Inverse of Elementary Matrices

Elementary Row Operation

- Every elementary row operation can be performed by matrix multiplication.
- 1. Interchange

elementary matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

- 2. Scaling

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$$

- 3. Adding *k* times row i to row j:

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ ka + c & kb + d \end{bmatrix}$$

Elementary Matrix

- Every elementary row operation can be performed by matrix multiplication.
- How to find elementary matrix?

elementary matrix

E.g. the elementary matrix that exchanges the 1st and 2nd rows

$$E\begin{bmatrix}1 & 4\\2 & 5\\3 & 6\end{bmatrix}=\begin{bmatrix}2 & 5\\1 & 4\\3 & 6\end{bmatrix}$$

$$E\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow \quad E=\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}$$

Elementary Matrix

- How to find elementary matrix?
- Apply the desired elementary row operation on Identity matrix

Exchange the 2nd and 3rd rows

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Multiply the 2nd row by -4

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Adding 2 times row 1 to row 3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Elementary Matrix

- How to find elementary matrix?
- Apply the desired elementary row operation on Identity matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$E_1 A =$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2 A =$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$E_3 A =$$

Inverse of Elementary Matrix

Reverse elementary row operation

Exchange the 2nd and 3rd rows

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

⟺

Exchange the 2nd and 3rd rows

$$E_1^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Multiply the 2nd row by -4

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

⟺

Multiply the 2nd row by -1/4

$$E_2^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Adding 2 times row 1 to row 3

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

⟺

Adding -2 times row 1 to row 3

$$E_3^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

RREF v.s. Elementary Matrix

- Let A be an mxn matrix with reduced row echelon form R.

$$E_k \cdots E_2 E_1 A = R$$

- There exists an invertible m x m matrix P such that PA=R

$$P = E_k \cdots E_2 E_1$$

$$P^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

Invertible

An n x n matrix A is invertible.

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The reduced row echelon form of A is I_n

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A is a product of elementary matrices

R=RREF(A)=I_n

$$E_k \cdots E_2 E_1 A = I_n$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n$$

$$= E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$