9.0 Laplace Transform

9.1 General Principles of Laplace Transform

Laplace Transform

- Eigenfunction Property

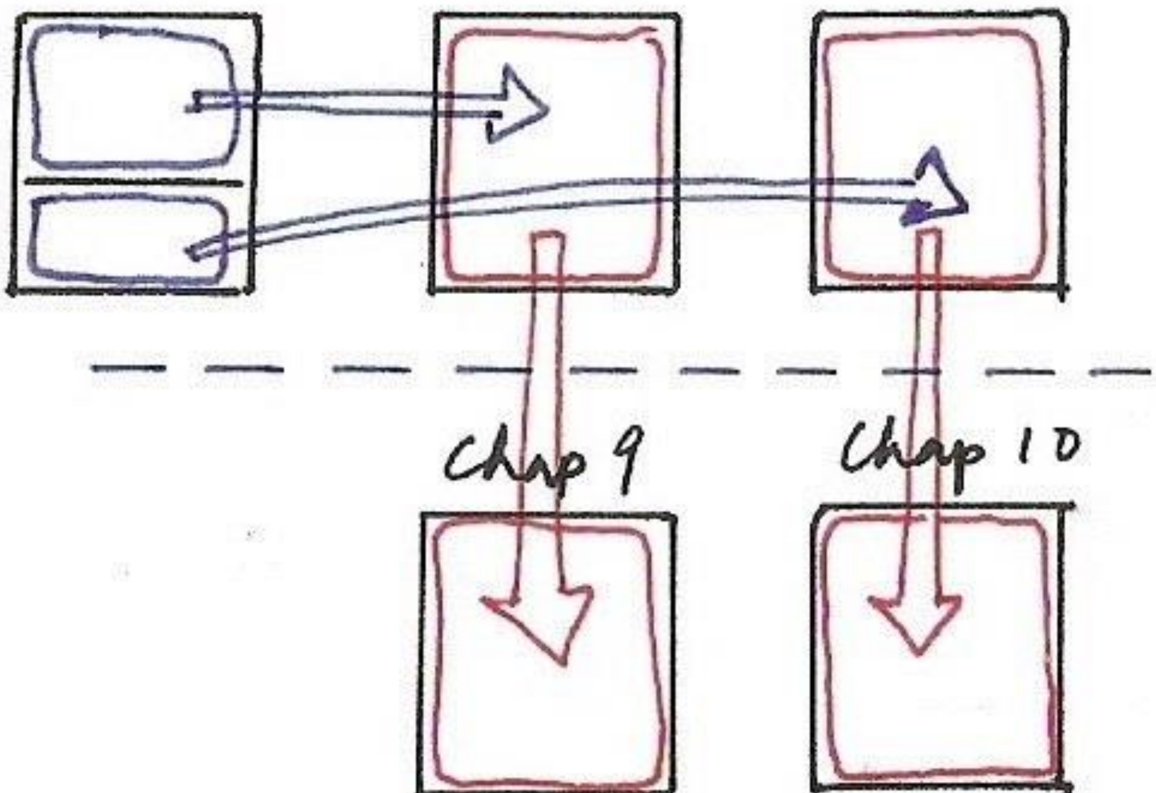
$$x(t) = e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = H(s)e^{st}$$

linear time-invariant

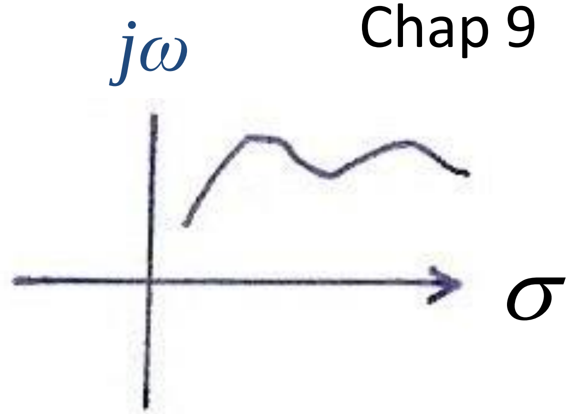
$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

Chapters 3, 4, 5, 9, 10

Chap 3 Chap 4 Chap 5



Chap 9



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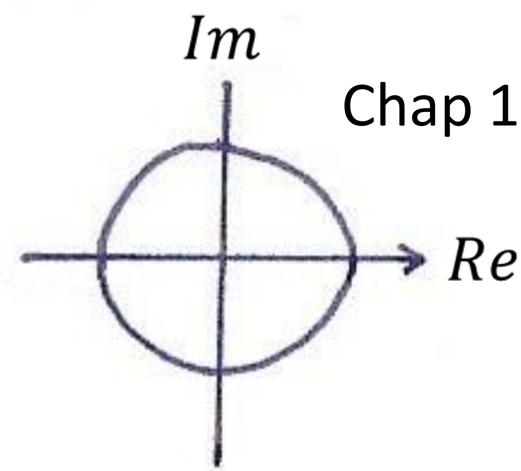


Laplace Transform

- Eigenfunction Property

  e^{st} eigenfunction of all linear time-invariant systems with unit impulse response $h(t)$

$$H(s) = \int_{-\infty}^{\infty} h(t)\ e^{-st}\, dt$$ eigenvalue

  – applies for all complex variables s

$$s = j\omega \qquad e^{st} = e^{j\omega t}$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$ Fourier Transform

$$s = \sigma + j\omega$$

$$H(\sigma + j\omega) = \int_{-\infty}^{\infty} h(t) e^{-(\sigma + j\omega)t} dt$$ Laplace Transform

Laplace Transform

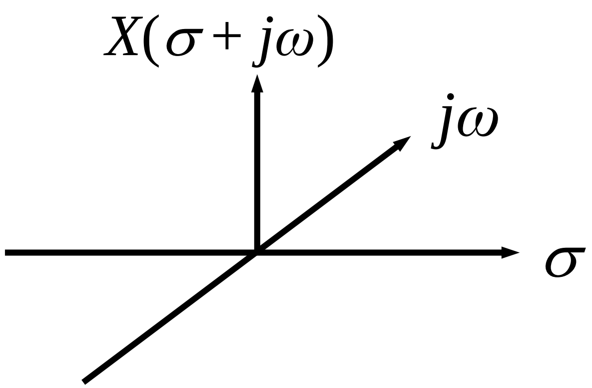
- Laplace Transform

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt, \ s = \sigma + j\omega$$

$$x(t) \xleftrightarrow{L} X(s)$$

- A Generalization of Fourier Transform

from $s = j\omega$ to $s = \sigma + j\omega$

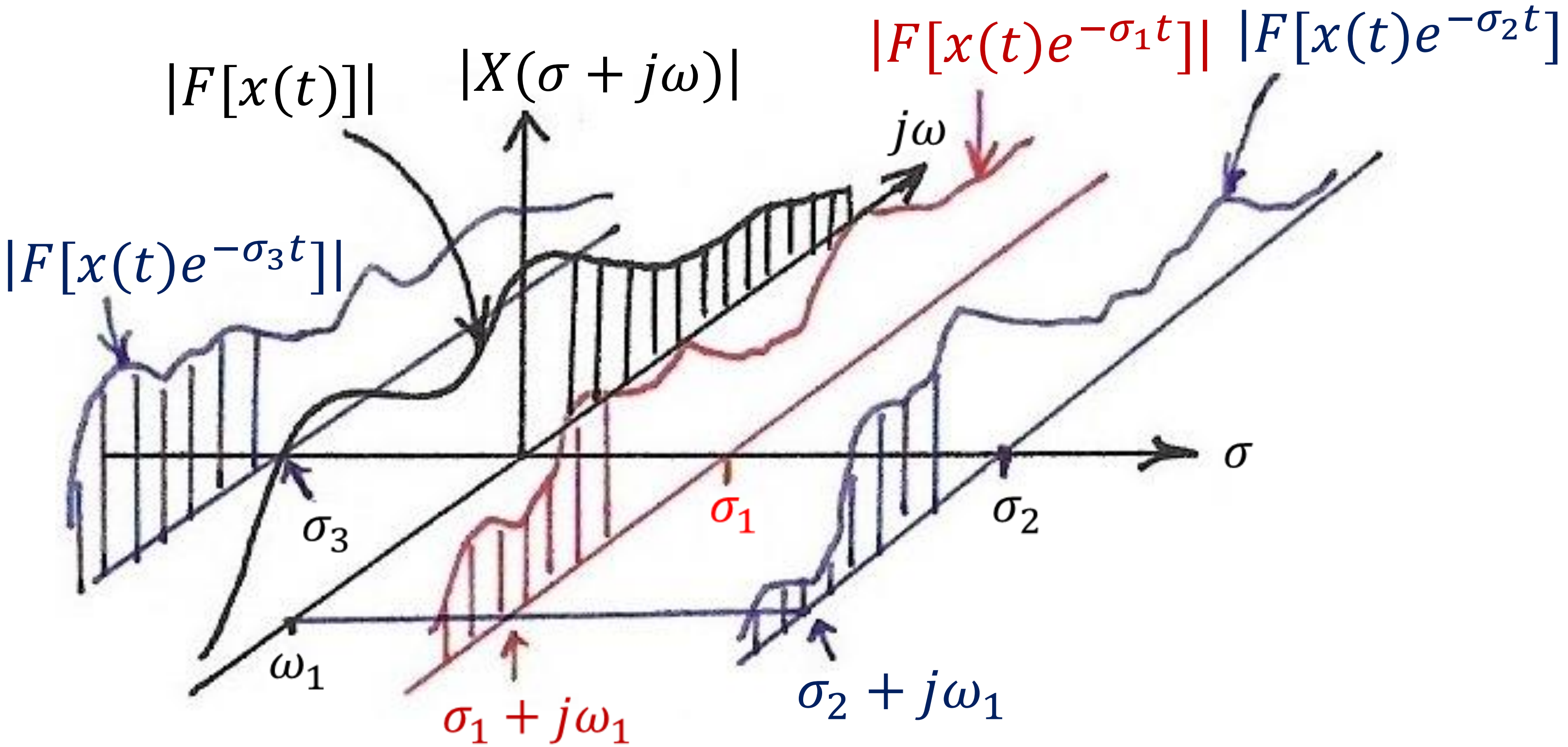


$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt$$

$$= \int_{-\infty}^{\infty} \left[x(t) \mathrm{e}^{-\sigma t} \right] e^{-j\omega t} dt$$

Fourier transform of $x(t) e^{-\sigma t}$

Laplace Transform



$(e^{j\omega_1 t}) \perp (e^{j\omega_2 t})$ orthogonal

$\left(e^{(\sigma_1 + j\omega_1)t}\right) \not\perp \left(e^{(\sigma_2 + j\omega_1)t}\right)$ Not orthogonal

Laplace Transform

- A Generalization of Fourier Transform
  - $X(s)$ may not be well defined (or converged) for all s
  - $X(s)$ may converge at some region of s-plane, while $x(t)$ doesn't have Fourier Transform
  - covering broader class of signals, performing more analysis for signals/systems

Laplace Transform

- Rational Expressions and Poles/Zeros

$$X(s) = \frac{N(s) \longrightarrow \text{roots} \longrightarrow \text{zeros}}{D(s) \longrightarrow \text{roots} \longrightarrow \text{poles}}$$

