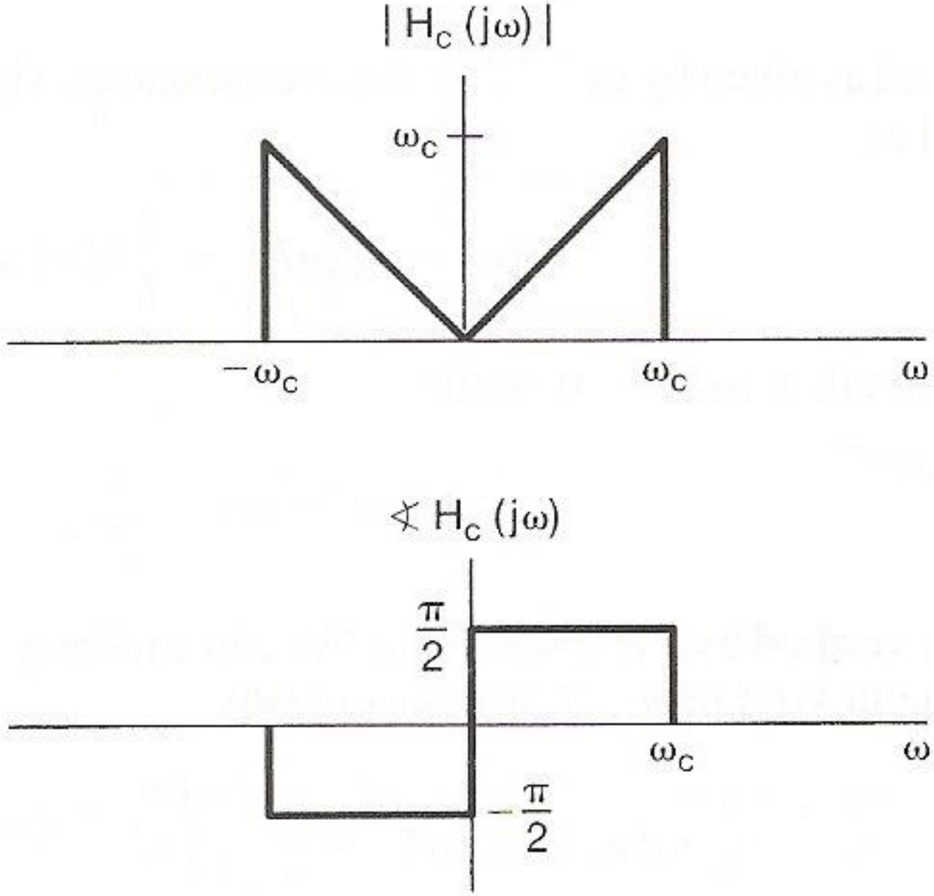


Figure 7.27 Frequency response of a continuous-time ideal band-limited differentiator $H_c(j\omega) = j\omega$, $|\omega| < \omega_c$.

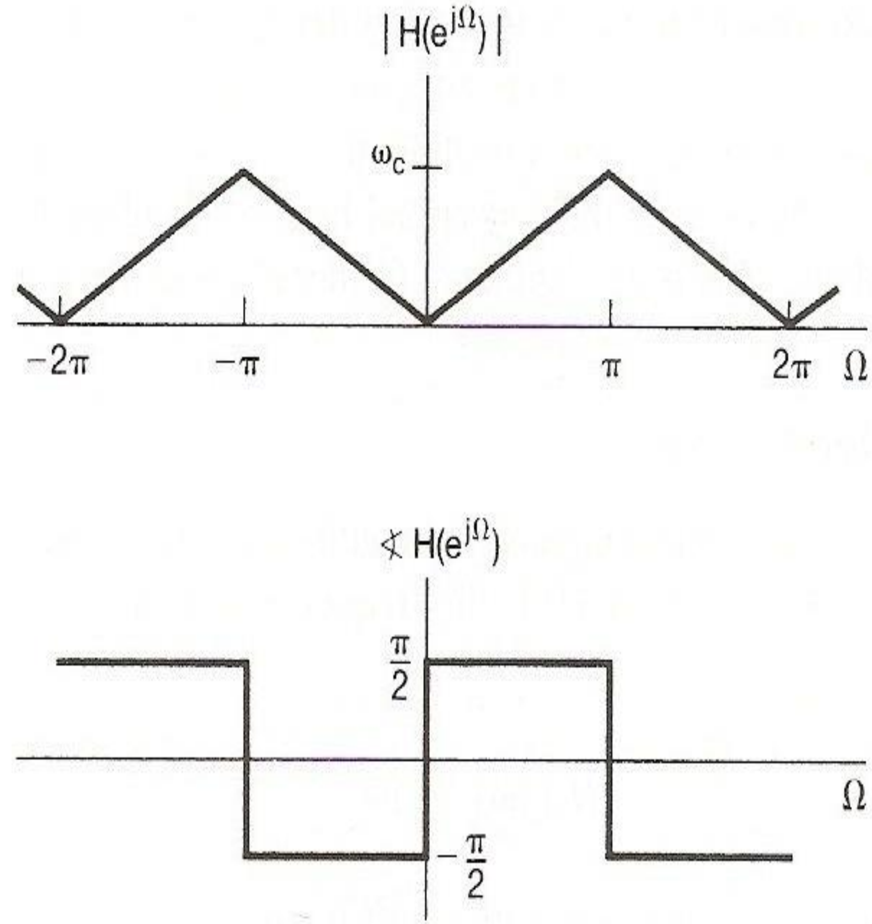


Figure 7.28 Frequency response of discrete-time filter used to implement a continuous-time band-limited differentiator.

Examples

- Delay
  - $y_c(t)=x_c(t\text{-}\Delta)$

$$H_c(j\omega) = e^{-j\omega\Delta}, \ |\omega| < \omega_c = \omega_s/2$$

$$0 \quad , \ |\omega| \geq \omega_c$$

  - discrete-time equivalent

$$H_d\left(e^{j\Omega}\right) = e^{-j\Omega\Delta/T}, \ |\Omega| < \pi$$

See Fig. 7.29, p.543 of text

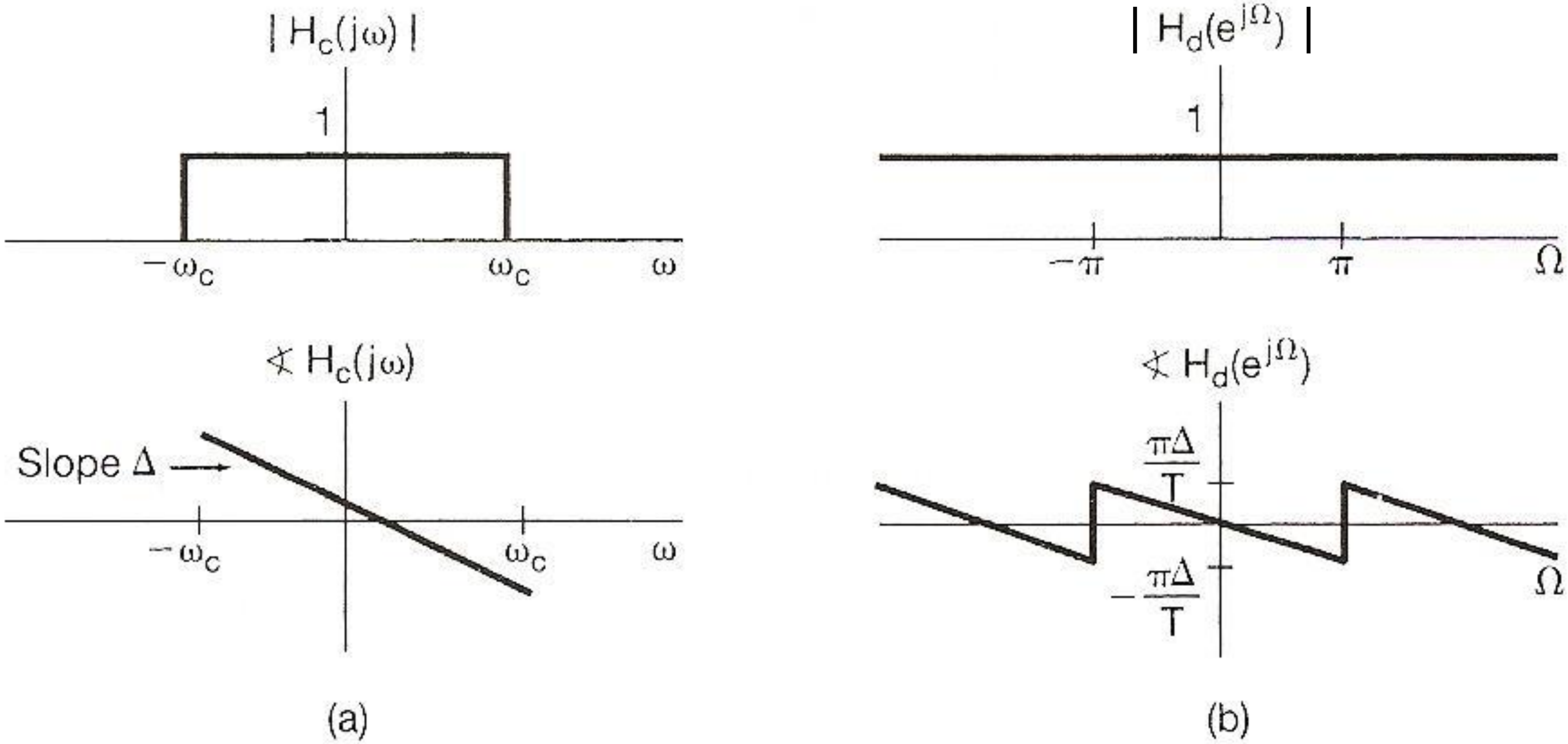


Figure 7.29 (a) Magnitude and phase of the frequency response for a continuous-time delay; (b) magnitude and phase of the frequency response for the corresponding discrete-time delay.

Examples

- Delay
  - Δ/T an integer

    $$y_d[n] = x_d[n - \Delta/T]$$

  - Δ/T not an integer

    $x_d[n - \Delta/T]$ undefined in principle

    but makes sense in terms of sampling if the sampling theorem is satisfied

    e.g. $\Delta/T = 1/2$, half-sample delay

$$y_d[n] = y_c(nT) = x_c\left(nT - \frac{1}{2}T\right)$$

See Fig. 7.30, p.544 of text

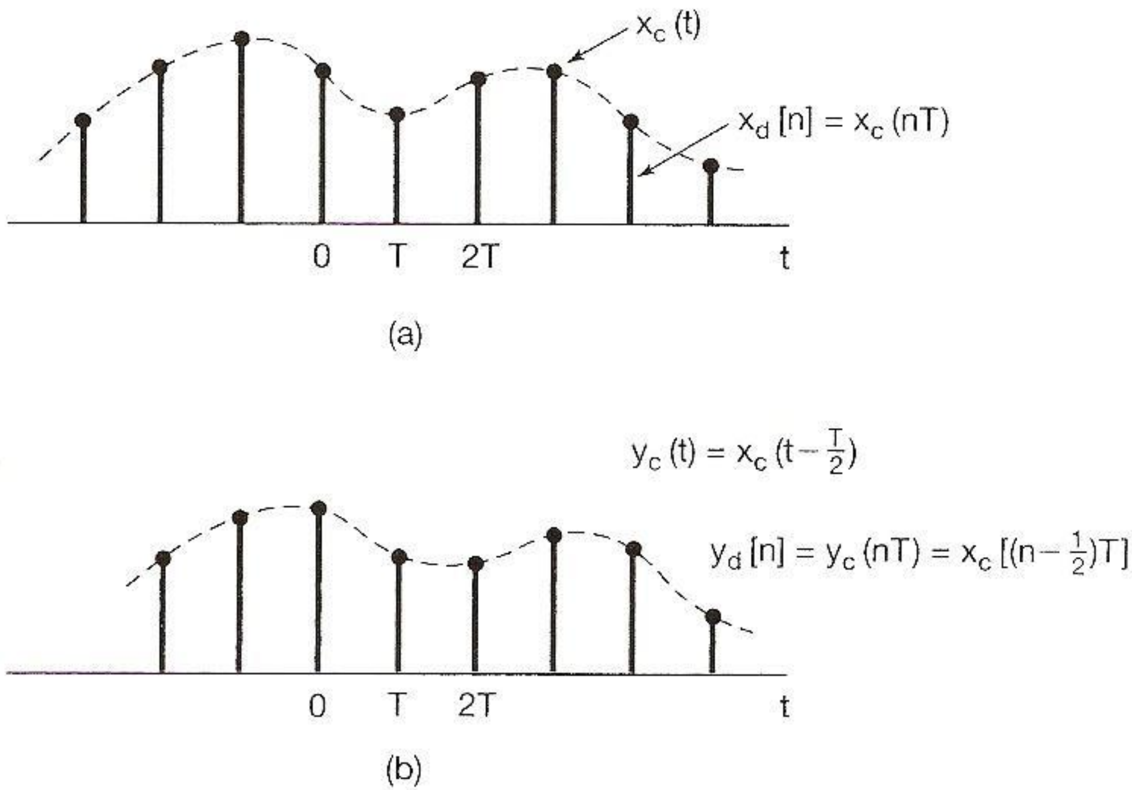
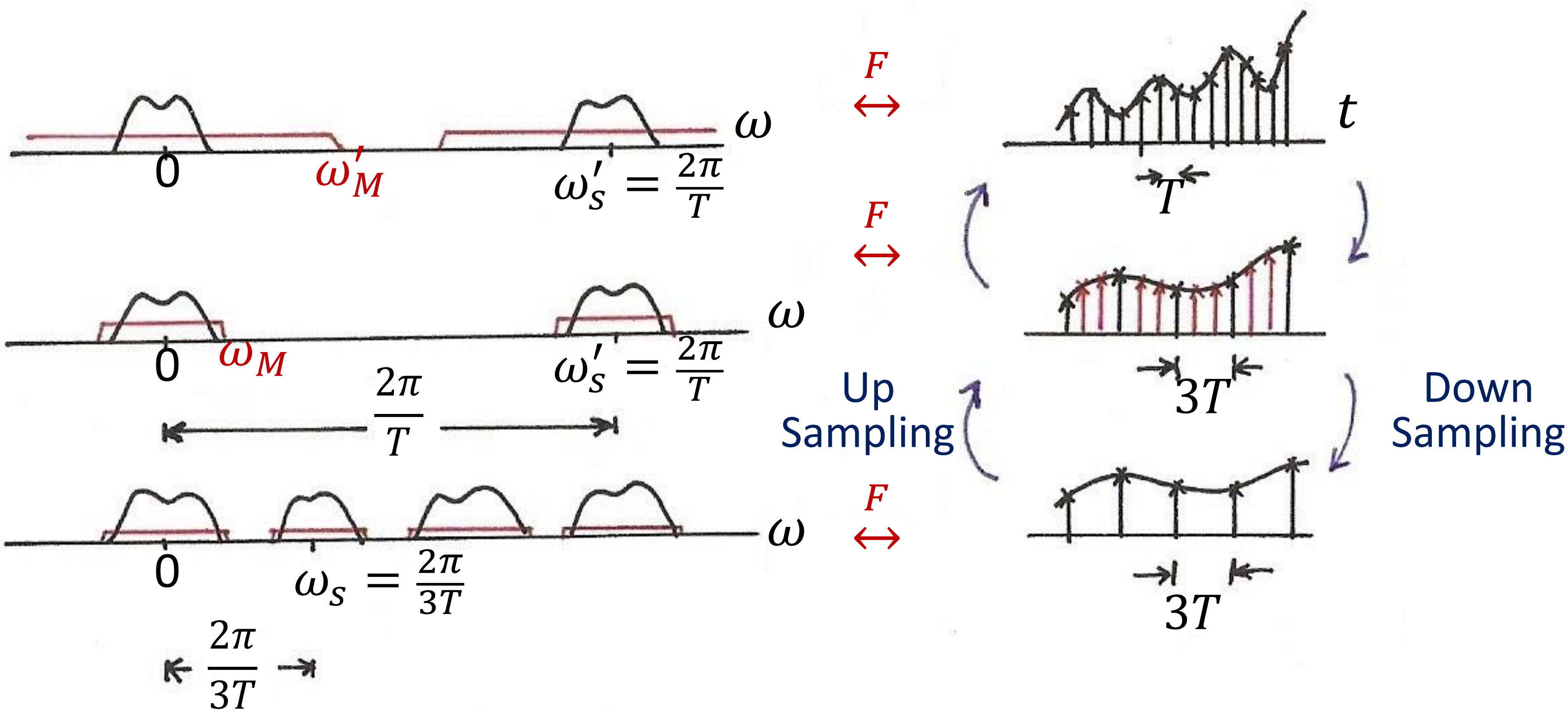


Figure 7.30 (a) Sequence of samples of a continuous-time signal $x_c(t)$; (b) sequence in (a) with a half-sample delay.

