

# Inverse of Elementary Matrices

# Elementary Row Operation

- Every elementary row operation can be performed by matrix multiplication.
- 1. Interchange

elementary matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

- 2. Scaling

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$$

- 3. Adding  $k$  times row  $i$  to row  $j$ :

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ ka + c & kb + d \end{bmatrix}$$

# Elementary Matrix


- Every elementary row operation can be performed by matrix multiplication.
- How to find elementary matrix?

elementary matrix

E.g. the elementary matrix that exchanges the 1<sup>st</sup> and 2<sup>nd</sup> rows

$$E \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 4 \\ 3 & 6 \end{bmatrix}$$

$$E \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$


$$E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Elementary Matrix

- How to find elementary matrix?
- Apply the desired elementary row operation on Identity matrix

Exchange the 2<sup>nd</sup> and 3<sup>rd</sup> rows

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Multiply the 2<sup>nd</sup> row by -4

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Adding 2 times row 1 to row 3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

# Elementary Matrix

- How to find elementary matrix?
- Apply the desired elementary row operation on Identity matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$E_1 A =$$

$$E_2 A =$$

$$E_3 A =$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

# Inverse of Elementary Matrix

Reverse elementary row operation

Exchange the 2<sup>nd</sup> and 3<sup>rd</sup> rows

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$



Exchange the 2<sup>nd</sup> and 3<sup>rd</sup> rows

$$E_1^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Multiply the 2<sup>nd</sup> row by -4

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Multiply the 2<sup>nd</sup> row by -1/4

$$E_2^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Adding 2 times row 1 to row 3

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$



Adding -2 times row 1 to row 3

$$E_3^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

# RREF v.s. Elementary Matrix

- Let  $A$  be an  $m \times n$  matrix with reduced row echelon form  $R$ .

$$E_k \cdots E_2 E_1 A = R$$

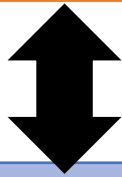
- There exists an invertible  $m \times m$  matrix  $P$  such that  $PA=R$

$$P = E_k \cdots E_2 E_1$$

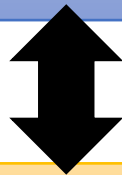
$$P^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

# Invertible

An  $n \times n$  matrix  $A$  is invertible.



The reduced row echelon form of  $A$  is  $I_n$



$A$  is a product of elementary matrices

$$R = \text{RREF}(A) = I_n$$

$$E_k \cdots E_2 E_1 A = I_n$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n$$

$$= E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$