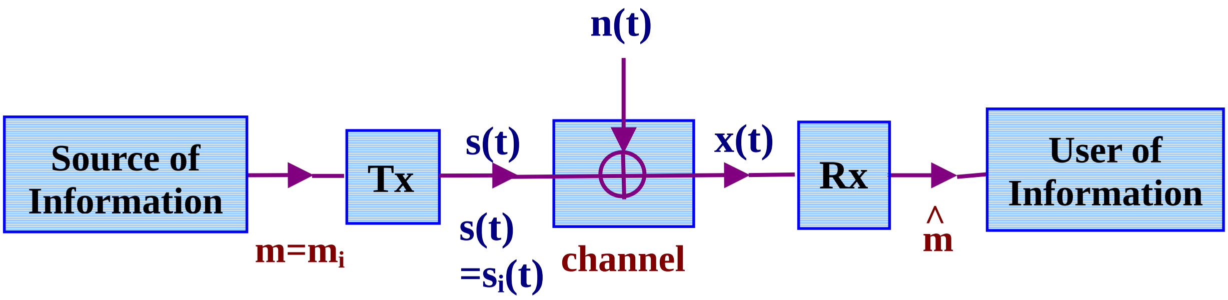
3.6 Vector Space Theory of Digital Communications

Problem Definition



- **Source of Information**

  a symbol $m = m_i \in \{m_1, m_2, \ldots m_M\}$ sent at a time in [0,T]

  $\text{Prob}[m = m_i] = P_i\ , \sum_{i=1}^{M} P_i = 1$

  - Example: $M = 2$

    $m_1 = 1,\ m_2 = 0$

    $M = 4$

    $m_1 = 00,\ m_2 = 01,\ m_3 = 10,\ m_4 = 11$

- **Transmitter**

  a signal pulse $s(t) = s_i(t) \in \{s_1(t), s_2(t), \ldots s_M(t)\}$ sent

  if $m = m_i$, $s_i(t)$ defined in [0,T]

- **Channel**

  ideal channel, $x(t) = s(t) + n(t)$ in [0,T]

  $n(t)$: white, Gaussian, zero-mean

  $S_n(\omega) = \dfrac{N_0}{2}$, all ω

  $R_n(\tau) = \dfrac{N_0}{2}\delta(\tau)$

Problem Definition

- **Receiver**

  a mapping relation

  $$Rx : x(t) \rightarrow \hat{m} \in \{m_1, m_2, \ldots m_M\}$$

- **Symbol Error Probability**

  $$P_{e,s} = \text{Prob}[\hat{m} \neq m] = \sum_{i=1}^{M} \text{Prob}[\hat{m} \neq m_i \mid m_i]\, P_i$$

- **Goal**

  Design the mapping relation in Receiver such that

  $$P_{e,s} = \min$$

Vector Space Representation of the Problem

- **Assume a set of basis functions**

$$\{\phi_j(t),\ j = 1, 2, \ldots, N\}$$

$$\int_0^T \phi_i(t)\ \phi_j(t)dt = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

- orthonormal basis

- **Assume the M signals $\{s_i(t),\ i = 1, 2, \ldots M\}$ can be expanded from these N basis functions**

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\ \phi_j(t)$$

$$s_{ij} = \int_0^T s_i(t)\phi_j(t)dt$$

$$= \int_0^T \Big[\sum_{k=1}^{N} s_{ik}\ \phi_k(t)\Big]\phi_j(t)dt$$

$$s_i(t) \longleftrightarrow \overline{s_i} = (s_{i1}, s_{i2}, \ldots s_{iN})$$

$$[s_i(t)] \cdot [s_j(t)] = \int_0^T s_i(t)s_j(t)dt = \overline{s_i} \cdot \overline{s_j} = \sum s_{ik}\ s_{jk}$$