7.3 Change of Sampling Frequency

Up/Down Sampling



$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad T: \text{sampling period}$$

$$\omega_s = \frac{2\pi}{T}: \text{sampling frequency}$$

$$x_p(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

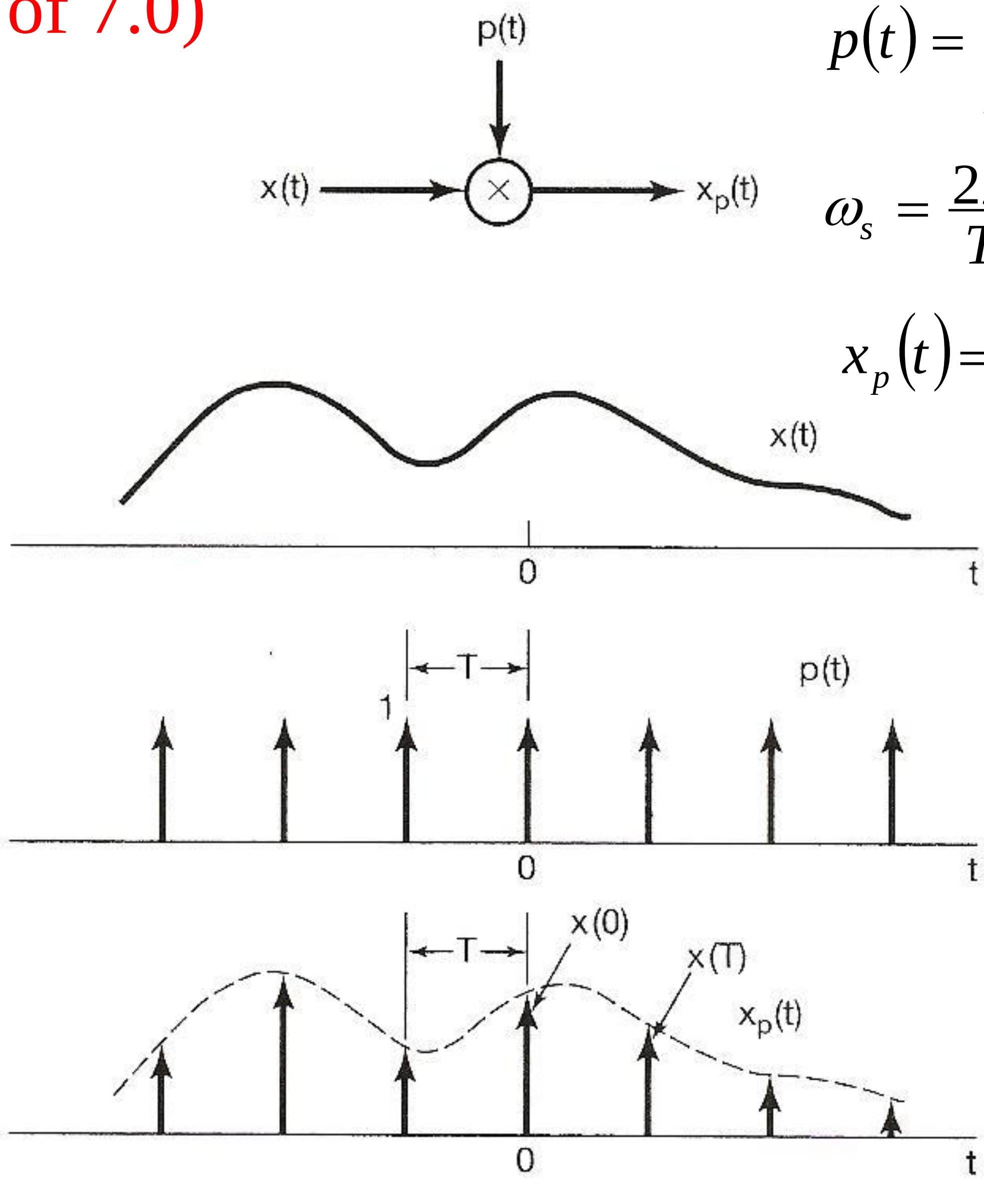


Figure 7.2 Impulse-train sampling.

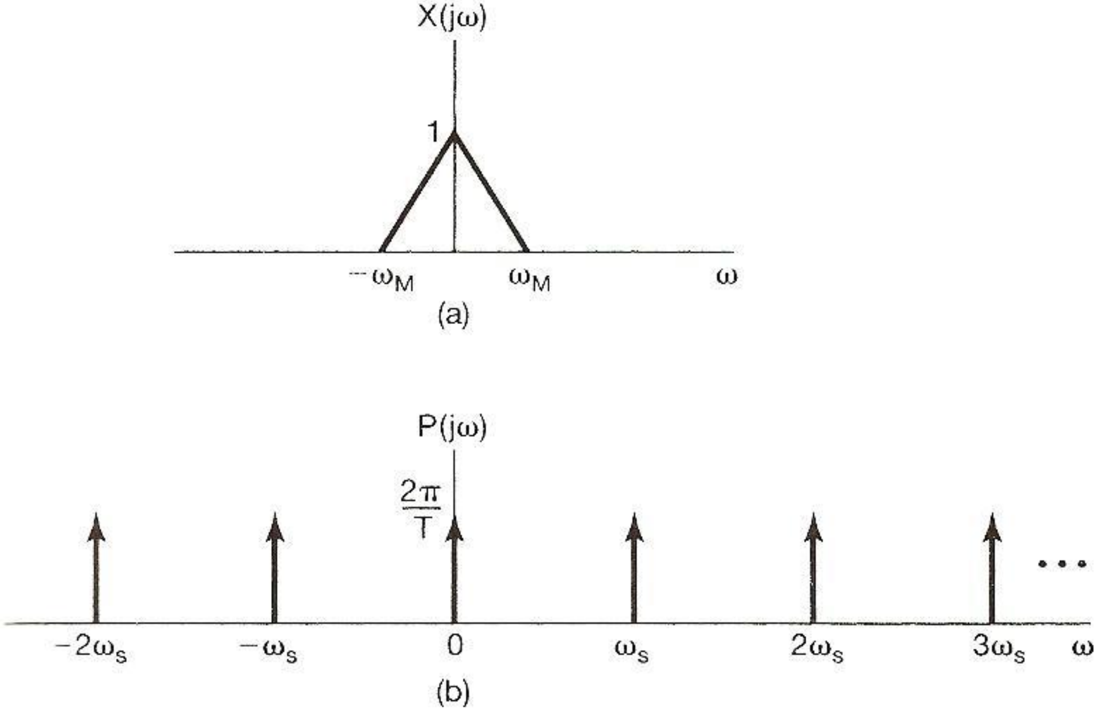


Figure 7.3 Effect in the frequency domain of sampling in the time domain: (a) spectrum of original signal; (b) spectrum of sampling function;

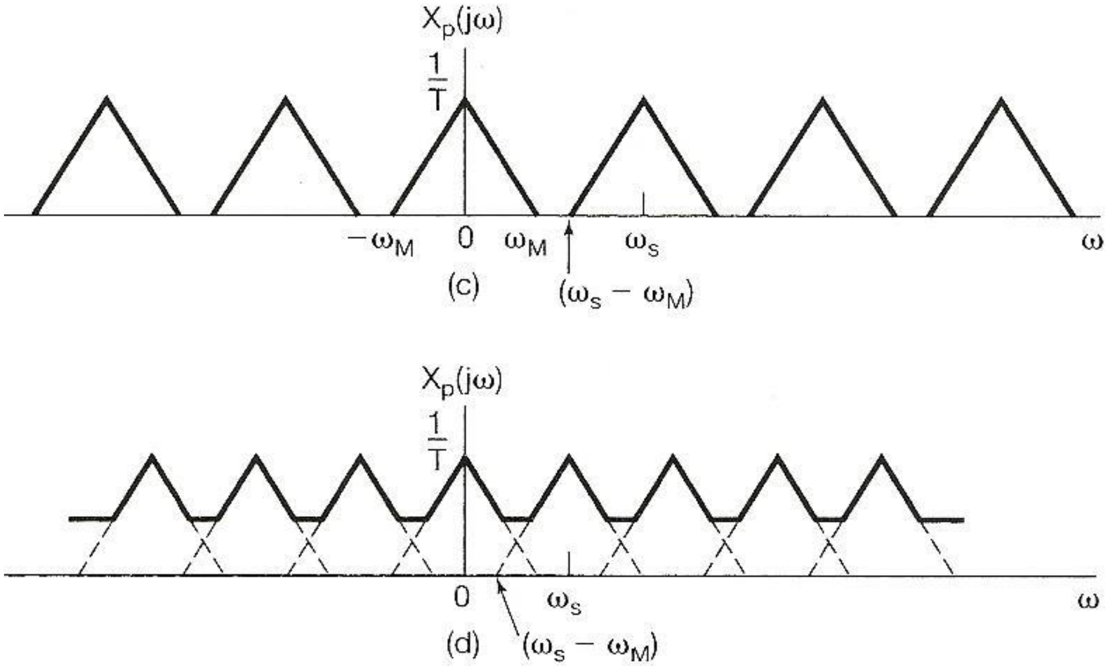


Figure 7.3 *Continued* (c) spectrum of sampled signal with $\omega_s > 2\omega_M$; (d) spectrum of sampled signal with $\omega_s < 2\omega_M$.

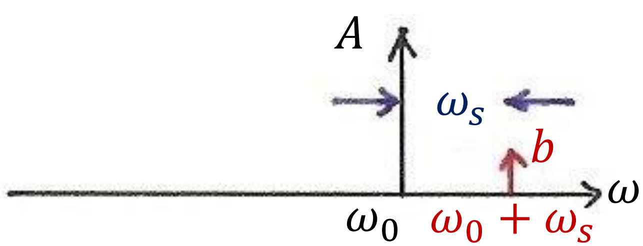
Aliasing Effect (P.13 of 7.0)

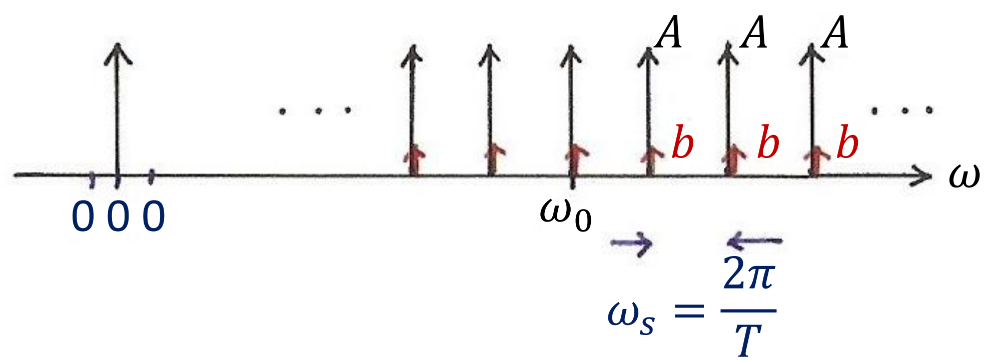
$x(t) = A\cos\omega_0 t$

$y(t) = b\cos(\omega_0 + \omega_s)t, \ \omega_s = \frac{2\pi}{T}$

$x(nT) = A\cos\omega_0 nT$

$$y(nT) = b\cos(\omega_0 + k\frac{2\pi}{T})nT$$
$$= b\cos\omega_0 nT$$





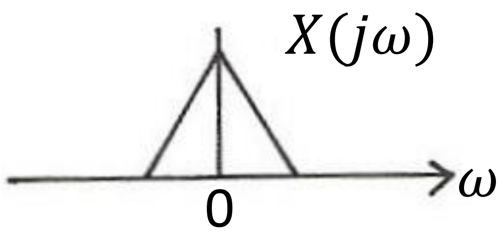
After sampling with $\omega_s = \frac{2\pi}{T}$, any two frequency components ω_1, ω_2 become indistinguishable, or sharing identical samples, or should be considered as identical frequency components if $|\omega_1 - \omega_2| = k\frac{2\pi}{T}$

$$\boxed{e^{j\left(\omega_0 + k\underset{\underset{\omega_s}{\|}}{\frac{2\pi}{T}}\right)nT} = e^{j\omega_0 nT} \qquad (T = 1 \text{ for discrete-time signals })}$$

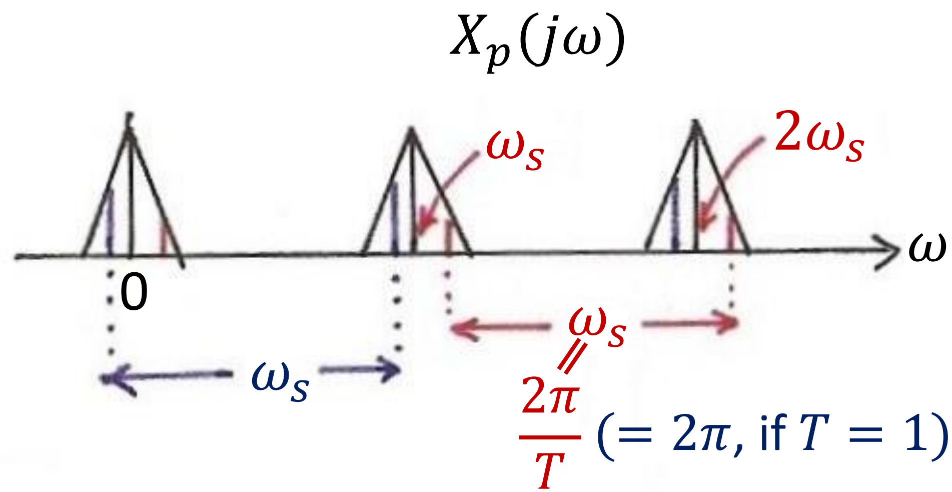
Sampling (P.15 of 7.0)

sampling

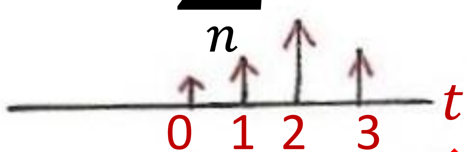
$$x(t) \overset{F}{\leftrightarrow} X(j\omega), \quad x_p(t) \overset{F}{\leftrightarrow} X_p(j\omega)$$