  – Pole-Zero Plots

  – specifying $X(s)$ except for a scale factor

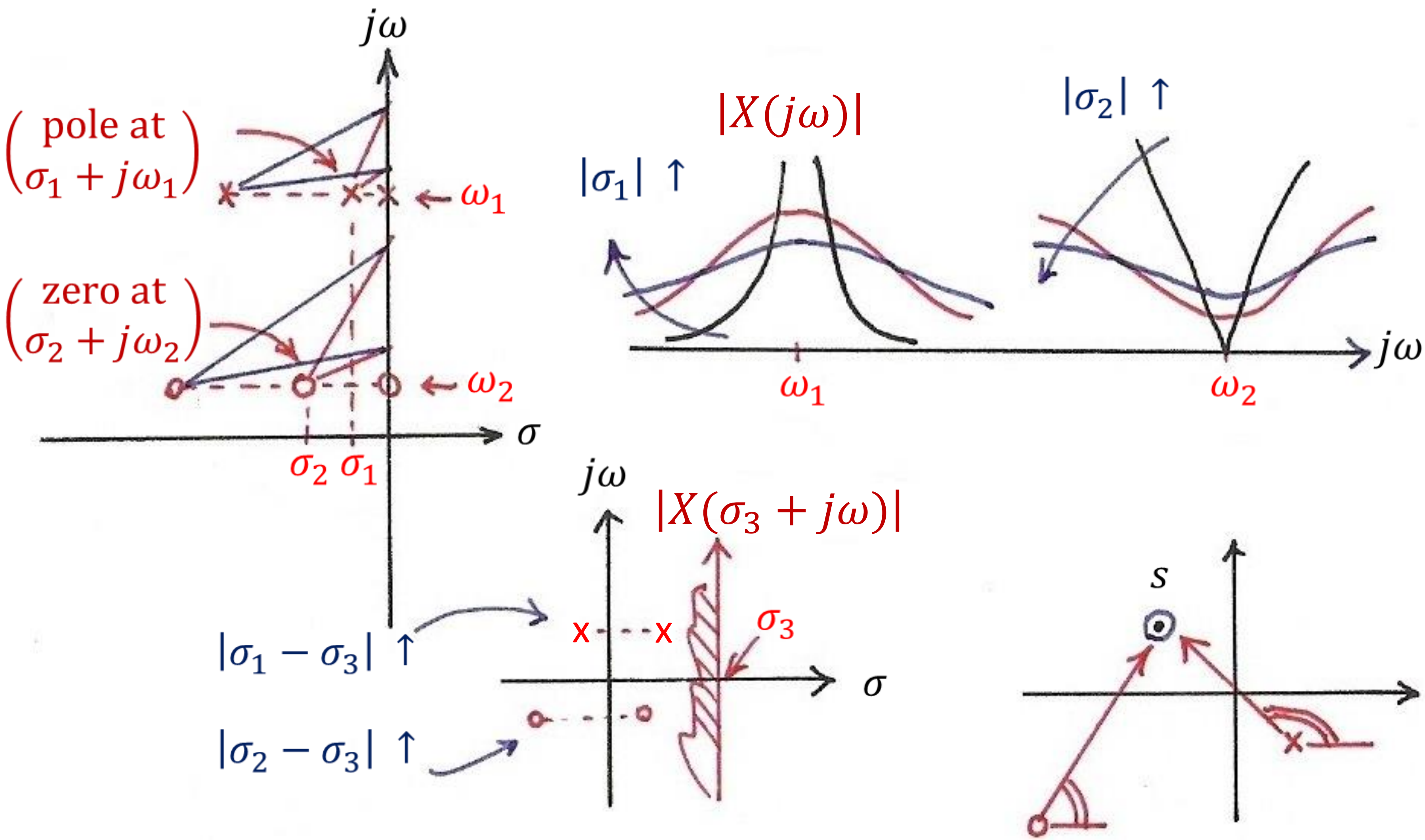
Laplace Transform

- Rational Expressions and Poles/Zeros
  - Geometric evaluation of Fourier/Laplace transform from pole-zero plots

$$X(s) = M \frac{\Pi_i \left(s - \beta_i\right)}{\Pi_j \left(s - \alpha_j\right)}$$

each term (s-β_i) or (s-α_j) represented by a vector with magnitude/phase

Poles & Zeros



$j\omega$
pole at $(\sigma_1 + j\omega_1)$
$\leftarrow \omega_1$
zero at $(\sigma_2 + j\omega_2)$
$\leftarrow \omega_2$
σ
σ_2 σ_1
$|X(j\omega)|$
$|\sigma_1| \uparrow$
$|\sigma_2| \uparrow$
$j\omega$
ω_1
ω_2
$|\sigma_1 - \sigma_3| \uparrow$
$|\sigma_2 - \sigma_3| \uparrow$
$j\omega$
$|X(\sigma_3 + j\omega)|$
σ_3
σ
s

Region of Convergence (ROC)

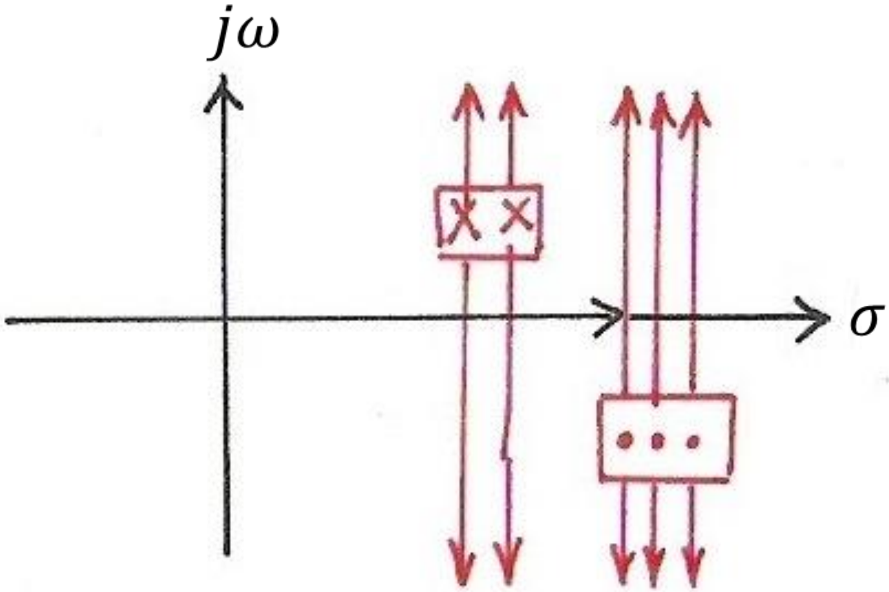
- Property 1 : ROC of $X(s)$ consists of strips parallel to the $j\omega$ -axis in the s-plane
  - For the Fourier Transform of $x(t)e^{-\sigma t}$ to converge

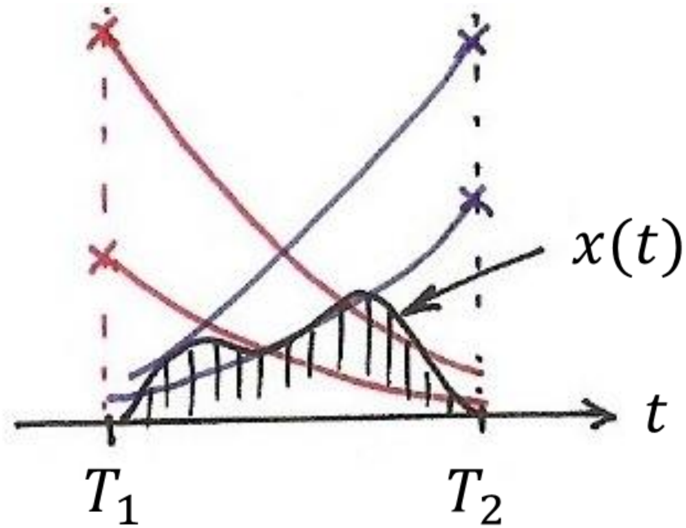
$$\int_{-\infty}^{\infty} \left| x(t)e^{-\sigma t} \right| dt < \infty$$

depending on σ only

- Property 2 : ROC of $X(s)$ doesn't include any poles

Property 1, 3





Region of Convergence (ROC)

- Property 3 : If $x(t)$ is of finite duration and absolutely integrable, the ROC is the entire s-plane

$$\int_{T_1}^{T_2} |\, x(t) \,|\, dt < \infty$$

$$[T_1, T_2]: \text{the finite duration}$$

$$\text{for } \sigma > 0, \quad \int_{T_1}^{T_2} \left| x(t) e^{-\sigma t} \right| dt < e^{-\sigma T_1} \int_{T_1}^{T_2} |\, x(t) \,|\, dt$$

$$\text{for } \sigma < 0, \quad \int_{T_1}^{T_2} \left| x(t) e^{-\sigma t} \right| dt < e^{-\sigma T_2} \int_{T_1}^{T_2} |\, x(t) \,|\, dt$$

Region of Convergence (ROC)

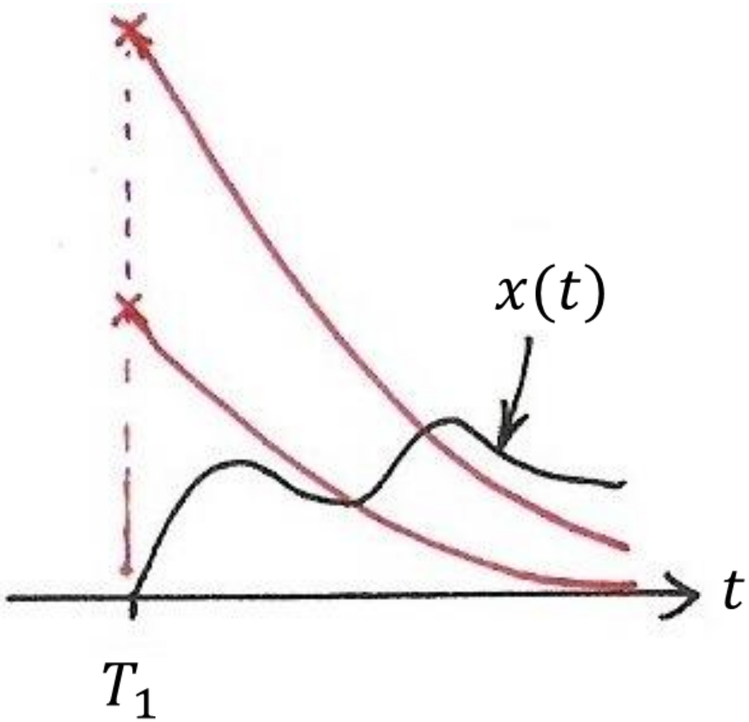
- Property 4 : If $x(t)$ is right-sided ($x(t)=0,\ t < T_1$), and $\{s \mid \mathrm{Re}[s] = \sigma_0\} \in$ ROC, then $\{s \mid \mathrm{Re}[s] > \sigma_0\} \in$ ROC, i.e., ROC includes a right-half plane

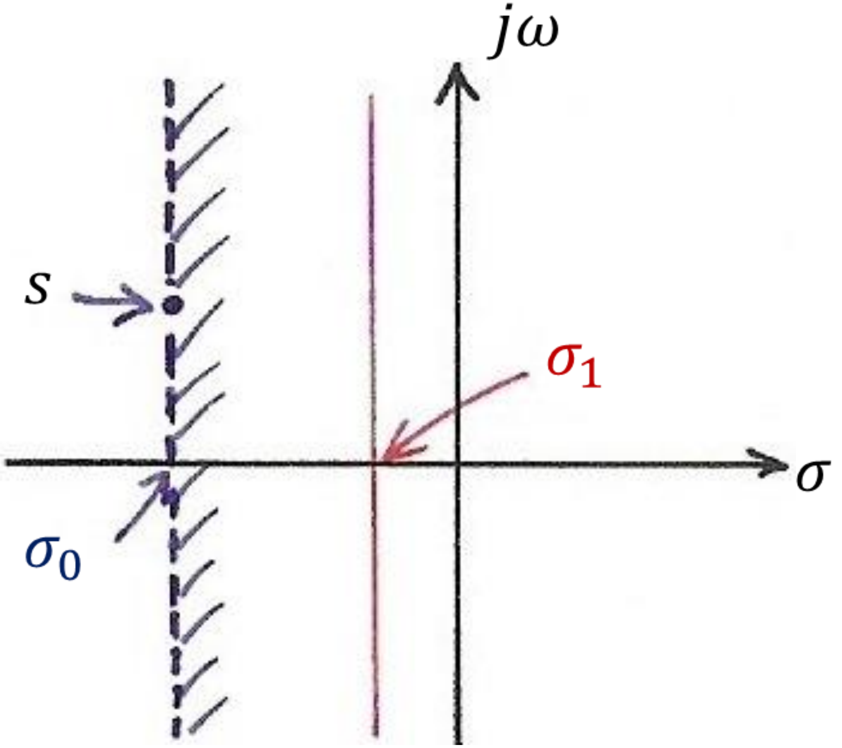
$$\int_{T_1}^{\infty} |\ x(t)\ |\ e^{-\sigma_0 t} dt < \infty$$

for $\sigma_1 > \sigma_0$

$$\int_{T_1}^{\infty} |x(t)|\, e^{-\sigma_1 t}\ dt < e^{-(\sigma_1-\sigma_0)T_1} \int_{T_1}^{\infty} |\ x(t)\ |\ e^{-\sigma_0 t} dt < \infty$$