$$\int_0^T [s_i(t)]^2 dt = \|\ \overline{s_i}\ \|^2 = E_i$$

- **Dimension N of the sub-space**

$$N \leq M$$

- Gram-Schmidt Orthogonalization Procedure

$$\phi_1(t) = s_1(t) / E_1^{1/2}$$

$$a_{21} = \int_0^T s_2(t)\ \phi_1(t)dt$$

$$\phi_2'(t) = s_2(t) - a_{21}\phi_1(t)$$

$$E_2' = \int_0^T [\phi_2'(t)]^2 dt$$

$$\phi_2(t) = \phi_2'(t) / (E_2')^{1/2}, \ \ldots\ldots$$

- M signals expand an N-dim subspace, $N \leq M$

Vector Space Representation of the Problem

- **Projection of n(t) in the N-dim subspace**

$$n(t) \rightarrow \bar{n} = (n_1, n_2, \ldots n_N)$$

$$n_j = \int_0^T n(t)\phi_j(t)dt$$

$$n(t) = \sum_{j=1}^{N} n_j \phi_j(t) + n'(t)$$

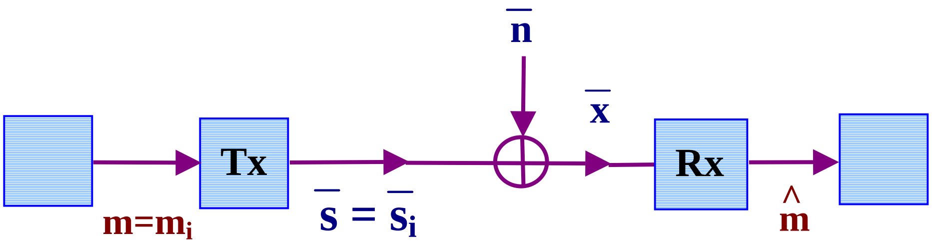
noise outside of the subspace or orthogonal to the subspace

- **Thm :**

$n'(t)$ is Irrelevant for optimal decision

optimal decision can be made solely in the subspace without considering $n'(t)$

- **Vector Space Representation**



$$\bar{x} = \bar{s} + \bar{n}$$

$$Rx : \bar{x} \rightarrow \hat{m} \in \{m_1, m_2, \ldots m_M\}$$

$x(t)$ reduced to a N-dim vector, $N \leq M$

Vector Space Representation of the Problem

- **Statistics of $\bar{n}$**

$$n_j = \int_0^T n(t)\phi_j(t)dt$$

n_j : Gaussian, zero-mean random variable

$$E[n_i n_j] = E[\int_0^T n(t)\,\phi_i(t)\,dt \int_0^T n(\tau)\phi_j(\tau)d\tau]$$

$$= \iint_0^T \phi_i(t)\phi_j(\tau)E[n(t)n(\tau)]dtd\tau$$

$$R_n(t-\tau) = \frac{N_0}{2}\,\delta(t-\tau)$$

$$= \frac{N_0}{2}\,\delta_{ij} = \begin{cases} \frac{N_0}{2}, & i = j \\ 0\,, & i \neq j \end{cases}$$

- all n_j are independent, identically distributed (iid), Gaussian, zero-mean with variance $\sigma^2 = \frac{N_0}{2}$

$$f_{n_j}(n) = \frac{1}{(2\pi)^{1/2}\sigma}\; e^{-n^2/2\sigma^2}$$

$$f_{\bar{n}}(\bar{n}) = \frac{1}{(2\pi\sigma^2)^{N/2}}\; e^{-|\bar{n}|^2/2\sigma^2}$$

spherically symmetry, isotropic

See Fig. 5.4, 5.7, p. 313, p. 323 of Haykin

Optimal Decision

- **Maximum A Posteriori Probability (MAP) Principle**

  assume $\bar{x}$ is received

  possibly m_1 was sent with $\text{Prob}[m_1|\bar{x}]$
  m_2 was sent with $\text{Prob}[m_2|\bar{x}]$
  ⋮
  m_k was sent with $\text{Prob}[m_k|\bar{x}] = \max$
  ⋮
  m_M was sent with $\text{Prob}[m_M|\bar{x}]$

  if $\text{Prob}[m_k|\bar{x}] > \text{Prob}[m_j|\bar{x}]$, $j \neq k$

  then let $\bar{x} \rightarrow \hat{m} = m_k$

- **Evaluation of $\text{Prob}[m_i|\bar{x}]$**

  $$\text{Prob}[m_i|\bar{x}] = \frac{\text{Prob}[\bar{x}|m_i]P_i}{\text{Prob}[\bar{x}]} = D_i(\bar{x}), \text{ decision function}$$