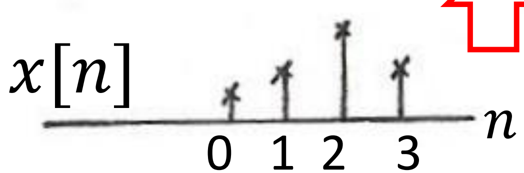


sampling



$$x(t) = \sum_n x[n]\,\delta(t-n)$$





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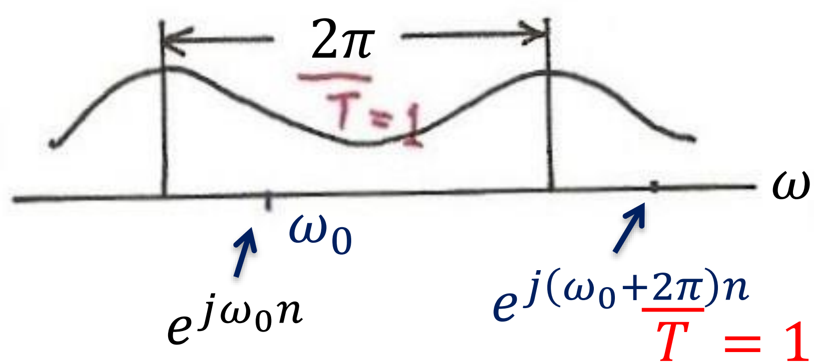
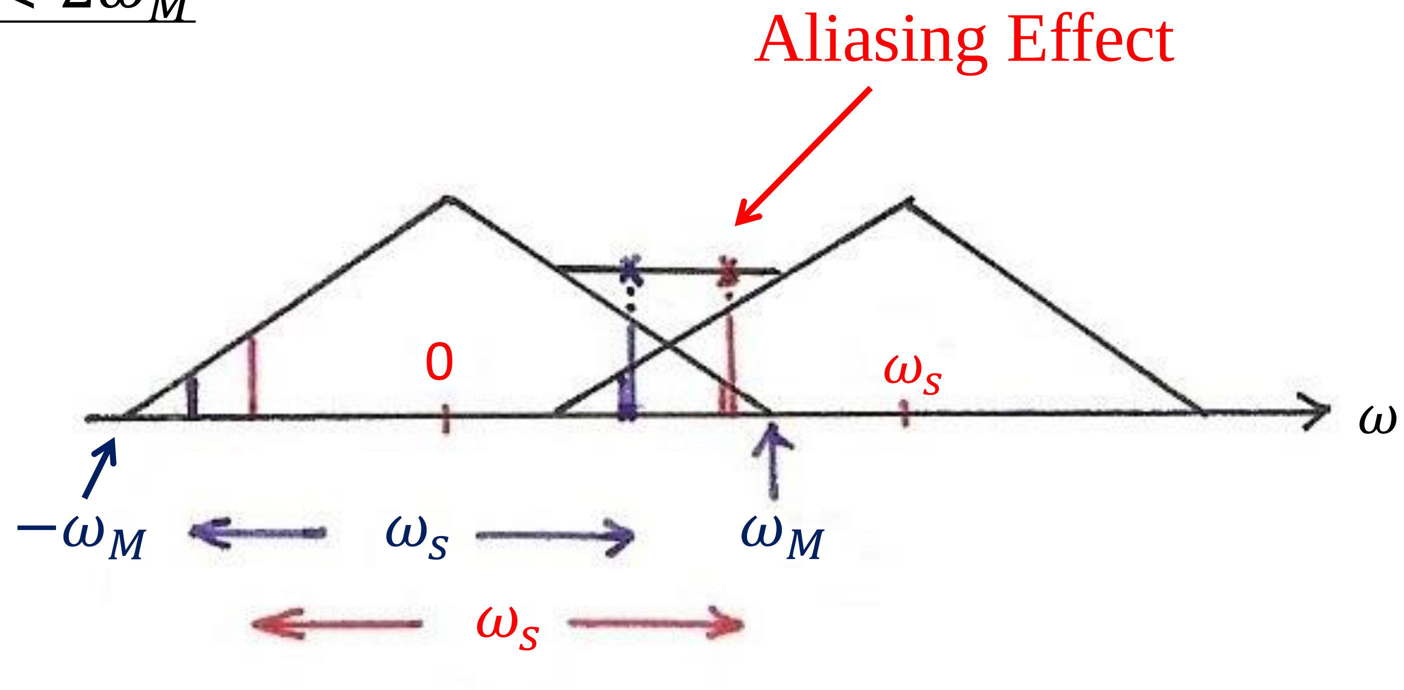


Aliasing Effect (P.16 of 7.0)

$$z(t) = x(t) + y(t) = A\cos\omega_0 t + b\cos(\omega_0 + \omega_s)t$$

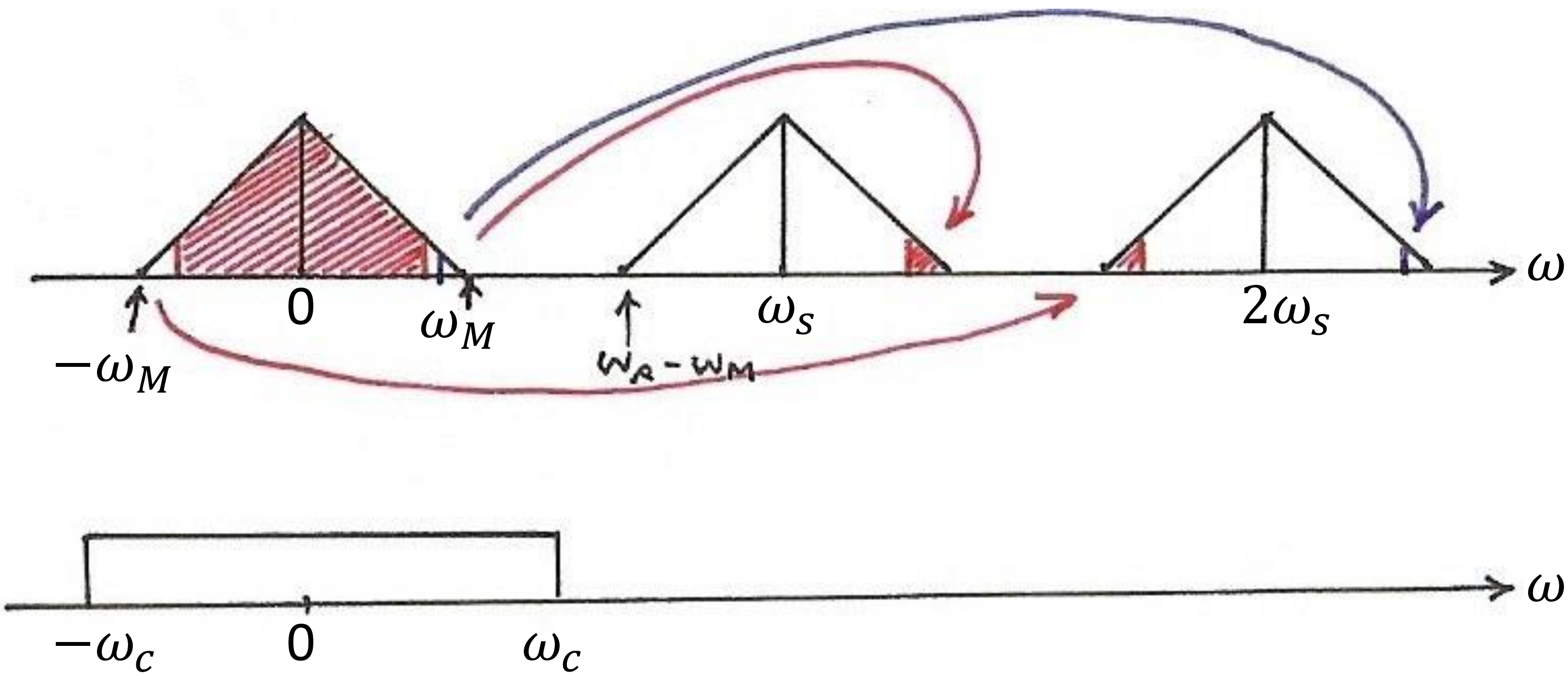
$$z(nT) = x(nT) + y(nT) = (A + b)b\cos\omega_0 nT$$

$\underline{\omega_s < 2\omega_M}$



<u>Sampling Thm</u> (p.17 of 7.0)

$\omega_s > 2\omega_M$



Aliasing for Discrete-time Signals

kN

$x[n] = Ae^{j\omega_0 n}$

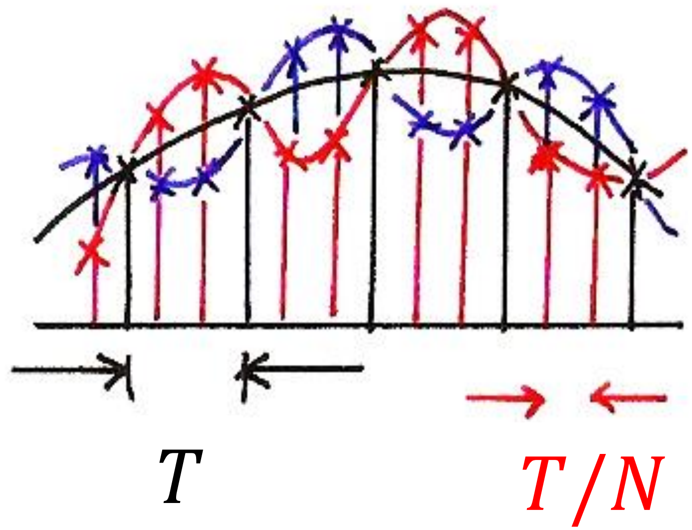
$k\cancel{N}$

$y[n] = be^{j(\omega_0+\frac{2\pi}{\cancel{N}})n}$

$z[n] = x[n] + y[n]$

$z[kN] = (A+b)e^{j\omega_0 kN}$

$$\boxed{e^{j\left(\omega_0+\frac{2\pi}{N}\right)kN} = e^{j\omega_0 kN}}$$



Impulse Train Sampling of Discrete-time Signals

- Completely in parallel with impulse train sampling of continuous-time signals

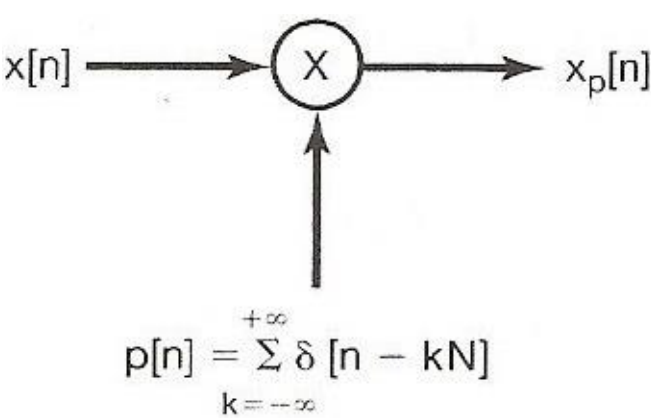
$$p[n] = \sum_{k=-\infty}^{\infty} \delta(n - kN), \quad N : \text{sampling period}$$

$$x_p[n] = x[n]p[n] = \sum_{k=-\infty}^{\infty} x[kN]\delta[n - kN]$$

$$= x[n] \quad \text{if } n \text{ is an integer multiple of } N$$

$$0 \qquad \text{else}$$

See Fig. 7.31, p.546 of text



$$p[n] = \sum_{k=-\infty}^{\infty} \delta(n - kN), \quad N: \text{sampling period}$$

$$x_p[n] = x[n]p[n] = \sum_{k=-\infty}^{\infty} x[kN]\delta[n - kN]$$

$$\begin{aligned} &= x[n] \quad &&\text{if } n \text{ is an integer multiple of } N \\ &\;\; 0 &&\text{else} \end{aligned}$$

See Fig. 7.31, p.546 of text

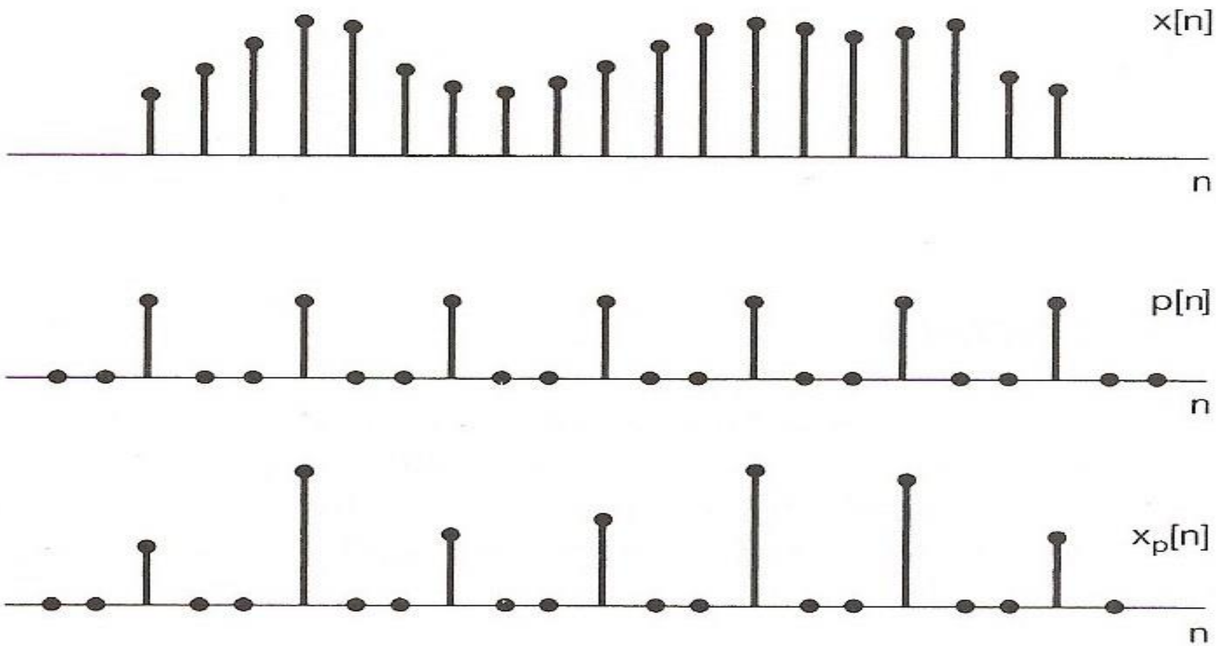


Figure 7.31 Discrete-time sampling.

Impulse Train Sampling of Discrete-time Signals

- Completely in parallel with impulse train sampling of continuous-time signals

$$X_p\left(e^{j\omega}\right)=\frac{1}{2\pi}\int_{2\pi}P\left(e^{j\theta}\right)X\left(e^{j(\omega-\theta)}\right)d\theta$$

$$P\left(e^{j\omega}\right)=\frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta\left(\omega-k\omega_s\right)$$

$$\omega_s=\frac{2\pi}{N}:\text{sampling frequency}$$

$$X_p\left(e^{j\omega}\right)=\frac{1}{N}\sum_{k=0}^{N-1}X\left(e^{j(\omega-k\omega_s)}\right)$$

See Fig. 7.32, p.547 of text

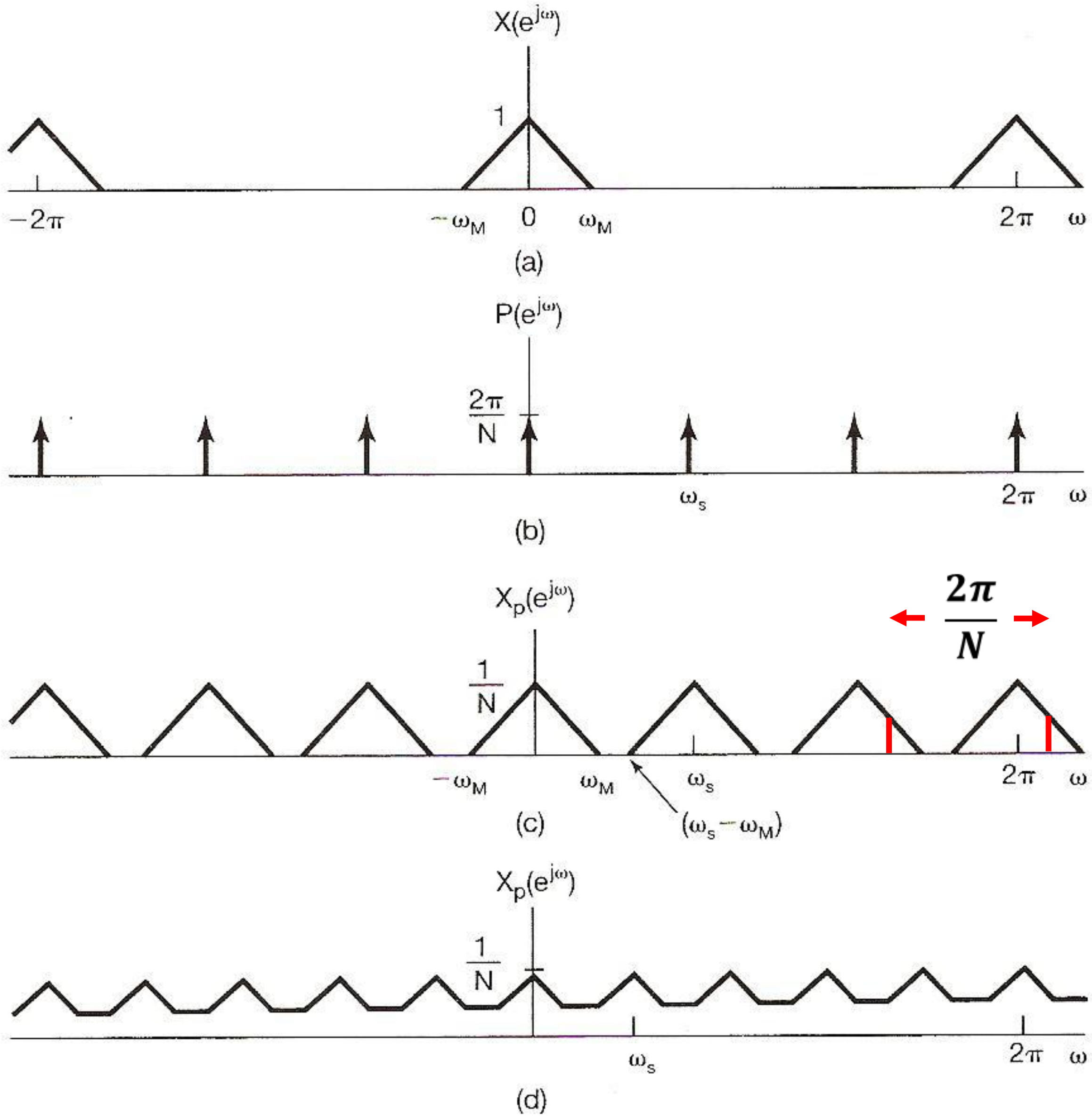


Figure 7.32 Effect in the frequency domain of impulse-train sampling of a discrete-time signal: (a) spectrum of original signal; (b) spectrum of sampling sequence; (c) spectrum of sampled signal with $\omega_s > 2\omega_M$; (d) spectrum of sampled signal with $\omega_s < 2\omega_M$. Note that aliasing occurs.