Property 4

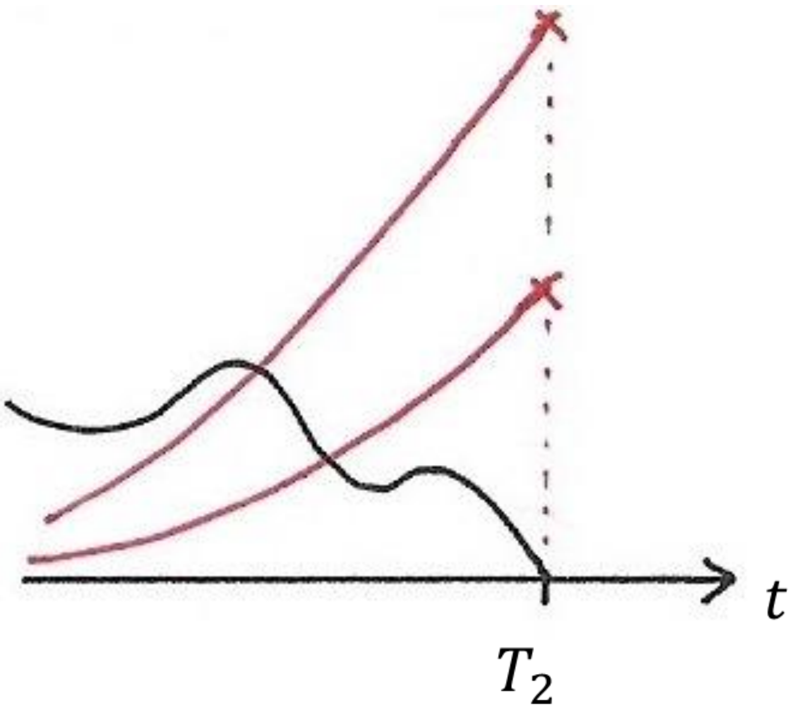


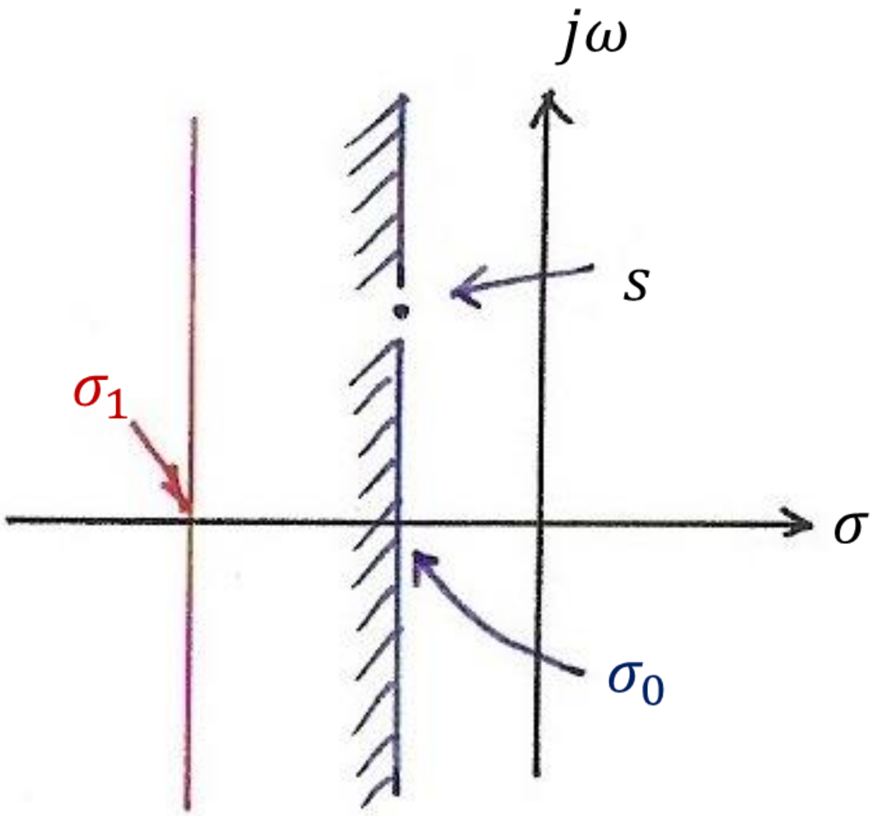


Region of Convergence (ROC)

- Property 5 : If $x(t)$ is left-sided ($x(t)=0,\ t > T_2$), and $\{s \mid \mathrm{Re}[s] = \sigma_0\} \in$ROC, then $\{s \mid \mathrm{Re}[s] < \sigma_0\} \in$ROC, i.e., ROC includes a left-half plane

Property 5





Region of Convergence (ROC)

- Property 6 : If $x(t)$ is two-sided, and $\{s \mid \mathrm{Re}[s] = \sigma_0\} \in$ ROC, then ROC consists of a strip in s-plane including $\{s \mid \mathrm{Re}[s] = \sigma_0\}$

$$x(t) = x_R(t) + x_L(t)$$

$$x_R(t): \text{right-sided}, \quad x_L(t): \text{left-sided}$$

$$\mathrm{ROC}[x(t)] = \mathrm{ROC}[x_R(t)] \cap \mathrm{ROC}[x_L(t)]$$

See Fig. 9.9, 9.10, p.667 of text

*note: ROC[$x(t)$] may not exist

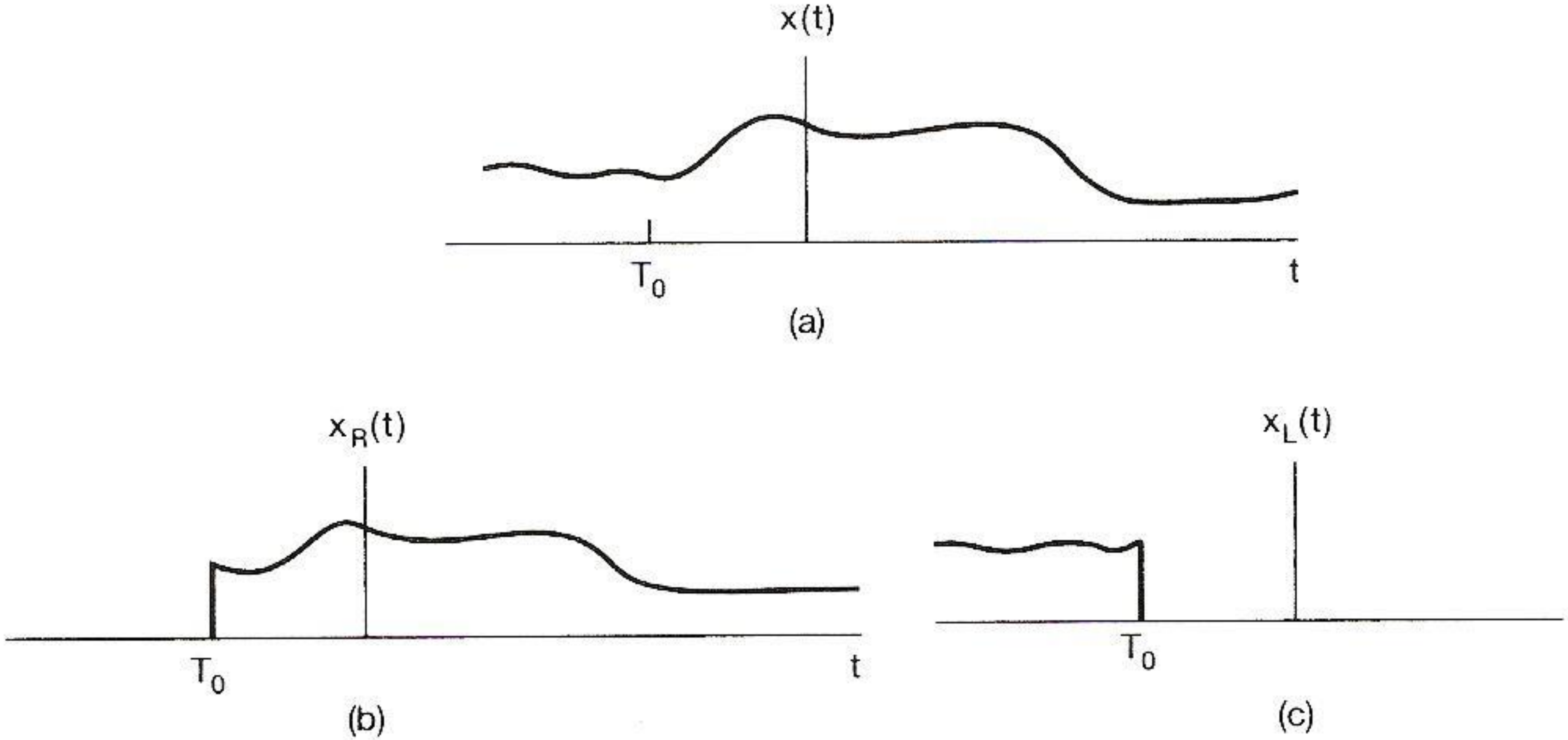


Figure 9.9 Two-sided signal divided into the sum of a right-sided and left-sided sig nal: (a) two-sided signal $x(t)$; (b) the right-sided signal equal to $x(t)$ for $t > T_0$ and equal to 0 for $t < T_0$; (c) the left-sided signal equal to $x(t)$ for $t < T_0$ and equal to 0 for $t > T_0$.