  Rx: $\bar{x} \rightarrow \hat{m} = m_k$ if $D_k(\bar{x}) = \max$ among all $D_i(\bar{x})$

  $$D_i(\bar{x}) = \text{Prob}[\bar{x}|m_i]P_i = \text{Prob}[\bar{x}|\bar{s}_i]P_i$$
  $$= \text{Prob}[\bar{n} = \bar{x} - \bar{s}_i]P_i$$
  $$= f_{\bar{n}}(\bar{x} - \bar{s}_i)P_i$$
  $$= \left[\frac{1}{(2\pi\sigma^2)^{N/2}} e^{-|\bar{x}-\bar{s}_i|^2/2\sigma^2}\right] \cdot P_i$$

  $$D_i(\bar{x}) = -|\bar{x} - \bar{s}_i|^2 + 2\sigma^2 \ln[P_i]$$

  - assign $\hat{m}$ to the signal vector NEAREST to $\bar{x}$, with distance adjusted by P_i
  - most probable noise corresponds to minimum distance

Optimal Decision

- **Decision Region**

  the N-dim subspace can be divided into M regions in advance, $R_i = \{\bar{x} | \bar{x} \rightarrow m_i\}$, i = 1, 2,…M

  - boundary between R_i and R_j defined by

$$D_i(\bar{x}) = D_j(\bar{x})$$

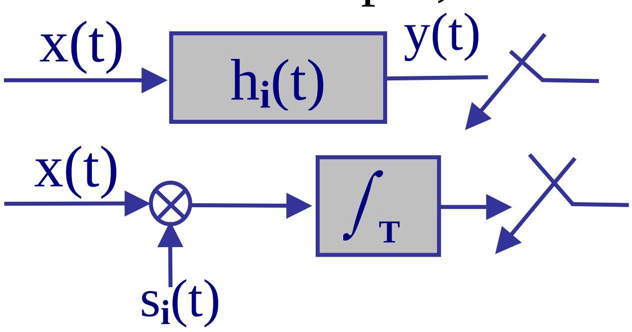
Receiver Structure

$$D_i(\bar{x}) = -\,|\,\bar{x} - \bar{s}_i\,|^2 + 2\sigma^2 \ln[P_i]$$
$$= -\,|\,\bar{x}\,|^2 - |\,\bar{s}_i\,|^2 + 2\bar{x} \bullet \bar{s}_i + 2\sigma^2 \ln[P_i]$$
$$D_i(\bar{x}) = \bar{x} \bullet \bar{s}_i + \sigma^2 \ln[P_i] - \frac{1}{2}\,|\,\bar{s}_i\,|^2$$
$$= \bar{x} \bullet \bar{s}_i + C_i$$

- **Receiver Structure 1**

$$\bar{x} \bullet \bar{s}_i = \int_0^T x(t) s_i(t) dt = y(t_1)$$
$$y(t) = \int_{-\infty}^{\infty} x(\tau) h_i(t-\tau) d\tau\ ,\ \ h_i(t) = s_i(t_1 - t)$$

  - matched filter output, correlator



  - a total of M arms needed in general

- **Receiver Structure 2**

$$\bar{x} = (x_1, x_2, \ldots x_N)\ ,\ \bar{x} \bullet \bar{s}_i = \sum_{j=1}^{N} x_j s_{ij}$$
$$x_j = \int_0^T x(t) \phi_j(t) dt\ ,\ j = 1, 2, \ldots N$$

  - a total of N arms needed, useful when N<M

  See Fig. 5.9 ,   p. 327 of Haykin

Error Probability Independent of Choice of Basis

- Decision Regions $\{R_i, i = 1, 2, \ldots M\}$ independent of basis
- Error probabilities depends on the constellation structure and distances among the signal vectors only
- $\bar{n}$ is spherically symmetric

  examples : $N = 2, \ M = 2$

  $N = 2, \ M = 4$

• **Minimum Energy Signal Set**

Given $\{\bar{s}_i, i = 1, 2, \ldots M\}$, find $\bar{a}$ such that

$$\sum_{i=1}^{M} \| \bar{s}_i - \bar{a} \|^2 P_i = \min$$

- $\bar{a} = \sum_{i=1}^{M} \bar{s}_i P_i$

Ref : 5.1, 5.2, 5.3, 5.4, 5.5, 5.6, 5.7 of Haykin