Impulse Train Sampling of Discrete-time Signals

- Completely in parallel with impulse train sampling of continuous-time signals
  - $\omega_s > 2\omega_M$, no aliasing, $\omega_s = \frac{2\pi}{N}$

    $x[n]$ can be exactly recovered from $x_p[n]$ by a lowpass filter

    With Gain N and cutoff frequency $\omega_M < \omega_c < \omega_s - \omega_M$

    See Fig. 7.33, p.548 of text
  - $\omega_s < 2\omega_M$, aliasing occurs

    filter output $x_r[n] \neq x[n]$

    but $x_r[kN] = x[kN]$, $k=0, \pm 1, \pm 2, \ldots\ldots$

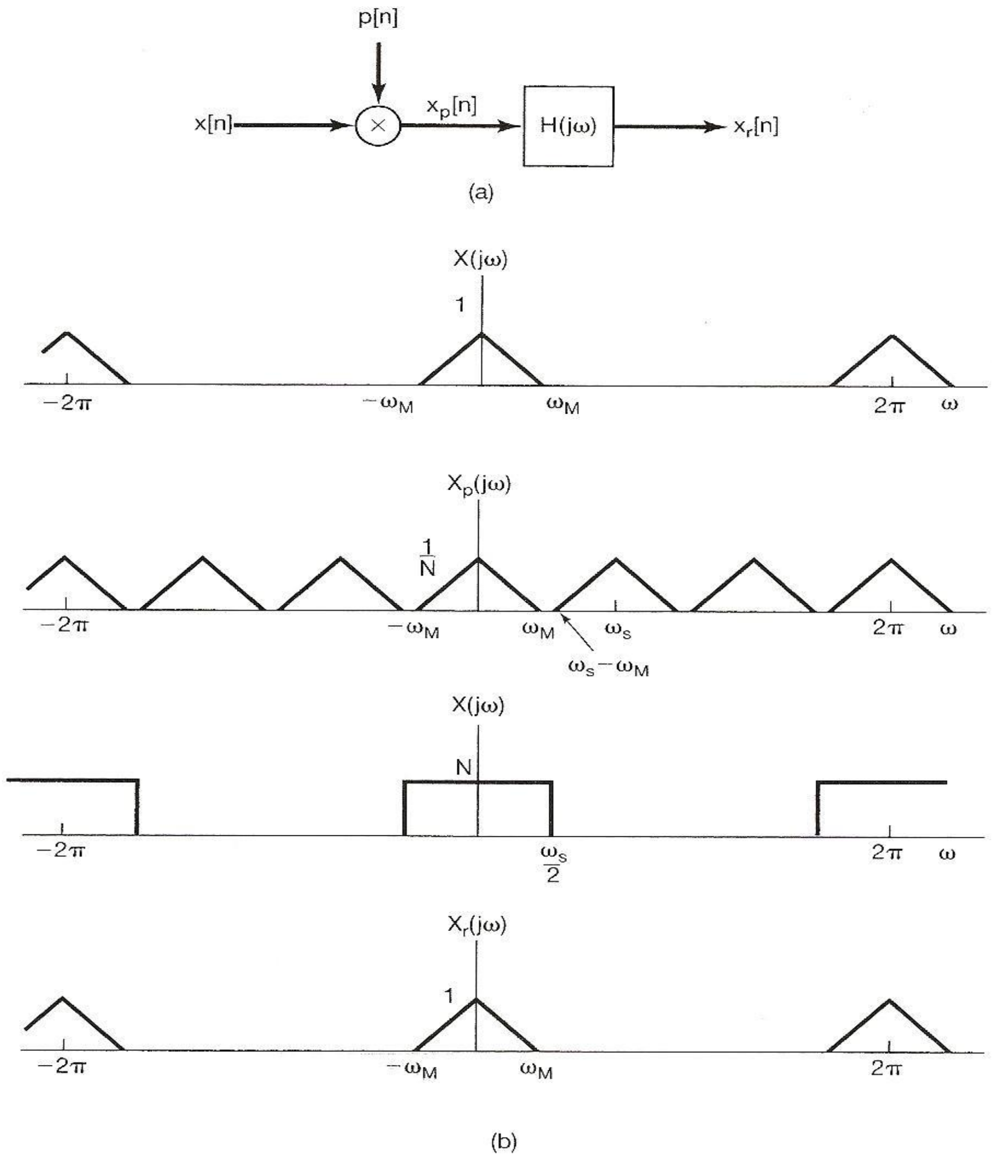


Figure 7.33 Exact recovery of a discrete-time signal from its samples using an ideal lowpass filter: (a) block diagram for sampling and reconstruction of a band-limited signal from its samples; (b) spectrum of the signal $x[n]$; (c) spectrum of $x_p[n]$; (d) frequency response of an ideal lowpass filter with cutoff frequency $\omega_s/2$; (e) spectrum of the reconstructed signal $x_r[n]$. For the example depicted here $\omega_s > 2\omega_M$ so that no aliasing occurs and consequently $x_r[n] = x[n]$.

Impulse Train Sampling of Discrete-time Signals

- Interpolation
  - $h[n]$: impulse response of the lowpass filter

$$h[n] = \frac{N\omega_c}{\pi}\frac{\sin \omega_c n}{\omega_c n}$$

$$x_r[n] = x_p[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[kN]\frac{N\omega_c}{\pi}\frac{\sin \omega_c(n-kN)}{\omega_c(n-kN)}$$

  - in general a practical filter $h_r[n]$ is used

$$x_r[n] = x_p[n] * h_r[n]$$

$$= \sum_{k=-\infty}^{\infty} x[kN]h_r[n-kN]$$

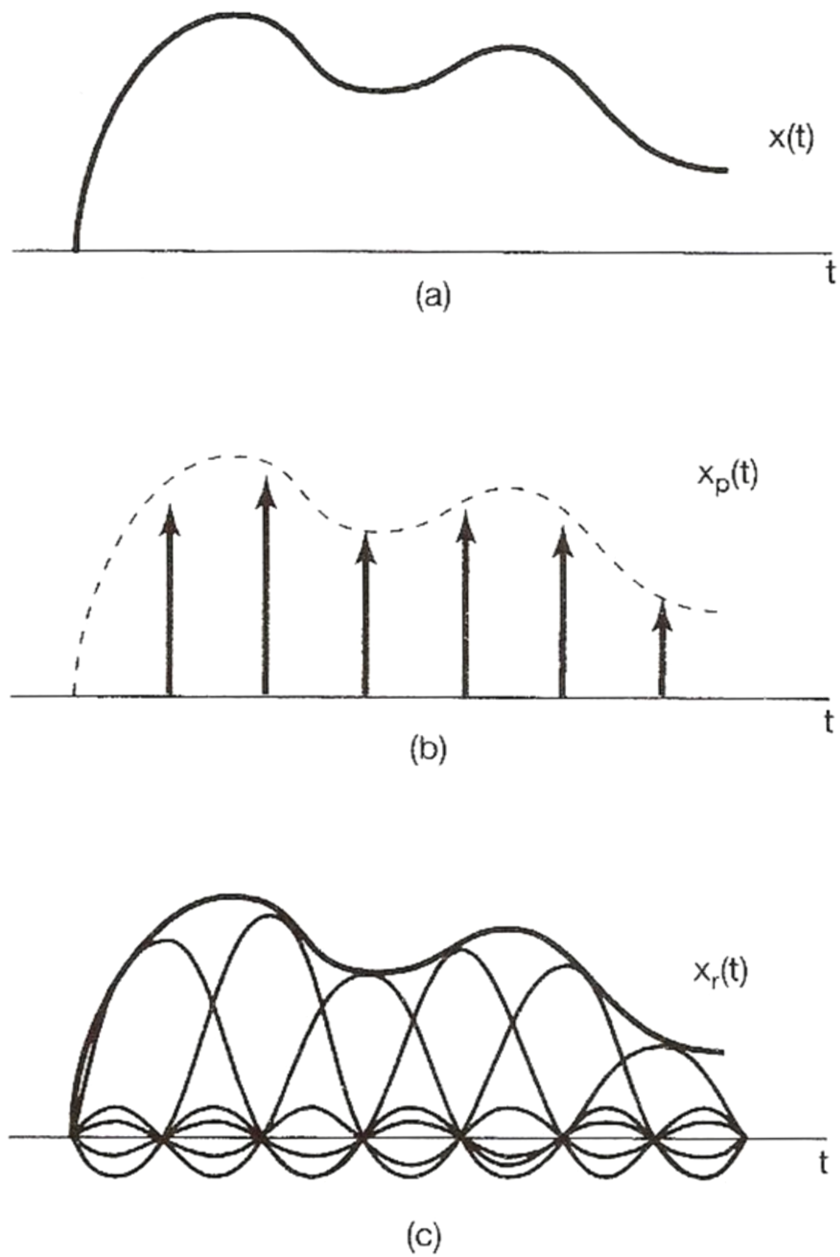


Figure 7.10 Ideal band-limited interpolation using the sinc function: (a) band-limited signal $x(t)$; (b) impulse train of samples of $x(t)$; (c) ideal band-limited interpolation in which the impulse train is replaced by a superposition of sinc functions [eq. (7.11)].

Decimation/Interpolation

- Decimation: reducing the sampling frequency by a factor of N, downsampling : two reversible steps

  – taking every N-th sample, leaving zeros in between

$$x_p[n] = \sum_{k=-\infty}^{\infty} x[kN]\,\delta[n-kN]$$

  – deleting all zero's between non-zero samples to produce a new sequence (inverse of time expansion property of discrete-time Fourier transform)

$$x_b[n] = x_p[nN] = x[nN]$$

  – both steps reversible in both time/frequency domains

See Fig. 7.34, p.550 of text

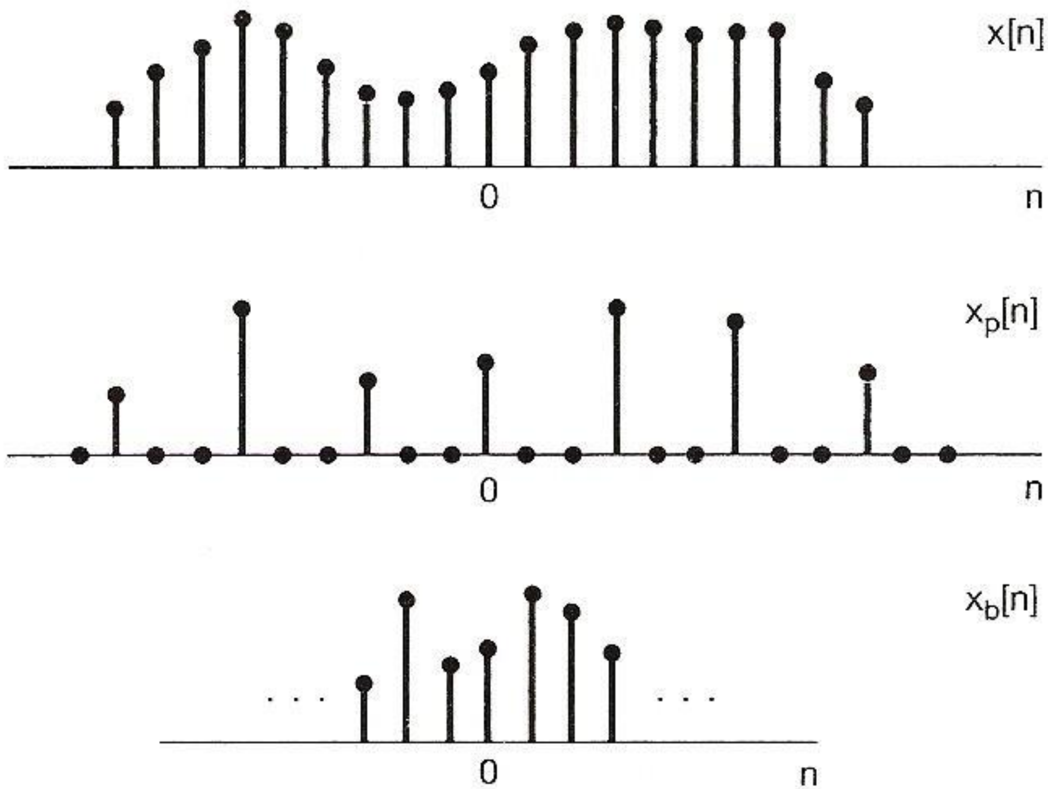


Figure 7.34 Relationship between $x_p[n]$ corresponding to sampling and $x_b[n]$ corresponding to decimation.

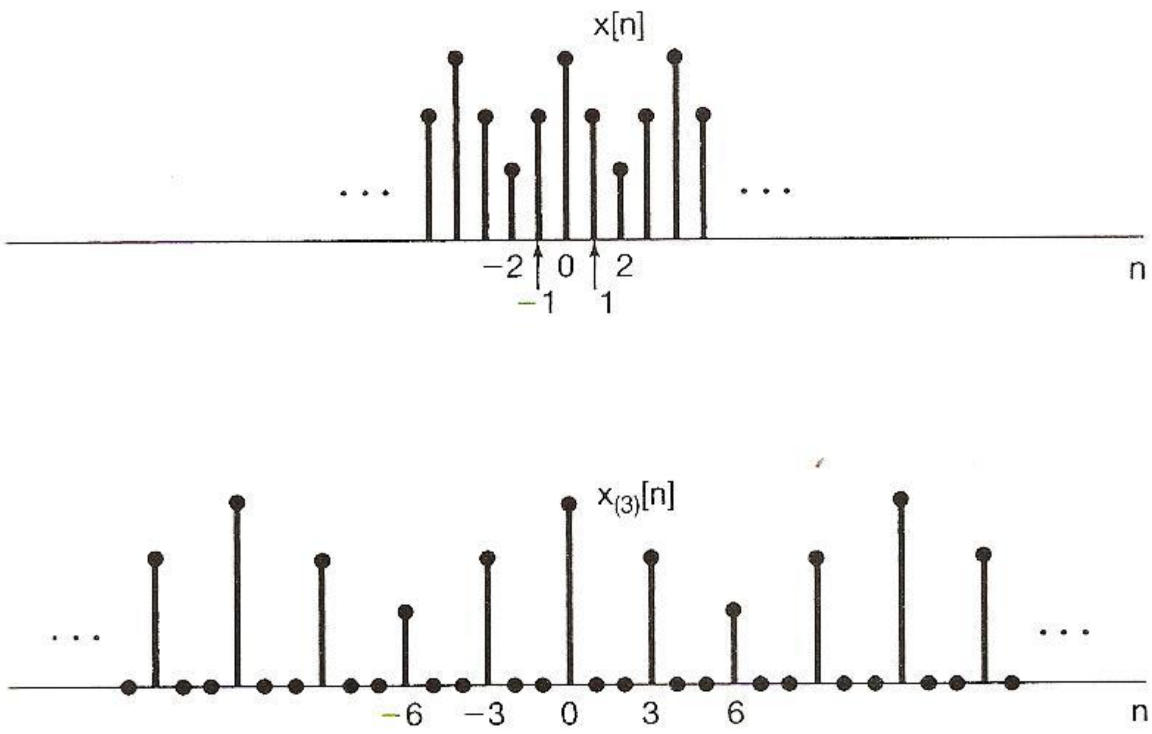


Figure 5.13 The signal $x_{(3)}[n]$ obtained from $x[n]$ by inserting two zeros between successive values of the original signal.