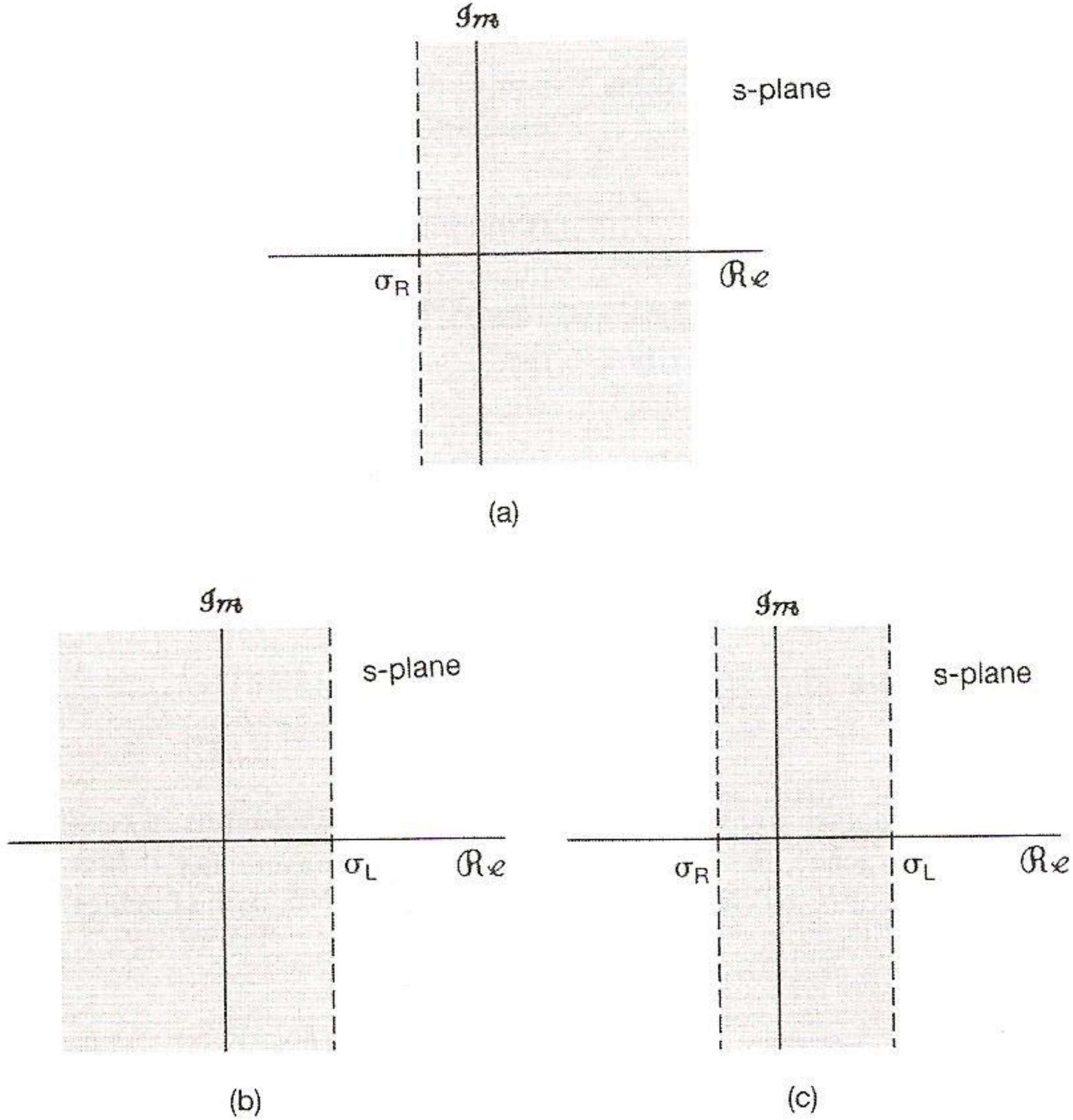


Figure 9.10 (a) ROC for $x_R(t)$ in Figure 9.9; (b) ROC for $x_L(t)$ in Figure 9.9; (c) the ROC for $x(t) = x_R(t) + x_L(t)$, assuming that the ROCs in (a) and (b) overlap.

Region of Convergence (ROC)

- A signal or an impulse response either doesn’t have a Laplace Transform, or falls into the 4 categories of Properties 3-6. Thus the ROC can be ϕ, s-plane, left-half plane, right-half plane, or a single strip

Region of Convergence (ROC)

- Property 7 : If $X(s)$ is rational, then its ROC is bounded by poles or extends to infinity
  - examples:

$$e^{-at}u(t) \xleftrightarrow{L} \frac{1}{s+a}, \mathrm{ROC} = \left\{ s \middle| \mathrm{Re}[s] > -a \right\}$$

$$-e^{-at}u(-t) \xleftrightarrow{L} \frac{1}{s+a}, \ \mathrm{ROC} = \left\{ s \middle| \mathrm{Re}[s] < -a \right\}$$

See Fig. 9.1, p.658 of text

  - partial-fraction expansion

$$X(s) = \sum_i \left(\frac{\beta_i}{s+a_i} \right)$$

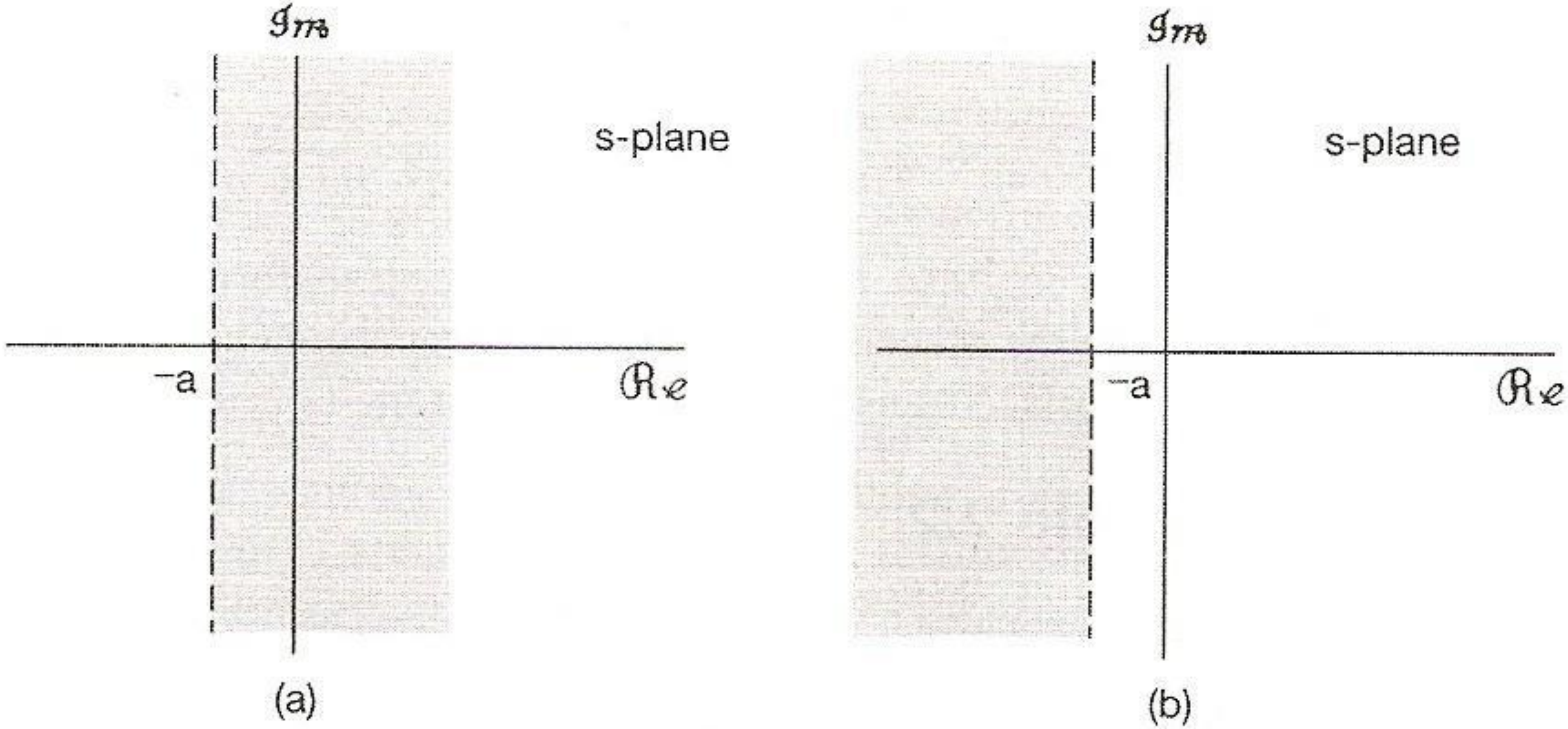
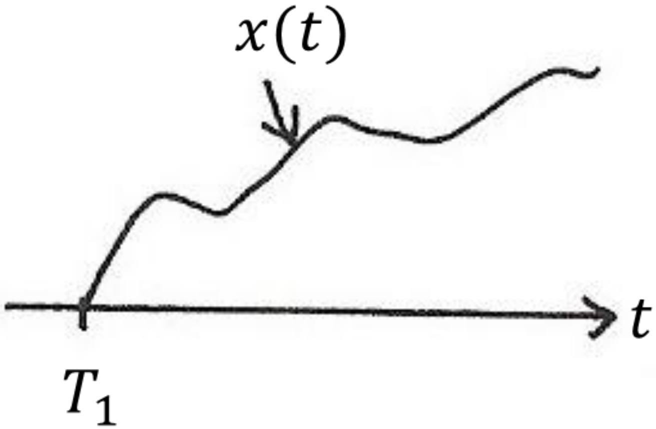


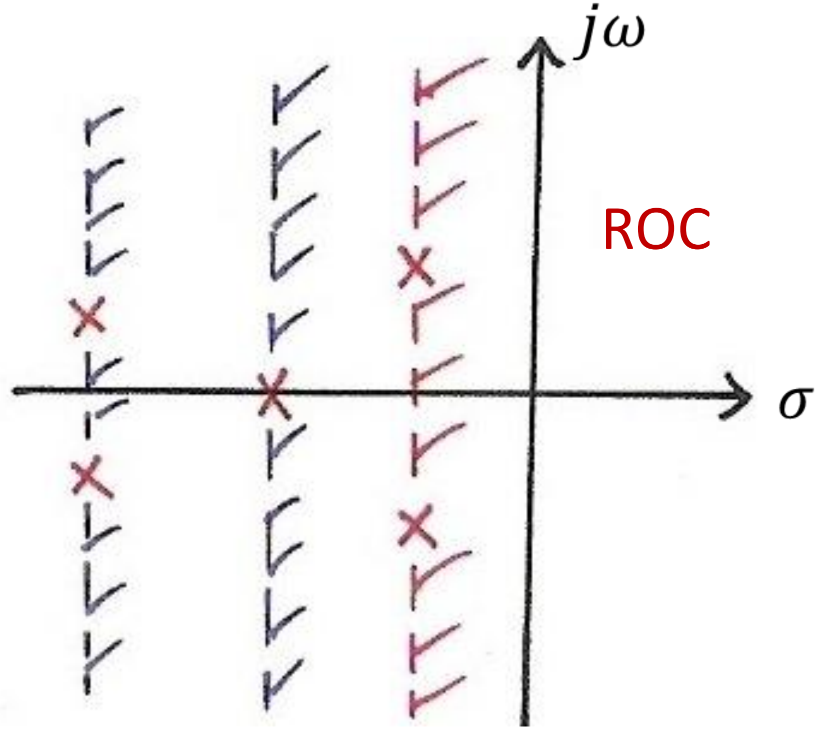
Figure 9.1 (a) ROC for Example 9.1; (b) ROC for Example 9.2.

Region of Convergence (ROC)

- Property 8 : If $x(s)$ is rational, then if $x(t)$ is right-sided, its ROC is the right-half plane to the right of the rightmost pole. If $x(t)$ is left-sided, its ROC is the left-half plane to the left of the leftmost pole.

Property 8





Region of Convergence (ROC)

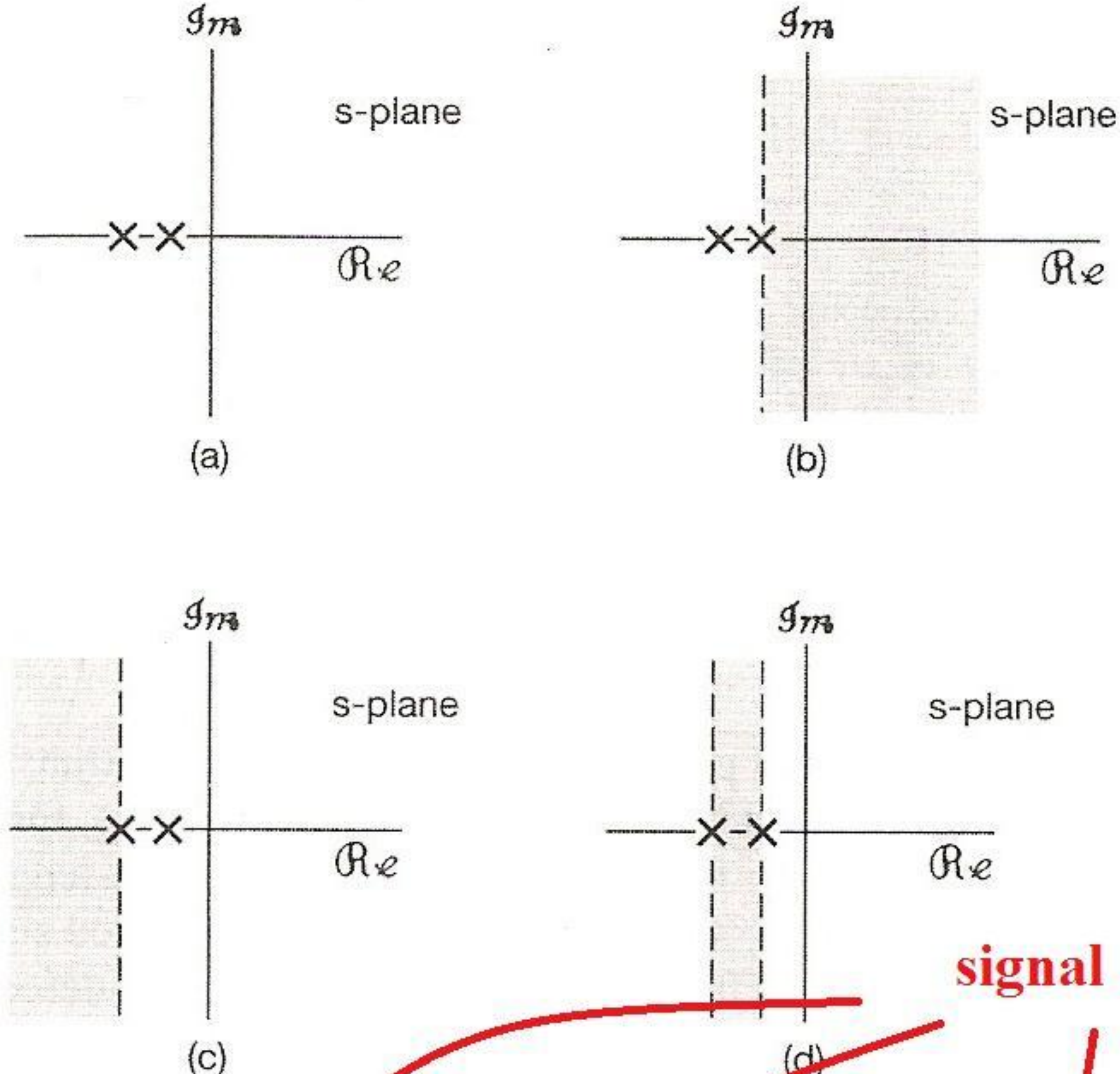
- An expression of $X(s)$ may corresponds to different signals with different ROC's.
  - an example:

$$X(s) = \frac{1}{(s+1)(s+2)}$$

See Fig. 9.13, p.670 of text

  - ROC is a part of the specification of $X(s)$

- The ROC of $X(s)$ can be constructed using these properties



signal

Figure 9.13 (a) Pole-zero pattern for Example 9.8; (b) ROC corresponding to a right-sided ~~sequence~~; (c) ROC corresponding to a left-sided ~~sequence~~; (d) ROC corresponding to a two-sided ~~sequence~~.

ROC and Poles/Zeros

- ROC: values of s for which $\int_{-\infty}^{\infty} x(t)e^{-st}\,dt$ converges
  - defined for a given $x(t)$, *not* for a given $X(s)$
  - very often in some region of s-plane this converges for $X(s)$, but such region may be out of ROC of $x(t)$
- Poles/Zeros: defined for a given $X(s)$
  - $X(s_1) = 0$ doesn't necessarily imply $\int_{-\infty}^{\infty} x(t)e^{-s_1 t}\,dt$ converges
  - Zeros may be out of ROC
- ROC and Poles/Zeros are related by the properties discussed in the textbook
  - one can define an $X(s)$ with given poles/zeros and then find $x(t)$, but such an $x(t)$ may not exist

Inverse Laplace Transform

$$x(t)e^{-\sigma_1 t} = F^{-1}\{X(\sigma_1 + j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma_1 + j\omega)e^{j\omega t}d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma_1 + j\omega)e^{(\sigma_1 + j\omega)t}d\omega$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} X(s)e^{st}ds \qquad ds = jd\omega$$

- integration along a line $\{s \mid \mathrm{Re}[s]=\sigma_1\} \in$ ROC for a fixed σ_1

Laplace Transform

(basis?)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma_1 + j\omega)\,\boxed{e^{(\sigma_1 + j\omega)t}}\,d\omega$$ (合成 ?)

$$X(\sigma_1 + j\omega) = \int_{-\infty}^{\infty} x(t)\,\boxed{e^{-(\sigma_1 + j\omega)t}}\,dt$$ (分析 ?)

$\neq$

$\neq \vec{A} \cdot \vec{v}$ (分析) $\left[e^{(\sigma_1 + j\omega)t}\right]^* = e^{\sigma_1 t} \cdot e^{-j\omega t}$

basis

$$x(t)e^{-\sigma_1 t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma_1 + j\omega)\,\boxed{e^{j\omega t}}\,dt$$ (合成)

$$X(\sigma_1 + j\omega) = \int_{-\infty}^{\infty} \left[x(t)e^{-\sigma_1 t}\right] \cdot e^{-j\omega t}\,dt$$ (分析)

Inverse Laplace Transform

- Practically in many cases : partial-fraction expansion works

$$X(s) = \sum_{i=1}^{m} \frac{A_i}{s + a_i} \quad , \text{ ROC}$$

for each term $\dfrac{A_i}{s + a_i}$

– ROC to the right of the pole at $s = -a_i$

$$\rightarrow A_i e^{-a_i t} u(t)$$

– ROC to the left of the pole at $s = -a_i$

$$\rightarrow -A_i e^{-a_i t} u(-t)$$

- Known pairs/properties practically helpful

9.2 Properties of Laplace Transform

$$x(t) \xleftrightarrow{L} X(s), \text{ ROC} = R$$

$$x_1(t) \xleftrightarrow{L} X_1(s), \text{ ROC} = R_1$$

$$x_2(t) \xleftrightarrow{L} X_2(s), \text{ ROC} = R_2$$

- Linearity

$$ax_1(t) + bx_2(t) \xleftrightarrow{L} aX_1(s) + bX_2(s), \text{ ROC} \supset (R_1 \cap R_2)$$

- Time Shift

$$x(t - t_0) \xleftrightarrow{L} e^{-st_0} X(s), \text{ ROC} = R$$

Time Shift

$$X(\sigma_1 + j\omega) = F[x(t)e^{-\sigma_1 t}]$$

$$F[x(t-t_0)\, e^{-\sigma_1 t}] = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-\sigma_1 t}\, e^{-j\omega t}\, dt$$

$$= \left(e^{-(\sigma_1+j\omega)t_0}\right) \cdot \int_{-\infty}^{\infty} x(t-t_0)\, e^{-\sigma_1(t-t_0)}\, e^{-j\omega(t-t_0)}\, d(t-t_0)$$

$$= e^{-(\sigma_1+j\omega)t_0} \cdot X(s)|_{s=\sigma_1+j\omega} = [e^{-st_0} \cdot X(s)]_{s=\sigma_1+j\omega}$$

- Shift in s-plane

$$e^{s_0 t}x(t) \xleftrightarrow{L} X(s-s_0), \ \text{ROC} = R + \text{Re}[s_0]$$

$$= \left\{ s + \text{Re}[s_0] \,\middle|\, s \in R \right\}$$

ROC shifted by $\text{Re}[s_0]$

See Fig. 9.23, p.685 of text

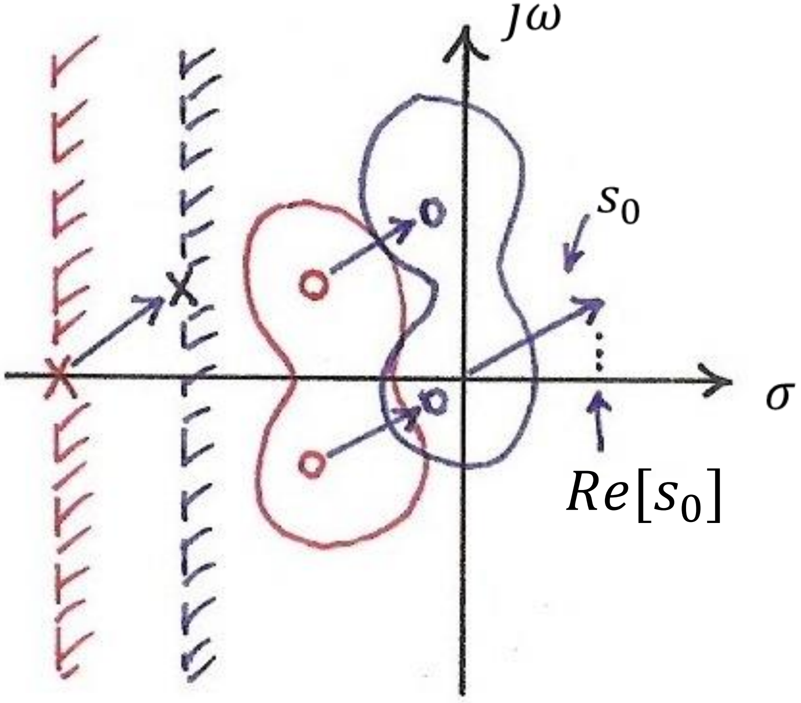
  – for $s_0 = j\omega_0$

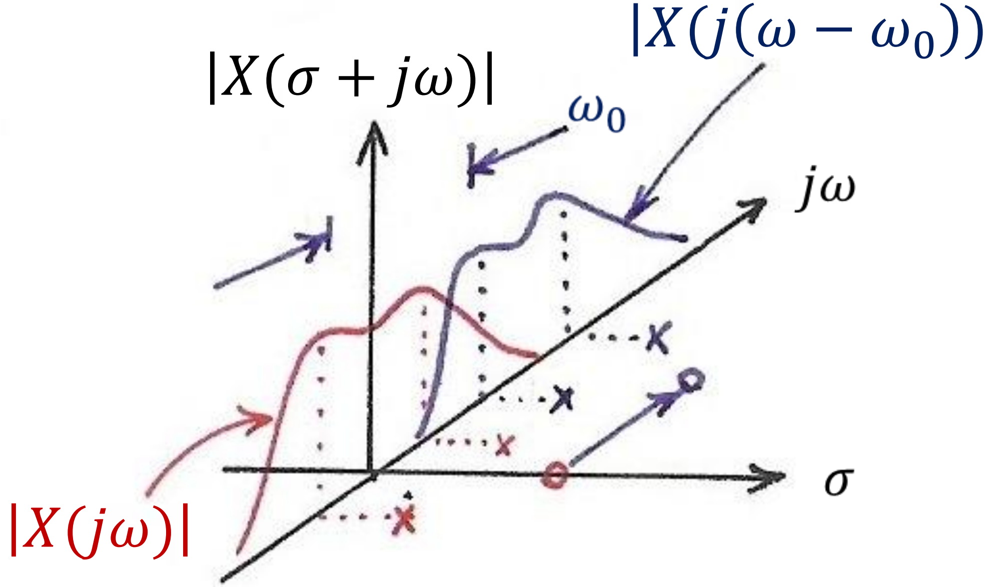
$$e^{j\omega_0 t}x(t) \xleftrightarrow{L} X(s-j\omega_0), \ \text{ROC} = R$$

