Summary

Is $b$ a linear combination of columns of $A$?  
Is $b$ in the span of the columns of $A$?

NO  
No solution

YES

The columns of $A$ are independent. 
Rank $A = n$ 
Nullity $A = 0$

Unique solution

The columns of $A$ are dependent. 
Rank $A < n$ 
Nullity $A > 0$

Infinite solution

$A: m \times n$  
$x \in \mathbb{R}^n$  
$b \in \mathbb{R}^m$
Rank

- Maximum number of Independent Columns
- Number of Pivot Column
- Number of Non-zero rows

\[ \text{Rank} A \leq \text{Number of columns} \]
\[ \leq \min(\text{Number of columns, Number of rows}) \]
\[ \leq \text{Number of rows} \]
73. Describe an $m \times n$ matrix with rank 0

79. What is the largest possible rank of an $m \times n$ matrix?

79. What is the smallest possible nullity of an $m \times n$ matrix?

82. Let $A$ be an $m \times n$ matrix and $b$ be a vector in $R^m$. What must be true about the rank of $A$ if $Ax = b$ has a unique solution?

Rank $A =$?
83. A system of linear equations is called *underdetermined* if it has fewer equations than variables. What can be said about the number of solutions of an *underdetermined* system?

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Summary

**Is \( b \) a linear combination of columns of \( A \)?**

- **NO**: The columns of \( A \) are \textit{independent}.
  - \( \text{Rank } A = n \)
  - \( \text{Nullity } A = 0 \)
  - Unique solution

- **YES**: The columns of \( A \) are \textit{dependent}.
  - \( \text{Rank } A < n \)
  - \( \text{Nullity } A > 0 \)
  - Infinite solution

**Is \( b \) in the span of the columns of \( A \)?**

- **NO**: No solution
- **YES**: Unique solution

\[ A: m \times n \quad x \in R^n \quad b \in R^m \]
Ax = b is consistent for every b

\[ A: m \times n \]

Every b is in the span of the columns of A = \([a_1, \ldots, a_n]\)

Every b belongs to \(\text{Span}\{a_1, \ldots, a_n\}\)

\(\text{Span}\{a_1, \ldots, a_n\} = R^m\)

RREF of [A b] cannot have a row whose only non-zero entry is at the last column

RREF of A cannot have zero row

Rank A = no. of rows
81. Let $A$ be a $4 \times 3$ matrix. Is it possible that $Ax = b$ is consistent for every $b$ in $R^4$?

85. Prove that if $A$ is an $m \times n$ matrix with rank $m$, then $Ax = b$ is consistent for every $b$ in $R^m$.

86. Prove that a matrix equation $Ax = b$ is consistent if and only if the ranks of $A$ and $[A \ b]$ are equal.
Chapter 1: Review
Let $A$ be an $m \times n$ matrix with reduced row echelon form $R$. Describe the reduced row echelon form of each of the following matrices.

(a) $[A \; 0]$

(b) $[a_1 \; a_2 \; \ldots \; a_k]$ for $k < n$

(c) $cA$, where $c$ is a nonzero scalar

(d) $[I_m \; A]$

(e) $[A \; cA]$, where $c$ is any scalar