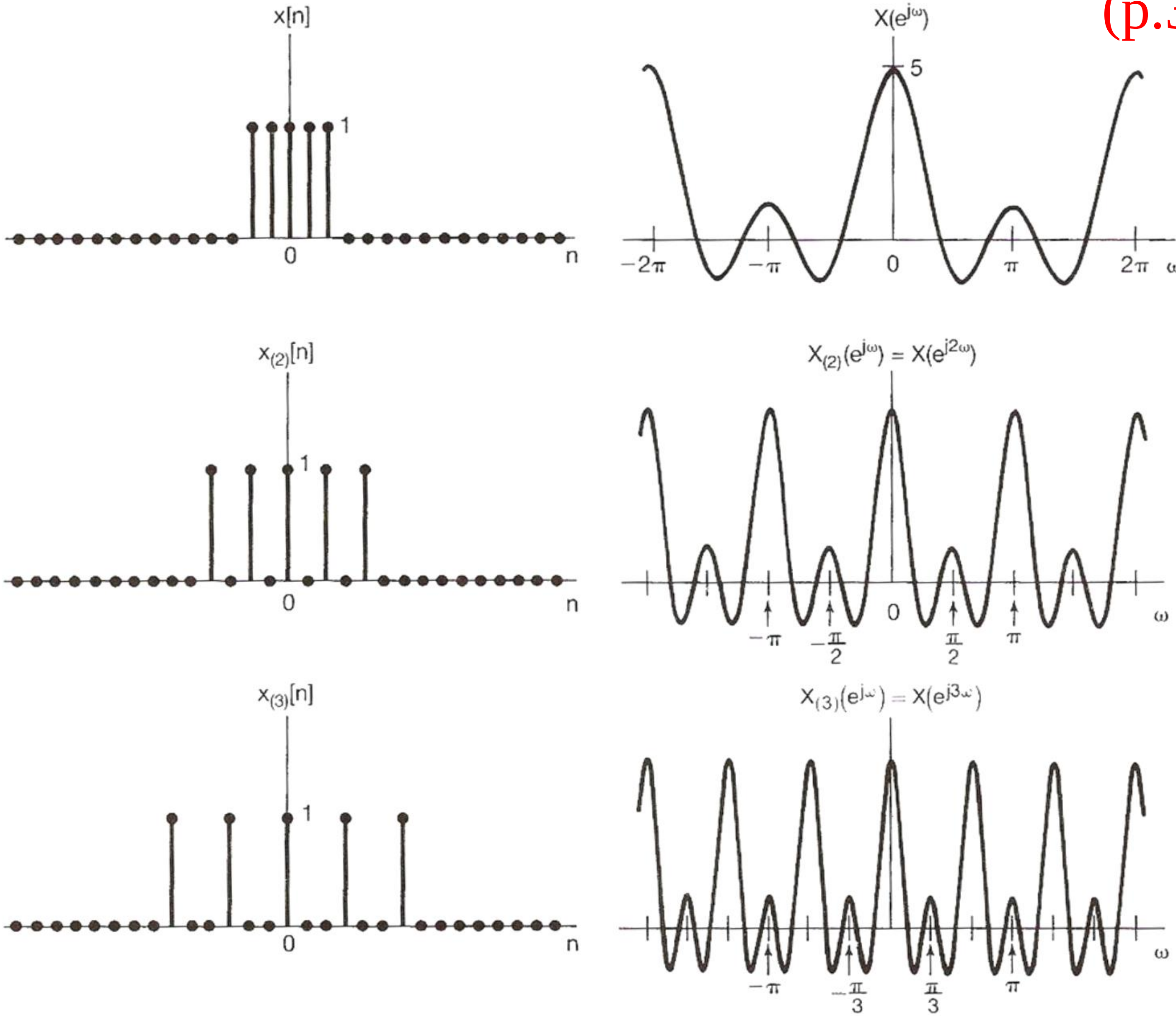


Figure 5.14 Inverse relationship between the time and frequency domains: As k increases, $x_{(k)}[n]$ spreads out while its transform is compressed.

$$x[n] \xleftrightarrow{F} X\left(e^{j\omega}\right)$$ (p.37 of 5.0)

- Time Expansion

  define $x_{(k)}[n] = x[n/k]$, If n/k is an integer,
  k: positive integer

  $= 0,$ else

  See Fig. 5.13, p.377 of text

$$x_{(k)}[n] \xleftrightarrow{F} X\left(e^{jk\omega}\right)$$

See Fig. 5.14, p.378 of text

Decimation/Interpolation

- Decimation:

$$X_b\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} x_b[k]e^{-j\omega k} = \sum_{k=-\infty}^{\infty} x_p[kN]e^{-j\omega k}$$

$$= \sum_{\substack{n=\text{integer} \\ \text{multiple of } N}} x_p[n]e^{-j\omega n/N} \qquad (k = n/N)$$

$$= \sum_{n=-\infty}^{\infty} x_p[n]e^{-j\omega n/N}$$

$$(x_p[n] = 0 \quad \text{if } n \text{ not integer multiple of } N)$$

$$= X_p\left(e^{j\omega/N}\right)$$

See Figs. 7.34, 7.35, p. 550, 551 of text

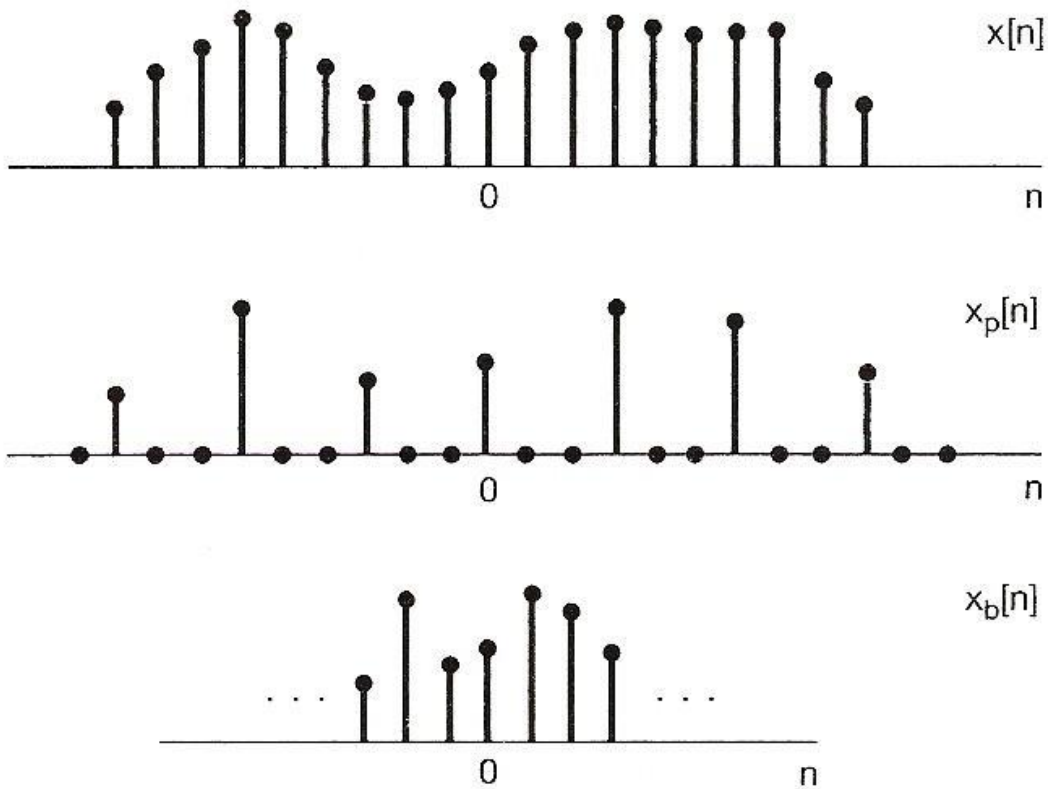


Figure 7.34 Relationship between $x_p[n]$ corresponding to sampling and $x_b[n]$ corresponding to decimation.

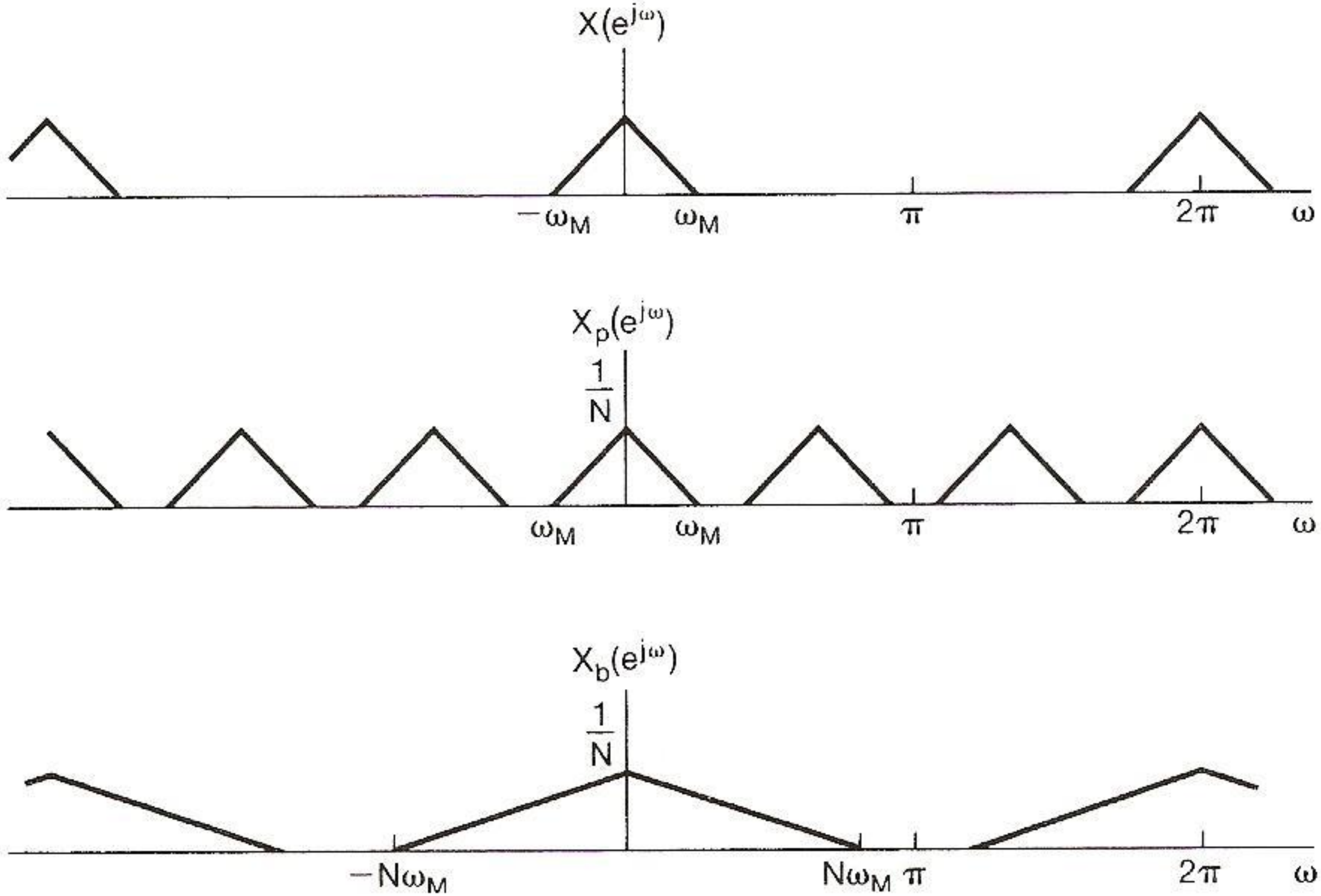


Figure 7.35 Frequency-domain illustration of the relationship between sampling and decimation.

Decimation/Interpolation

- Decimation
  - decimation without introducing aliasing requires oversampling situation

    See an example in Fig. 7.36, p. 552 of text

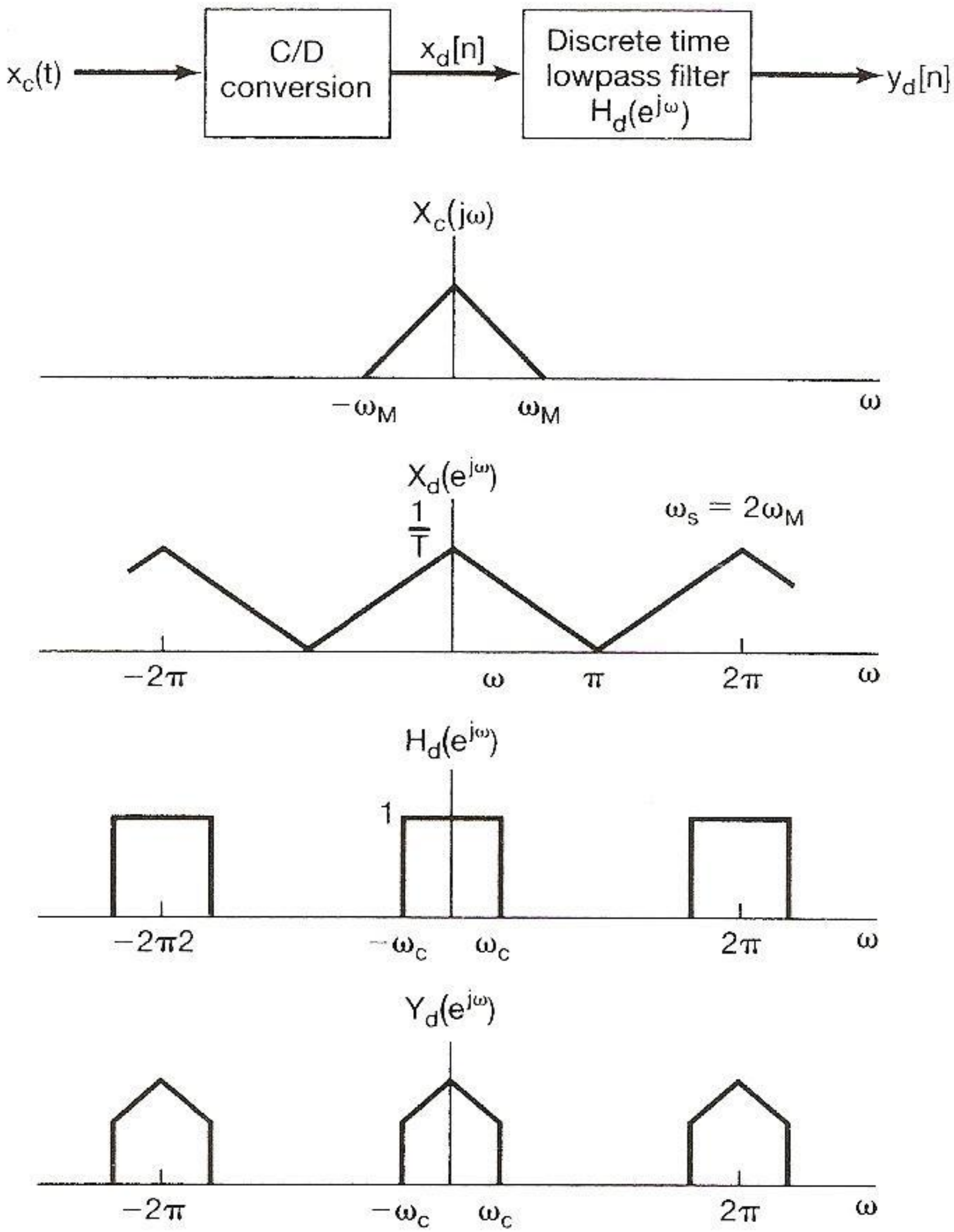


Figure 7.36 Continuous-time signal that was originally sampled at the Nyquist rate. After discrete-time filtering, the resulting sequence can be further downsampled. Here $X_c(j\omega)$ is the continuous-time Fourier transform of $x_c(t)$, $X_d(e^{j\omega})$ and $Y_d(e^{j\omega})$ are the discrete-time Fourier transforms of $x_d[n]$ and $y_d[n]$ respectively, and $H_d(e^{j\omega})$ is the frequency response of the discrete-time lowpass filter depicted in the block diagram.