shift along the $j\omega$ axis

Shift in *s*-plane

$$s_0 = j\omega_0$$





$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad \text{with ROC} = R,$$

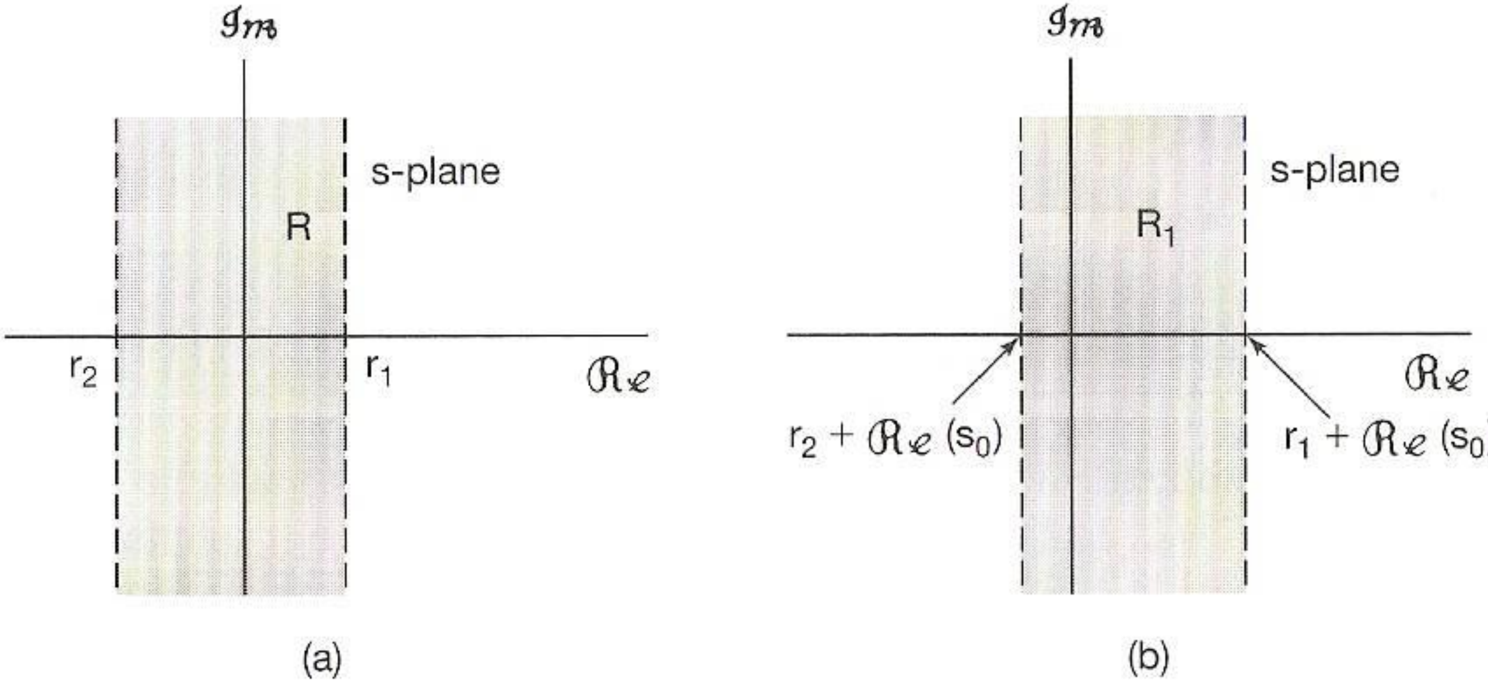


Figure 9.23 Effect on the ROC of shifting in the s-domain: (a) the ROC of $X(s)$; (b) the ROC of $X(s - s_0)$.

- Time Scaling (error on text corrected in class)

$$x(at) \xleftrightarrow{L} \frac{1}{|a|} X\left(\frac{s}{a}\right), \ \text{ROC} = aR = \left\{as \mid s \in R\right\}$$

- Expansion (?) of ROC if $a > 1$

  Compression (?) of ROC if $1 > a > 0$

  reversal of ROC about *jw*-axis if $a < 0$

  (right-sided → left-sided, etc.)

  See Fig. 9.24, p.686 of text

$$x(-t) \xleftrightarrow{L} X(-s), \ \text{ROC} = -R = \left\{-s \mid s \in R\right\}$$

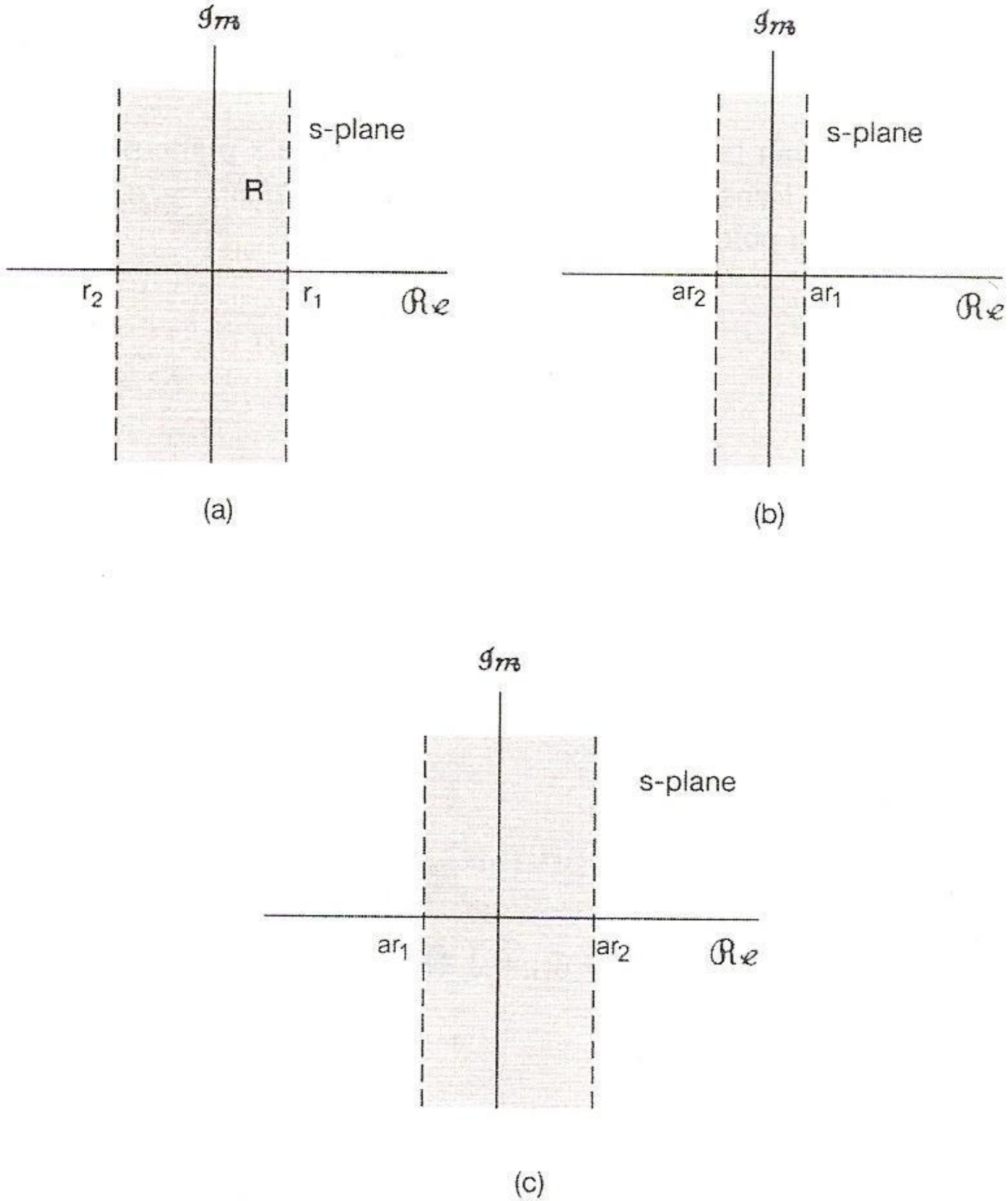


Figure 9.24 Effect on the ROC of time scaling: (a) ROC of $X(s)$; (b) ROC of $(1/|a|)X(s/a)$ for $0 < a < 1$; (c) ROC of $(1/|a|)X(s/a)$ for $0 > a > -1$.

- Conjugation

$$x^*(t) \xleftrightarrow{L} X^*(s^*), \ \text{ROC} = R$$

$$X(s) = X^*(s^*) \text{ if } x(t) \text{ real}$$

- if $x(t)$ is real, and $X(s)$ has a pole/zero at $s = s_0$

  then $X(s)$ has a pole/zero at $s = s_0^*$

- Convolution

$$x_1(t) * x_2(t) \xleftrightarrow{L} X_1(s)X_2(s), \ \text{ROC} \supset (R_1 \cap R_2)$$

- ROC may become larger if pole-zero cancellation occurs

- Differentiation

$$\frac{dx(t)}{dt} \xleftrightarrow{L} sX(s), \ \text{ROC} \supset R$$

ROC may become larger if a pole at $s = 0$ cancelled

$$-tx(t) \xleftrightarrow{L} \frac{dX(s)}{ds}, \ \text{ROC} = R$$

$$\left(-jtx(t) \overset{F}{\leftrightarrow} \frac{d}{d\omega} X(j\omega) \right)$$

- Integration in time Domain

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{\;L\;} \frac{1}{s}X(s),\ \text{ROC} \supset \left(R \cap \left\{ s \,\middle|\, \mathrm{Re}[s] > 0 \right\}\right)$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau = x(t) * u(t),$$

$$u(t) \xleftrightarrow{\;L\;} \frac{1}{s},\ \text{ROC} = \left\{ s \,\middle|\, \mathrm{Re}[s] > 0 \right\}$$

$$\left(\int_{-\infty}^{t} x(\tau)d\tau = x(t) * u(t) \overset{F}{\leftrightarrow} \frac{X(j\omega)}{j\omega} + \pi X(j0)\delta(\omega)\right)$$

$$\left(u(t) \overset{F}{\leftrightarrow} \frac{1}{j\omega} + \pi\delta(\omega)\right)$$

- Initial/Final – Value Theorems

  $x(t) = 0,\ t < 0$

  $x(t)$ has no impulses or higher order singularities at $t = 0$

$$x\left(o^{+}\right) = \lim_{s \to \infty} sX(s) \qquad \text{Initial - value Theorem}$$

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s) \qquad \text{Final - value Theorem}$$

- Tables of Properties/Pairs

  See Tables 9.1, 9.2, p.691, 692 of text

TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$	$X(s)$	R
		$x_1(t)$	$X_1(s)$	R_1
		$x_2(t)$	$X_2(s)$	R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\lvert a \rvert}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathcal{R}e\{s\} > 0\}$

Initial- and Final-Value Theorems

9.5.10 If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \longrightarrow \infty$, then

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\mathcal{Re}\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\mathcal{Re}\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\mathcal{Re}\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\mathcal{Re}\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\mathcal{Re}\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\mathcal{Re}\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{Re}\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{Re}\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos \omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathcal{Re}\{s\} > 0$
12	$[\sin \omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathcal{Re}\{s\} > 0$
13	$[e^{-\alpha t}\cos \omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\mathcal{Re}\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin \omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\mathcal{Re}\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\mathcal{Re}\{s\} > 0$

9.3 System Characterization with Laplace Transform

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

$$X(s) \qquad H(s) \qquad Y(s) = X(s)H(s)$$

system function

transfer function

- Causality

  – A causal system has an $H(s)$ whose ROC is a right-half plane

  $$h(t) \text{ is right-sided}$$

  – For a system with a rational $H(s)$, causality is equivalent to its ROC being the right-half plane to the right of the rightmost pole

  – Anticausality

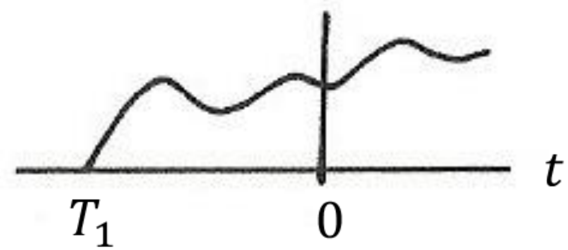
  a system is anticausal if $h(t) = 0,\ t > 0$

  an anticausal system has an $H(s)$ whose ROC is a left-half plane, etc.