Decimation/Interpolation

- Interpolation: increasing the sampling frequency by a factor of N, upsampling
  - reverse the two-step process in decimation

    from $x_b[n]$ construct $x_p[n]$ by inserting N-1 zero's

    from $x_p[n]$ construct $x[n]$ by lowpass filtering

    See Fig. 7.37, p. 553 of text

- Change of sampling frequency by a factor of N/M: first interpolating by N, then decimating by M

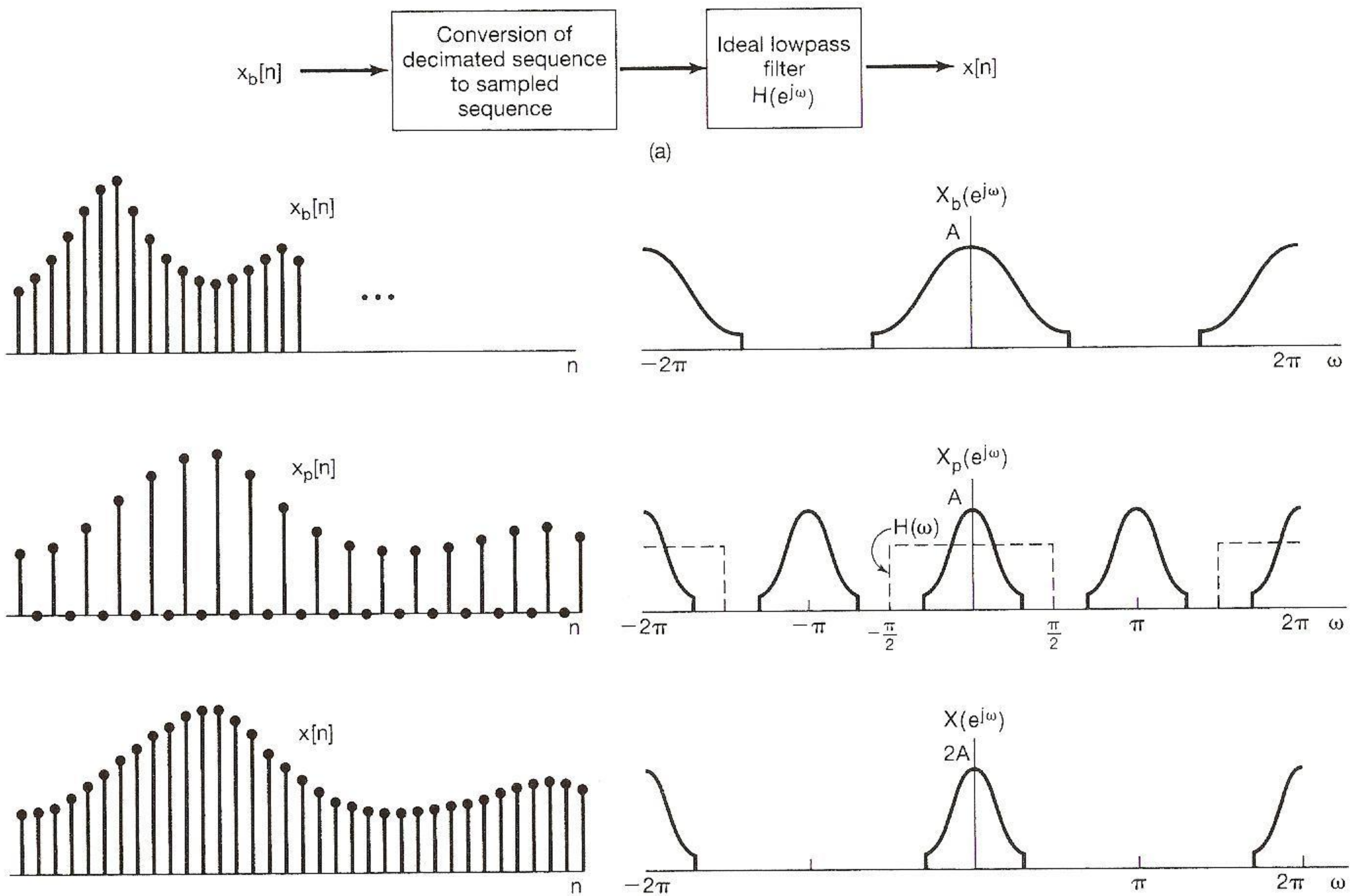
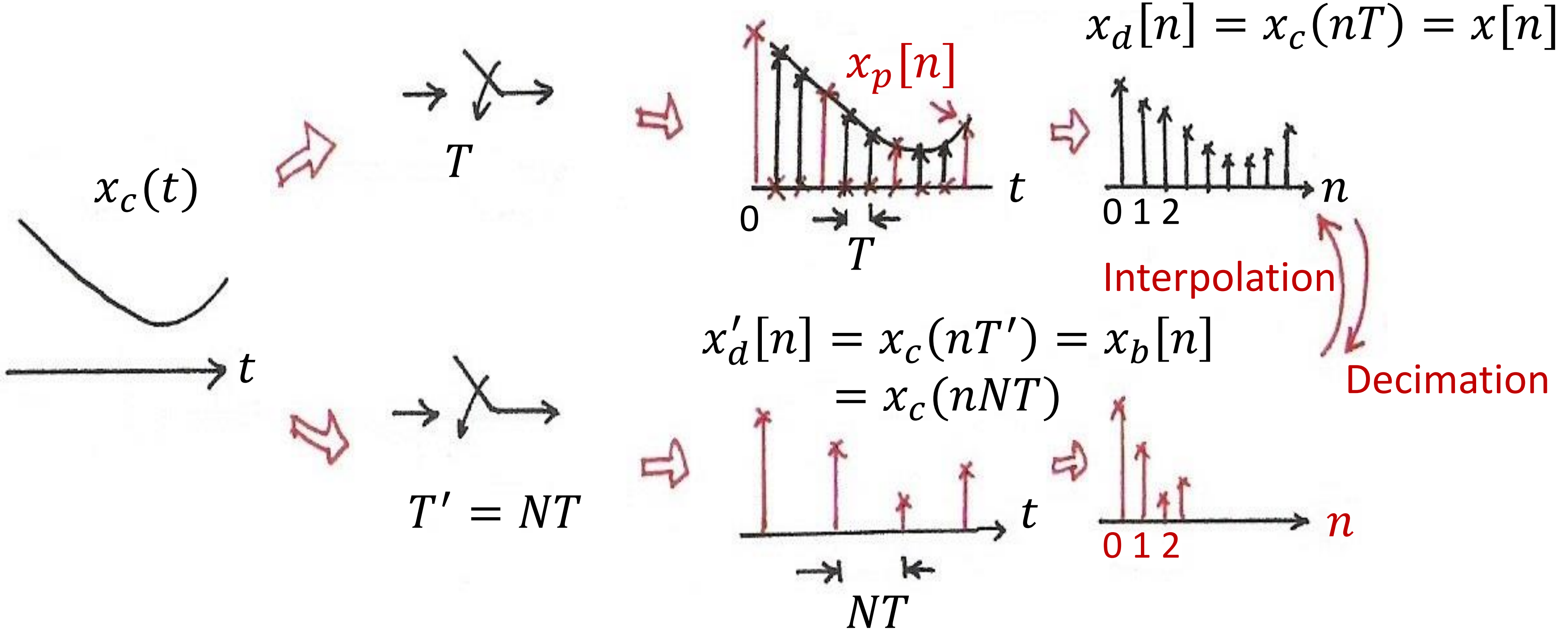
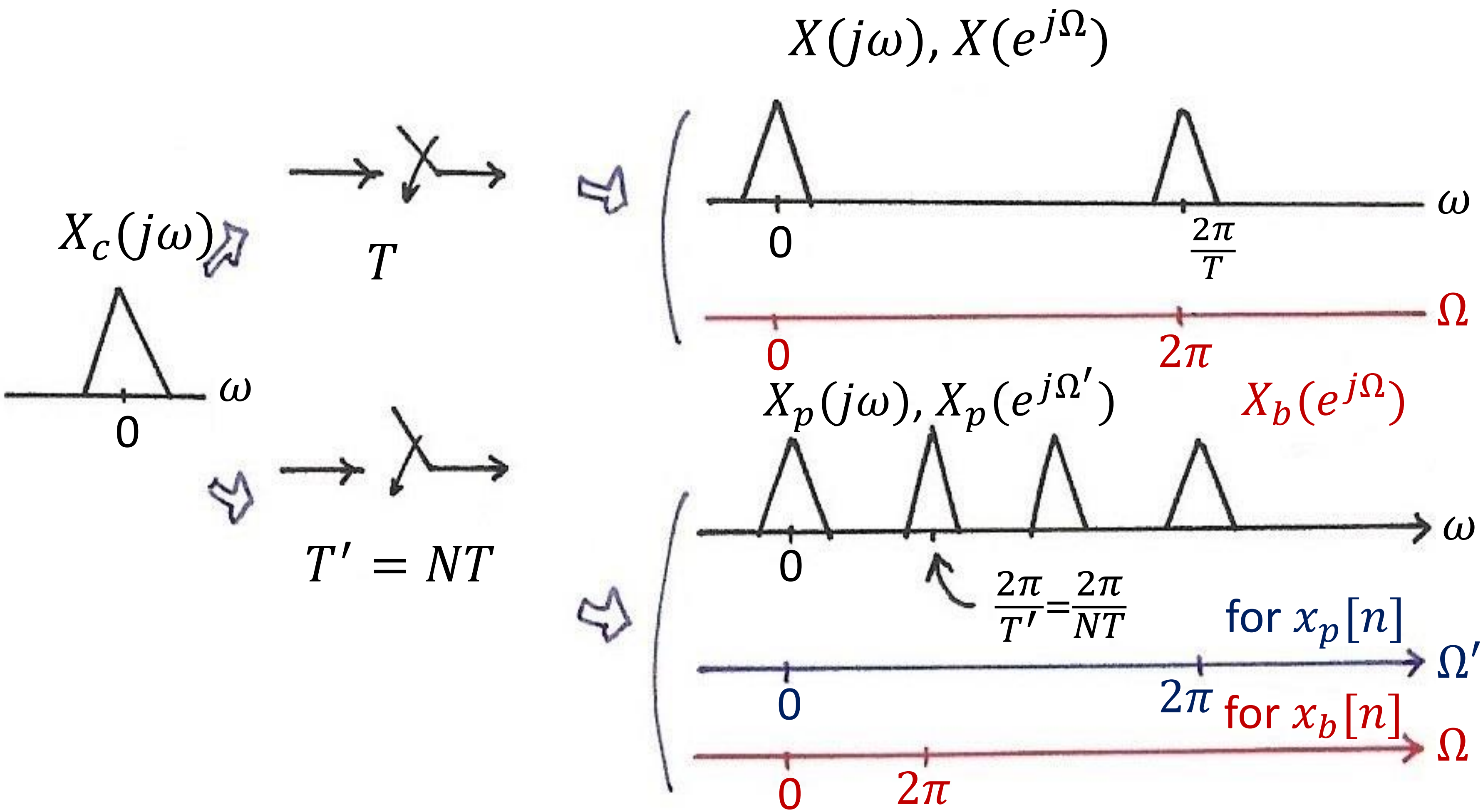


Figure 7.37 Upsampling: (a) overall system; (b) associated sequences and spectra for upsampling by a factor of 2.

Decimation/Interpolation



Decimation/Interpolation



Examples

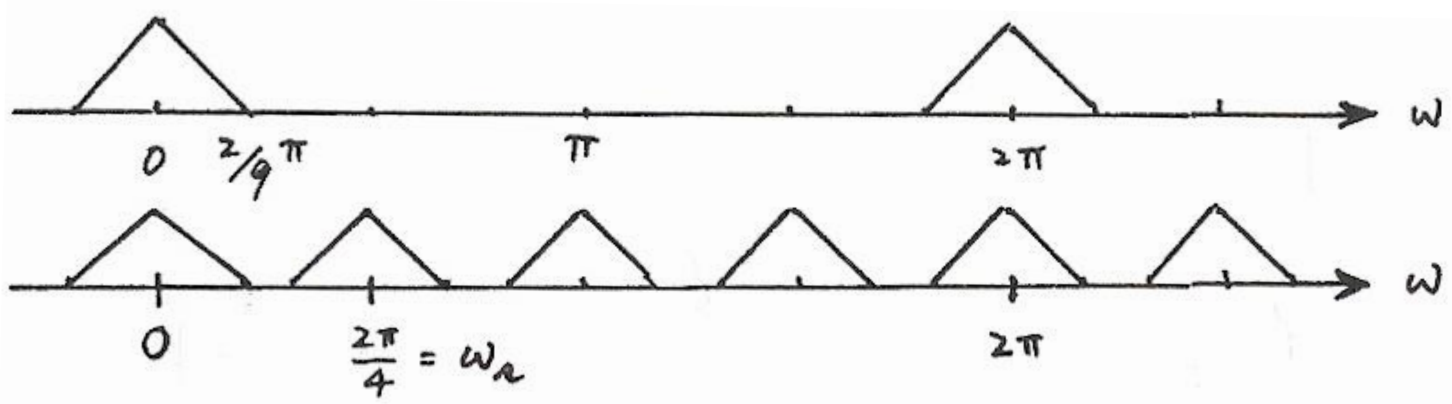
- Example 7.4/7.5, p.548, p.554 of text

$$x[n] \leftrightarrow X\left(e^{j\omega}\right), \quad X\left(e^{j\omega}\right) = 0 \text{ for } \frac{2\pi}{9} \leq |\omega| \leq \pi$$

sampling $x[n]$ without aliasing

$$\omega_s = \frac{2\pi}{N} > 2\omega_M = 2\left(\frac{2\pi}{9}\right), \quad \therefore \mathrm{N} \leq 9/2$$

$$N_{\max} = 4, \quad \omega_s = \frac{2\pi}{N_{\max}} = \frac{2\pi}{4} = \frac{\pi}{2}$$



Examples

- Example 7.4/7.5, p.548, p.554 of text

$$x[n] \leftrightarrow X\left(e^{j\omega}\right), \quad X\left(e^{j\omega}\right) = 0 \text{ for } \frac{2\pi}{9} \le |\omega| \le \pi$$

maximum possible downsampling: using full band [-π, π]

$$x[n] \xrightarrow[4:1]{} x_b[n] \qquad (N_{\max} = 4)$$

$$x[n] \xrightarrow[1:2]{} x_u[n] \xrightarrow[9:1]{} x_{ub}[n] \qquad (\mathrm{N/M} = 9/2)$$

Examples

- Example 7.4/7.5, p.548, p.554 of text

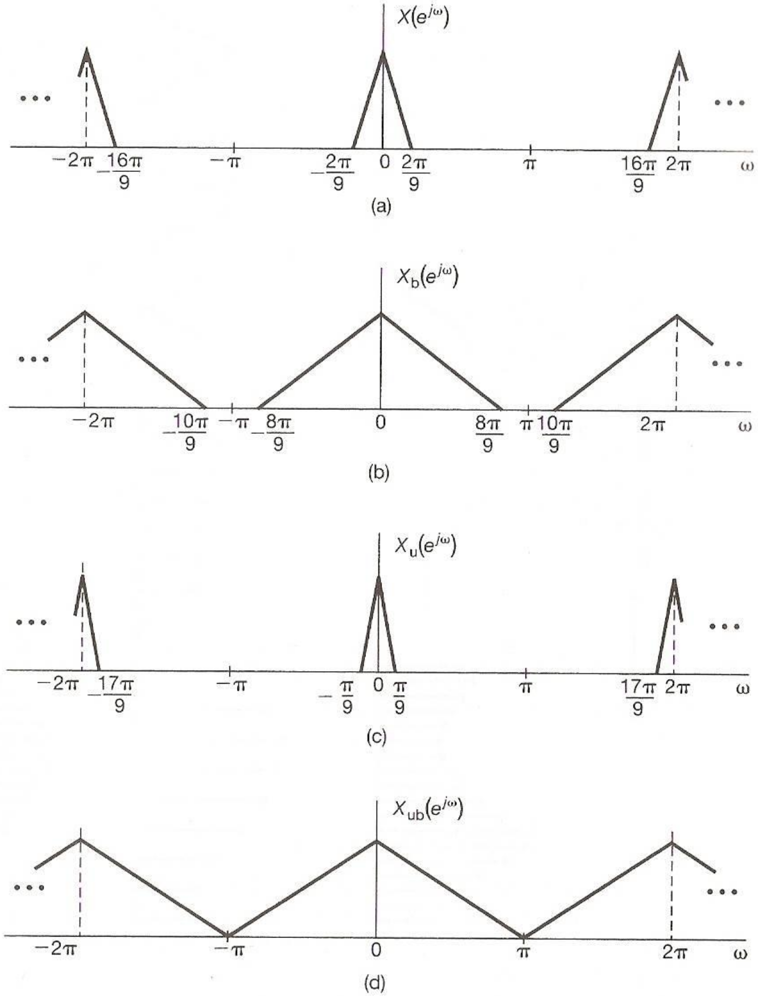


Figure 7.38 Spectra associated with Example 7.5. (a) Spectrum of $x[n]$; (b) spectrum after downsampling by 4; (c) spectrum after upsampling $x[n]$ by a factor of 2; (d) spectrum after upsampling $x[n]$ by 2 and then downsampling by 9.