Causality

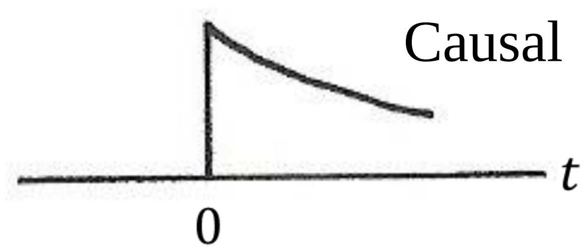
ROC is right half Plane $\Longleftrightarrow$ right-sided $\Longleftarrow$ Causal

right-sided $\not\rightarrow$ Causal



$$X(s) = \sum_i \frac{A_i}{s + a_i} \ , \qquad \frac{A_i}{s + a_i} \rightarrow A_i e^{-a_i t} u(t)$$

Causal



- Stability
  - A system is stable if and only if ROC of $H(s)$ includes the $j\omega$ -axis

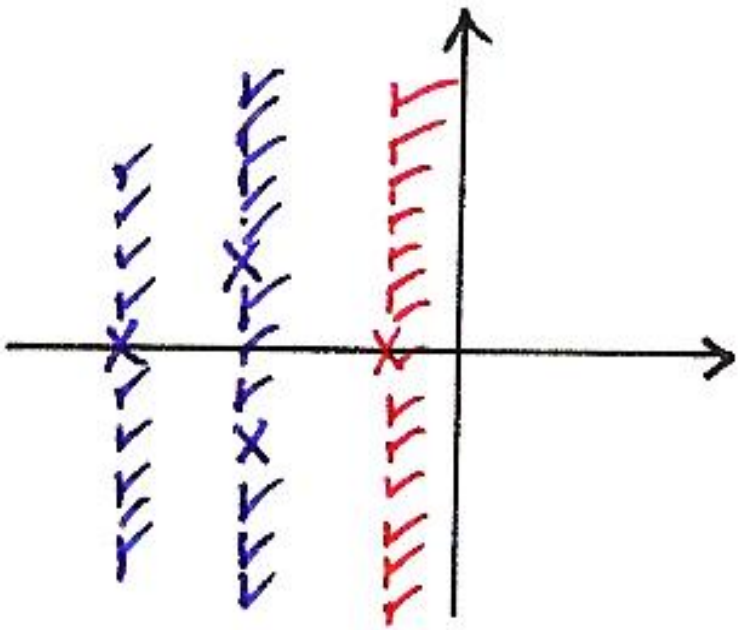
    $h(t)$ absolutely integrable, or Fourier transform converges

    See Fig. 9.25, p.696 of text

  - A causal system with a rational $H(s)$ is stable if and only if all poles of $H(s)$ lie in the left-half of s-plane

    ROC is to the right of the rightmost pole

Stability



$$\int_{-\infty}^{\infty} |h(t)| \; dt \;\; < \;\; B$$

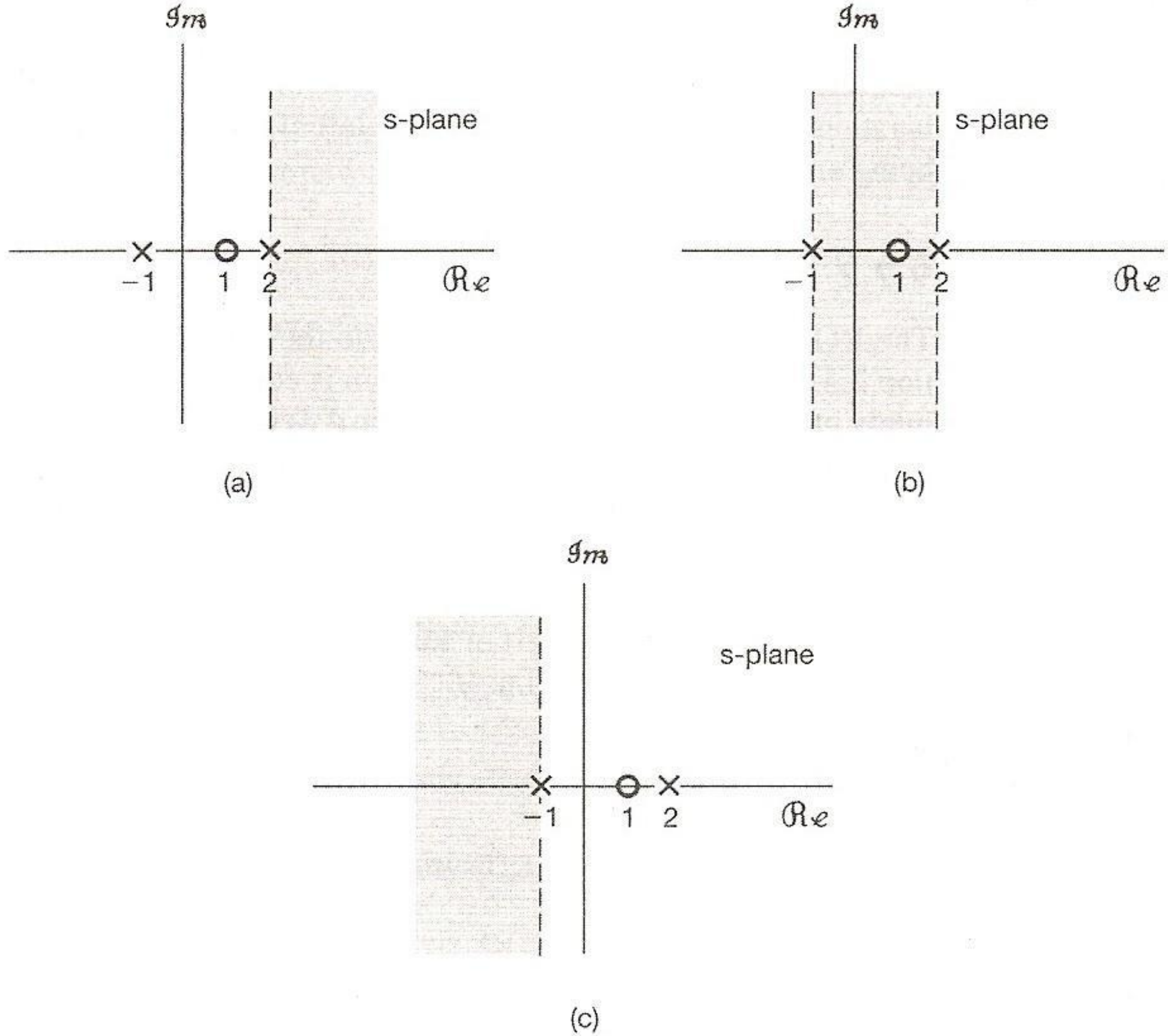


Figure 9.25 Possible ROCs for the system function of Example 9.20 with poles at $s = -1$ and $s = 2$ and a zero at $s = 1$: (a) causal, unstable system; (b) noncausal, stable system; (c) anticausal, unstable system.

- Systems Characterized by Differential Equations

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$\left(\sum_{k=0}^{N} a_k s^k\right) Y(s) = \left(\sum_{k=0}^{M} b_k s^k\right) X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s_k}$$

$\longrightarrow$ zeros

$\longrightarrow$ poles

- System Function Algebra
  - Parallel

$$h(t) = h_1(t) + h_2(t)$$

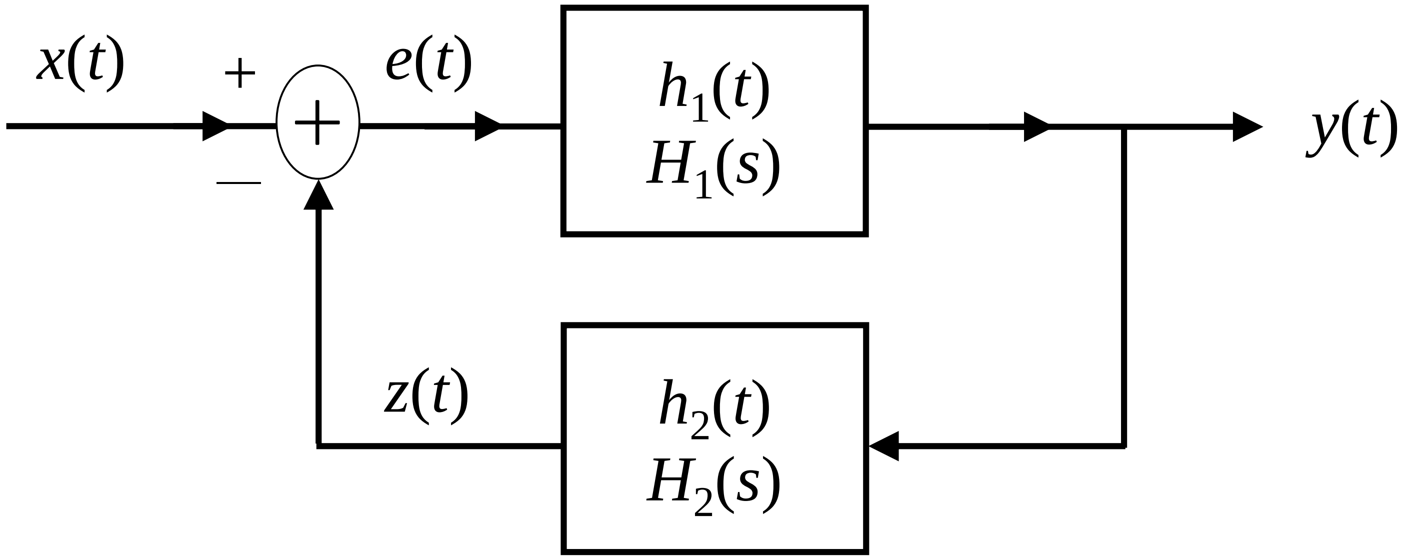
$$H(s) = H_1(s) + H_2(s)$$

  - Cascade

$$h(t) = h_1(t) \ast h_2(t)$$

$$H(s) = H_1(s) \cdot H_2(s)$$

- System Function Algebra
  - Feedback



$$Y(s) = H_1(s)\left[X(s) - H_2(s)Y(s)\right]$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

9.4 Unilateral Laplace Transform

$$X(s)_u = \int_{0^-}^{\infty} x(t)e^{-st}dt \quad \text{unilateral Laplace Transform}$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \quad \text{bilateral Laplace Transform}$$

impulses or higher order singularities at $t = 0$
included in the integration

$$x(t) \overset{L}{\longleftrightarrow} X(s)_u$$

– ROC for $X(s)_u$ is always a right-half plane

– a causal $h(t)$ has $H(s)_u = H(s)$

– two signals differing for $t < 0$

but identical for $t \geq 0$ have identical unilateral Laplace transforms

– similar properties and applications

Examples

- Example 9.7, p.668 of text

$$x(t) = e^{-b|t|} = e^{-bt}u(t) + e^{bt}u(-t)$$

$$e^{-bt}u(t) \xleftrightarrow{L} \frac{1}{s+b}, \ \mathrm{Re}\{s\} > -b$$

$$e^{bt}u(-t) \xleftrightarrow{L} \frac{-1}{s-b}, \ \mathrm{Re}\{s\} < +b$$

$$e^{-b|t|} \xleftrightarrow{L} \frac{1}{s+b} - \frac{1}{s-b} = \frac{-2b}{s^2 - b^2}, \ -\mathrm{b} < \mathrm{Re}\{s\} < +b, \quad \mathrm{b} > 0$$

$$\text{No Laplace Transform}, \quad \mathrm{b} \le 0$$

Examples

- Example 9.7, p.668 of text

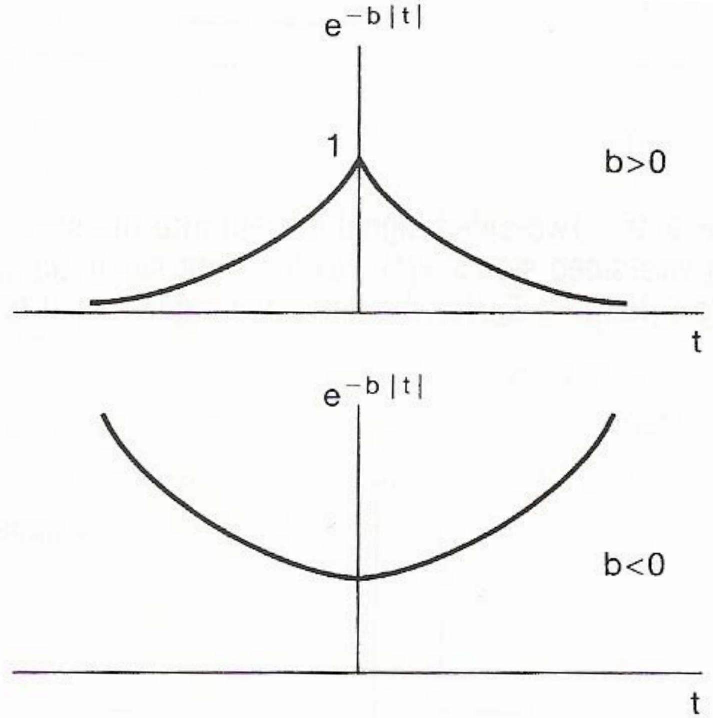


Figure 9.11 Signal $x(t) = e^{-b|t|}$ for both $b > 0$ and $b < 0$.

b > 0

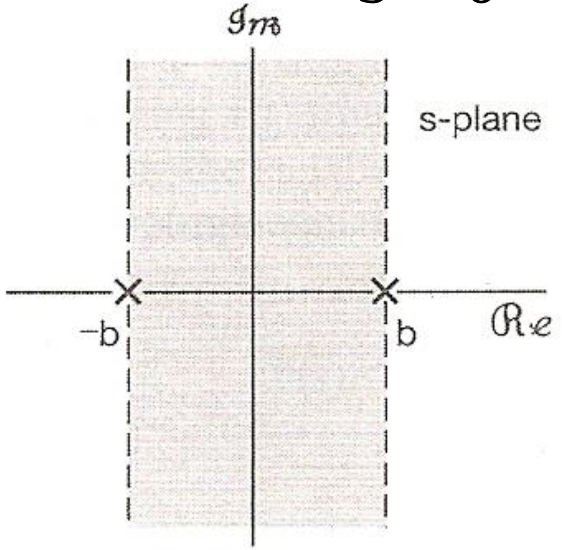


Figure 9.12 Pole-zero plot and ROC for Example 9.7.