Problem 7.6, p.557 of text

$$X_1(j\omega) = 0,\ |\omega| \geq \omega_1$$

$$X_2(j\omega) = 0,\ |\omega| \geq \omega_2$$

$$w(t) = x_1(t)x_2(t)$$

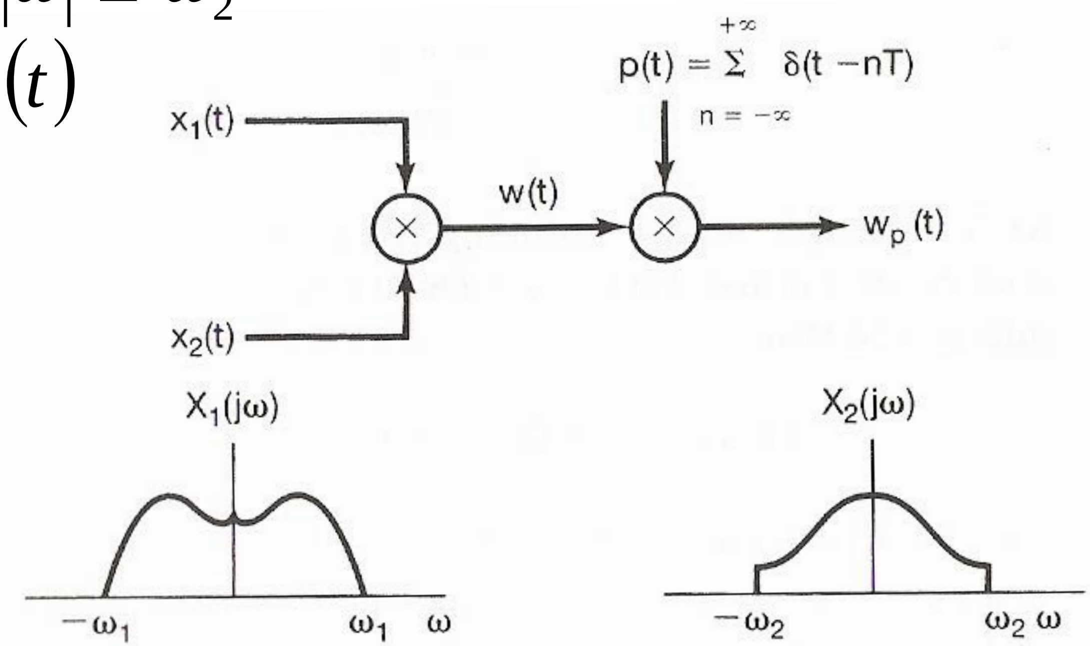


Figure P7.6

$$W(j\omega) = \frac{1}{2\pi}\left[X_1(j\omega) * X_2(j\omega)\right]$$

$$W(j\omega) = 0,\ \ |\omega| \geq (\omega_1 + \omega_2),\ \ \therefore\ \omega_s = \frac{2\pi}{T} > 2(\omega_1 + \omega_2)$$

Problem 7.20, p.560 of text

S_A : inserting one zero after each sample

S_B : decimation 2:1, extracting every second sample

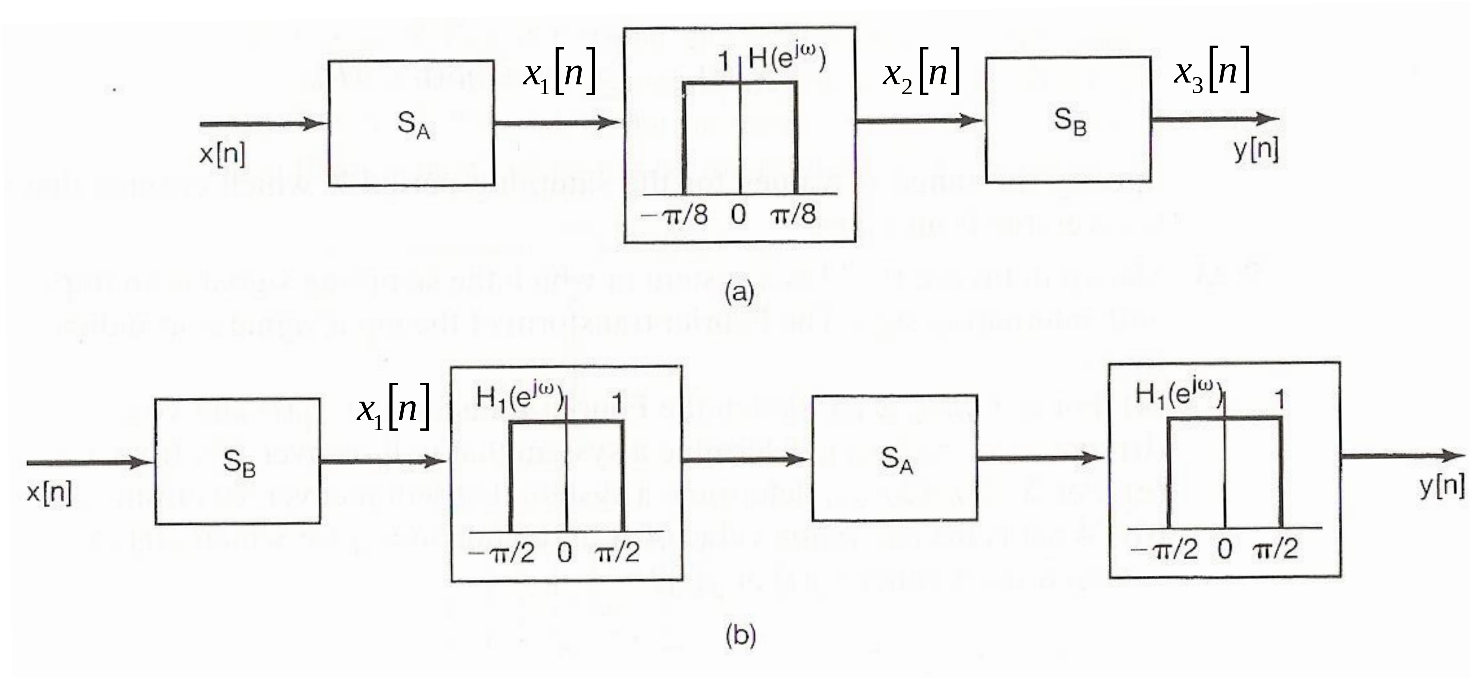
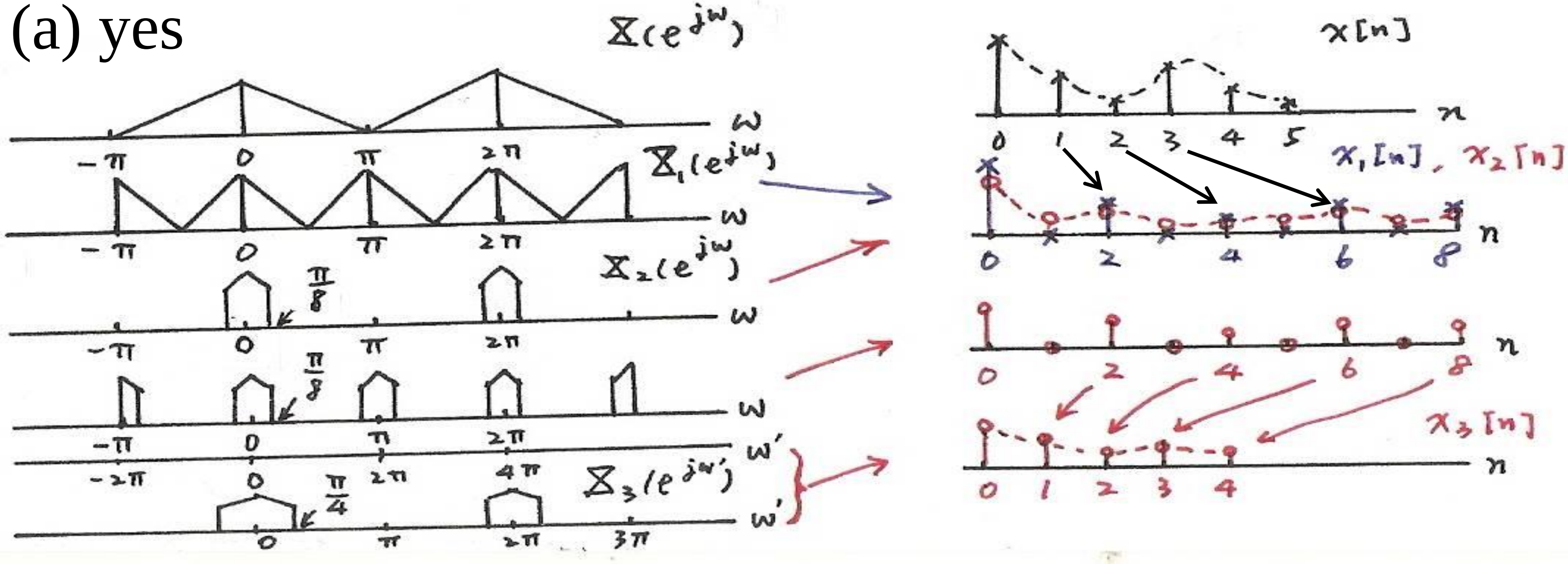


Figure P7.20

Which of (a)(b) corresponds to low-pass filtering with $\omega_c = \pi/4$?

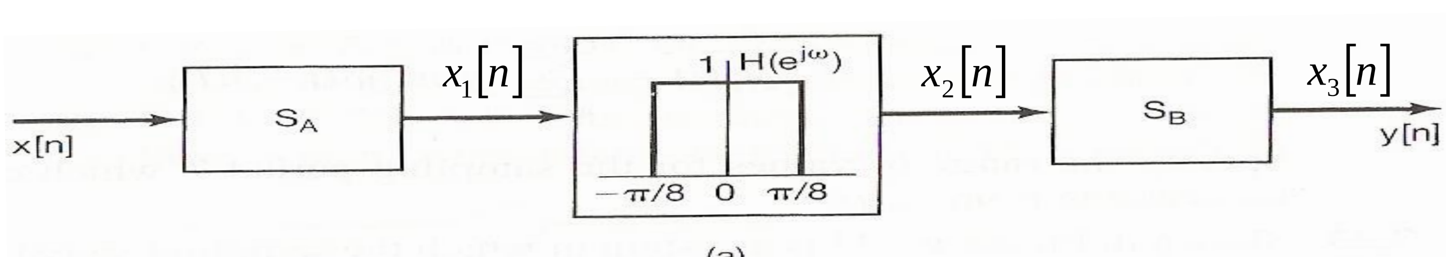
Problem 7.20, p.560 of text

(a) yes



S_A : inserting one zero after each sample

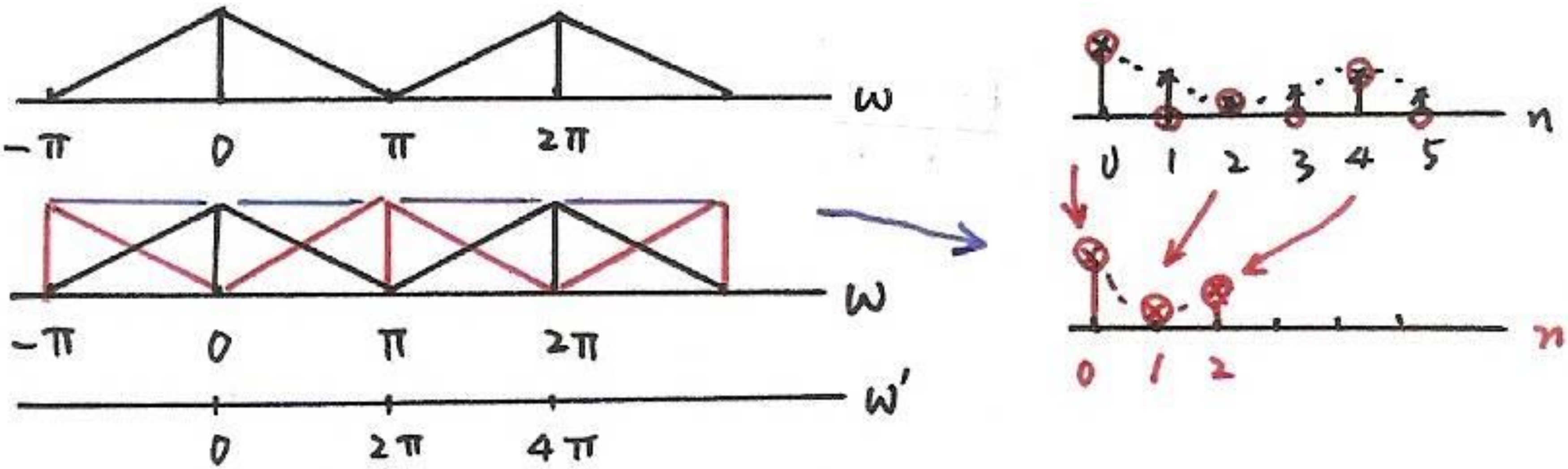
S_B : decimation 2:1, extracting every second sample



(a)

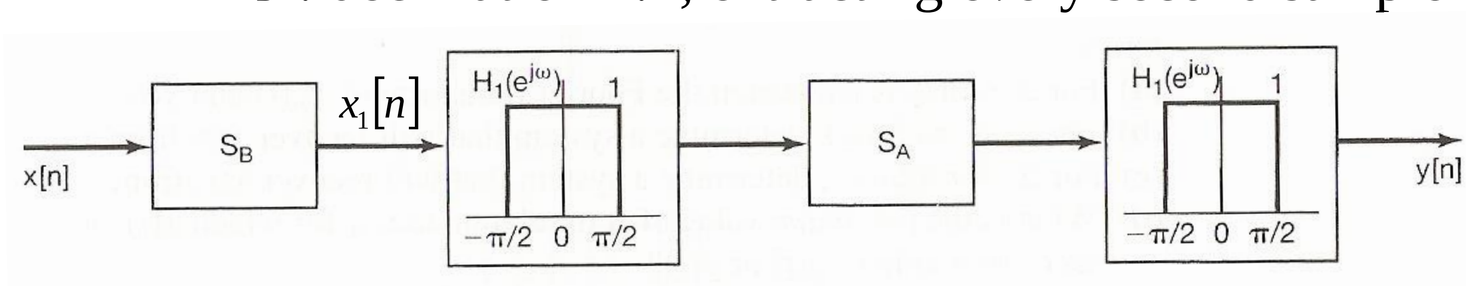
Problem 7.20, p.560 of text

(b) no



S_A : inserting one zero after each sample

S_B : decimation 2:1, extracting every second sample



(b)