Examples

- Example 9.9/9.10/9.11, p.671-673 of text

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$\mathrm{Re}\{s\} > -1, \quad x(t) = \left[e^{-t} - e^{-2t}\right]u(t)$$

$$\mathrm{Re}\{s\} < -2, \quad x(t) = \left[-e^{-t} + e^{-2t}\right]u(-t)$$

$$-2 < \mathrm{Re}\{s\} < -1, \quad x(t) = -e^{-t}u(-t) - e^{-2t}u(t)$$

Examples

- Example 9.9/9.10/9.11, p.671-673 of text

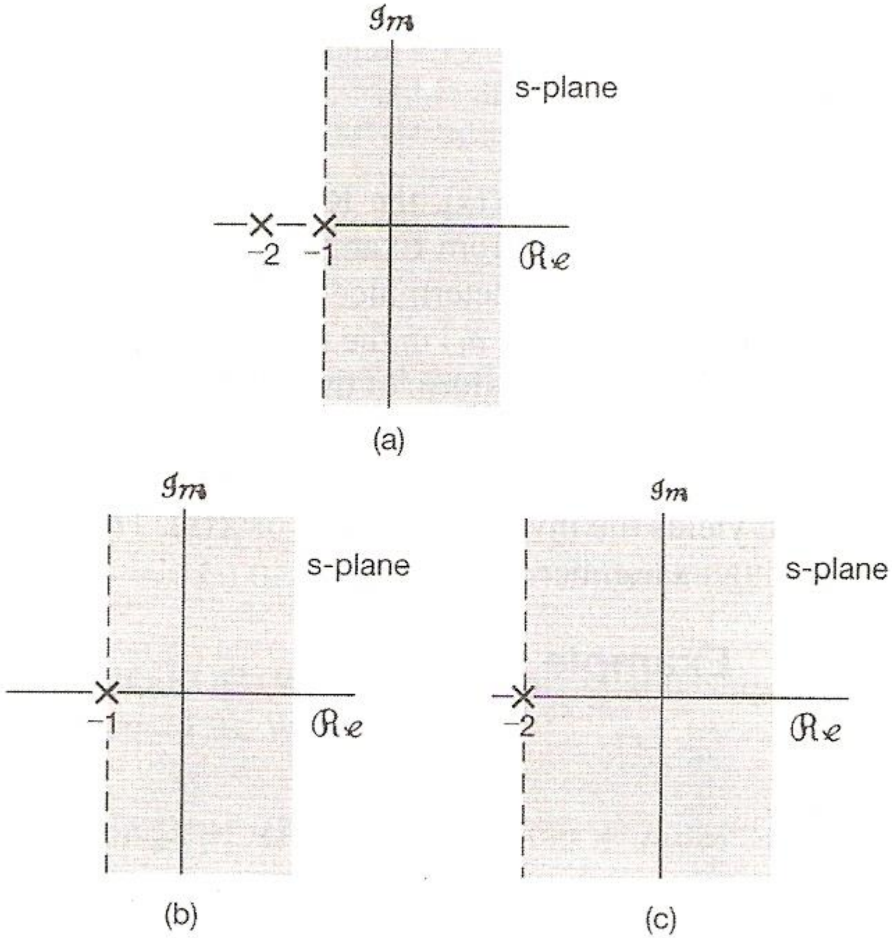


Figure 9.14 Construction of the ROCs for the individual terms in the partial-fraction expansion of $X(s)$ in Example 9.8: (a) pole-zero plot and ROC for $X(s)$; (b) pole at $s = -1$ and its ROC; (c) pole at $s = -2$ and its ROC.

Examples

- Example 9.25, p.701 of text

$$x(t) = e^{-3t}u(t) \to y(t) = \left[e^{-t} - e^{-2t}\right]u(t)$$

$$X(s) = \frac{1}{s+3}, \quad \mathrm{Re}\{s\} > -3$$

$$Y(s) = \frac{1}{(s+1)(s+2)}, \quad \mathrm{Re}\{s\} > -1$$

$$\therefore H(s) = \frac{Y(s)}{X(s)} = \frac{s+3}{(s+1)(s+2)}, \quad \mathrm{Re}\{s\} > -1$$

$$\left(ROC_Y = ROC_X \cap ROC_H, \text{ poles at } s = -1, s = -2\right)$$

ROC of $H(s)$ is to the right of the rightmost pole

$\to H(s)$ is causal

All poles in the left-half plane

$\to H(s)$ is stable

Problem 9.60, p.737 of text

Echo in telephone communication

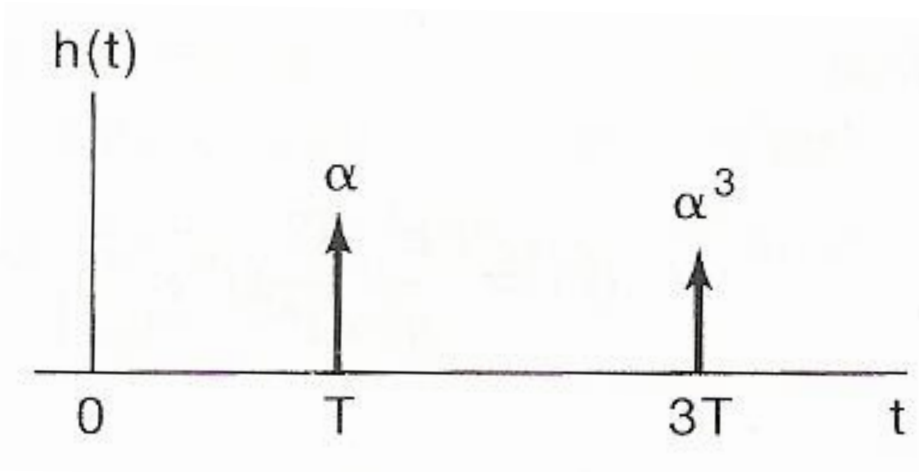


Figure P9.60

$$h(t) = \alpha\delta(t - T) + \alpha^3\delta(t - 3T)$$

$$H(s) = \alpha e^{-sT} + \alpha^3 e^{-3sT}$$

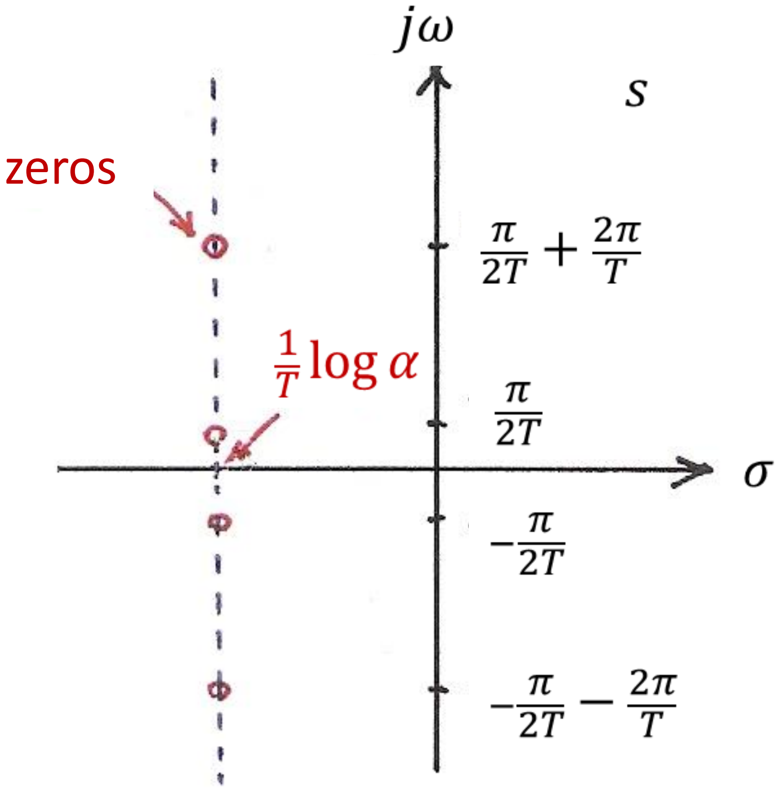
$$= \alpha e^{-sT}\left(1 + \alpha^2 e^{-2sT}\right), \text{ all } s, \text{ no poles}$$

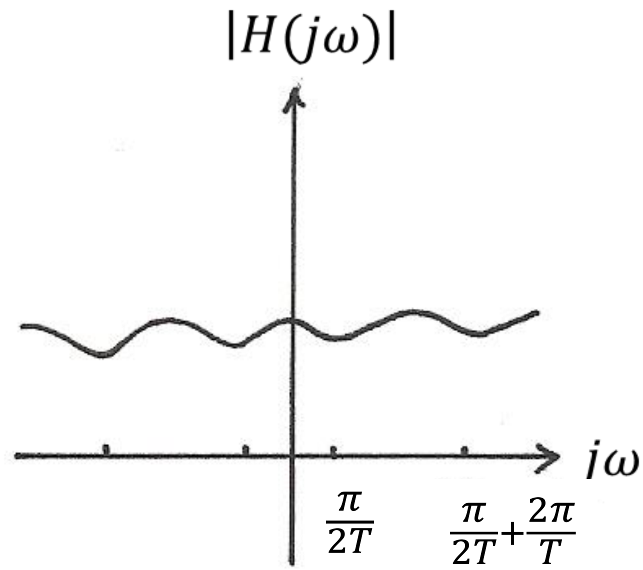
<u>Problem 9.60</u>, p.737 of text

To find zeros of $H(s)$

$$1 + \alpha^2 e^{-2sT} = 0, \ \alpha e^{-sT} = \pm j = e^{j(\pm\frac{\pi}{2} \pm 2m\pi)}$$

$$s = -\frac{1}{T}\log\alpha + j(\pm\frac{\pi}{2T} \pm \frac{2m\pi}{T})$$





Problem 9.60, p.737 of text

for $s_0 = \frac{1}{T} \log \alpha + j\left(\frac{\pi}{2T}\right) = \sigma_0 + j\omega_0$

$e^{s_0 t} = e^{\left(\frac{1}{T} \log \alpha + j\frac{\pi}{2T}\right)t} = -\alpha^2$ when $t = 2T$

$H(s_0) = 0$, eigenvalue

Signal generated by $\alpha\delta(t - T)$ cancels that by $\alpha^3\delta(t - 3T)$

Problem 9.44, p.733 of text

$$\mathrm{x}(\mathrm{t}) = \sum_{n=0}^{\infty} e^{-nT}\delta(t - nT) \qquad \left(\mathrm{e}^{\text{-T}} = \alpha\right)$$

$$\mathrm{X}(\mathrm{s}) = \sum_{n=0}^{\infty} e^{-nT}e^{-snT} = \frac{1}{1 - e^{-(1+s)T}}$$

to find poles

$$e^{-(1+s)T} = 1 = e^{jm(2\pi)}, \ \mathrm{s} = \text{-}1 + jm\left(\frac{2\pi}{T}\right)$$

$$\text{for } s_0 = \text{-}1 + j\frac{2\pi}{T} = \sigma_0 + j\omega_0$$

$$e^{s_0 t} = e^{(-1+j\frac{2\pi}{T})t} = e^{-T} = \alpha \quad \text{when } t = T$$