Problem 7.23, p.562 of text

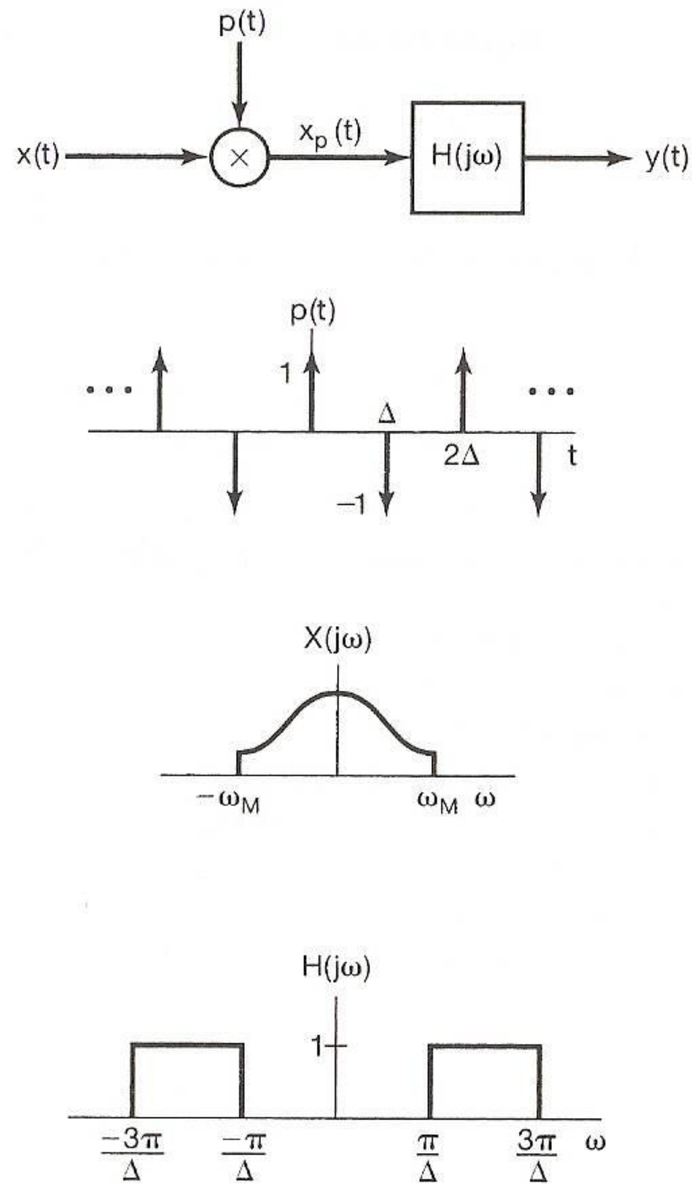


Figure P7.23

Problem 7.23, p.562 of text

$$p(t) = p_1(t) - p_1(t-\Delta)$$

$$p_1(t) = \sum_{k=-\infty}^{\infty} \delta(t - 2k\Delta)$$

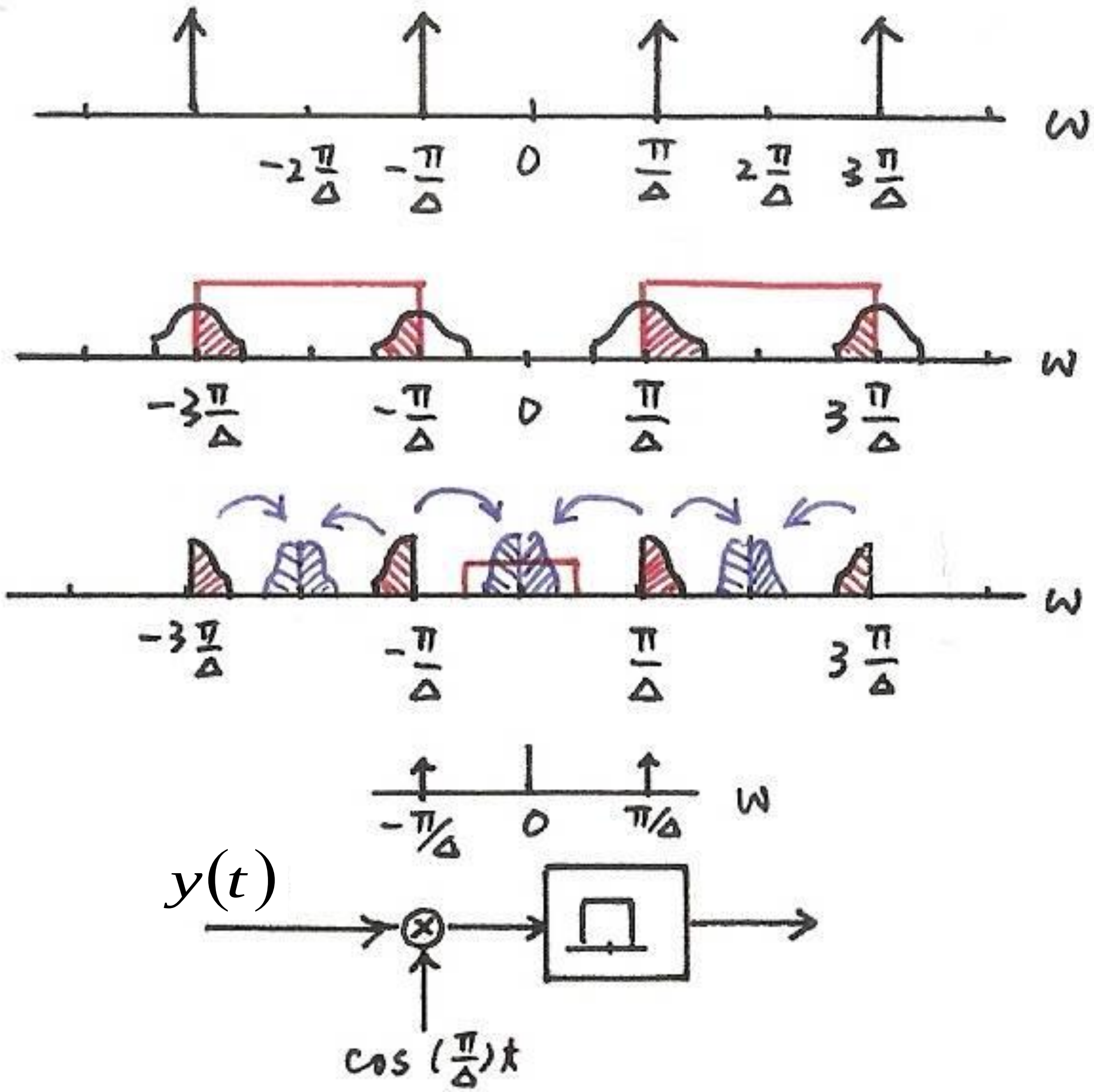
$$P_1(j\omega) = \sum_{k=-\infty}^{\infty} \frac{\pi}{\Delta} \delta\left(\omega - k\frac{\pi}{\Delta} \right)$$

$$P(j\omega) = P_1(j\omega) - e^{-j\omega\Delta} P_1(j\omega)$$

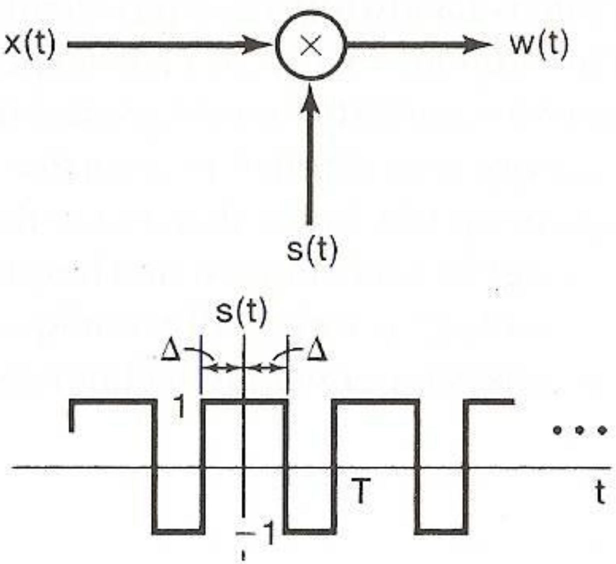
$$\text{for } \omega = (2m)\frac{\pi}{\Delta}, \ \text{m : integer}$$

$$e^{-j\omega\Delta} = 1, \text{etc.}$$

Problem 7.23, p.562 of text



Problem 7.24, p.562 of text



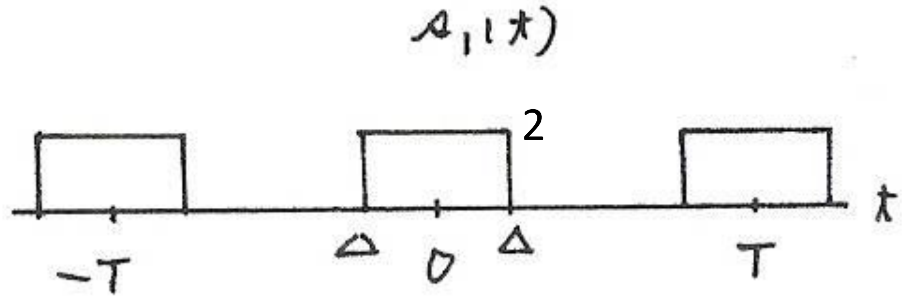


Figure P7.24

$$s(t) = s_1(t) - 1$$

$$S(j\omega) = \sum_{k=-\infty}^{\infty} a_k \, \delta\left(\omega - k\frac{\pi}{\Delta}\right) - 2\pi\delta(\omega)$$

Problem 7.41, p.572 of text

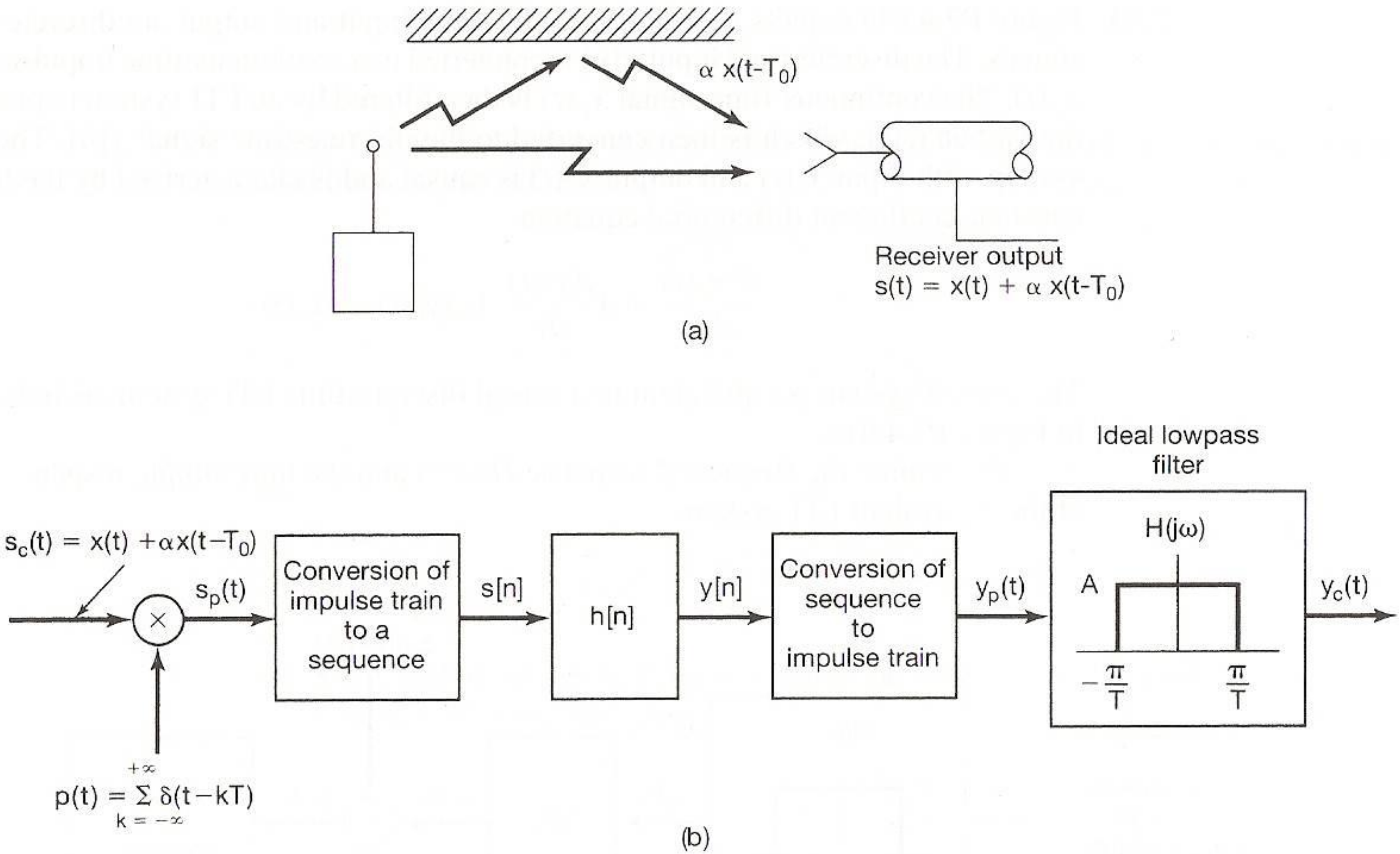


Figure P7.41

Problem 7.41, p.572 of text

$$s_c(t) = x_c(t) + \alpha\, x_c(t - T_0), \quad \mathrm{T} = T_0$$

$$s[n] = s_c(nT) = x[n] + \alpha\, x[n-1]$$

$$\mathrm{S}(e^{j\Omega}) = (1 + \alpha e^{-j\Omega})X(e^{j\Omega})$$

$$\mathrm{H}(e^{j\Omega}) = \frac{1}{1 + \alpha e^{-j\Omega}}$$

$$\text{difference equation}: y[n] + \alpha y[n-1] = s[n]$$

Problem 7.52, p.580 of text

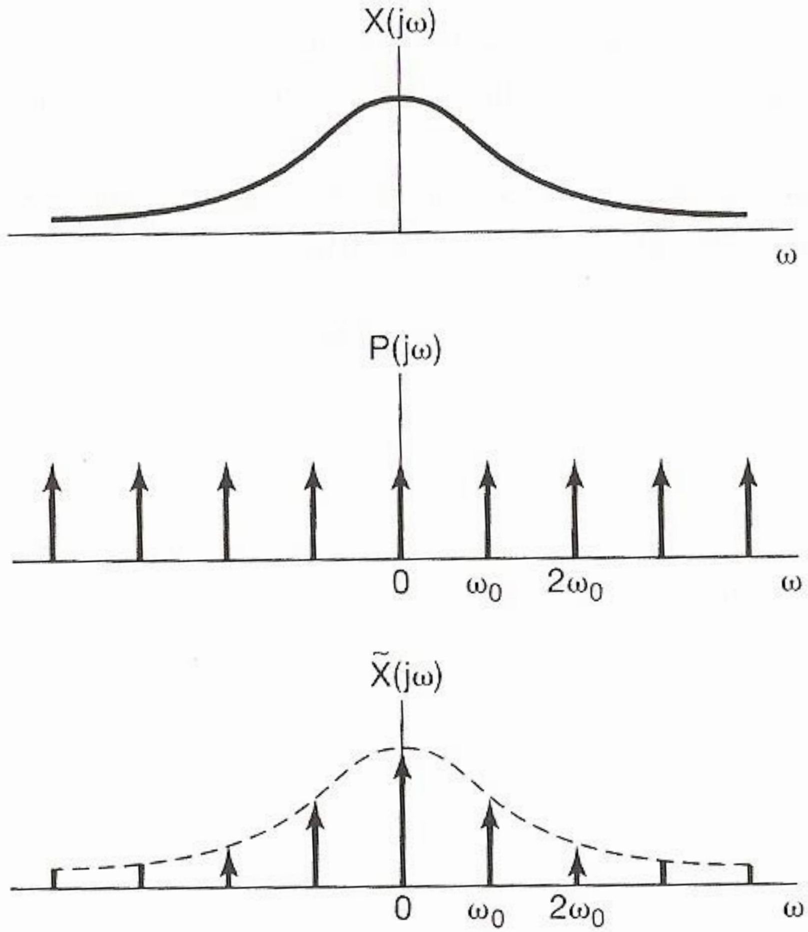


Figure P7.52

dual problem for frequency domain sampling