RREF v.s Solution

Solving system of linear equation

A ***complex*** system of linear equations

A ***simple*** system of linear equations

$A\mathbf{x} = \mathbf{b}$

equivalent

$R'\mathbf{x} = \mathbf{b}'$

$A' = [\, A \;\; \mathbf{b} \,]$ A'' A''' … $R = [\, R' \; \mathbf{b}' \,]$

…

Reduced Row Echelon Form (RREF)

elementary row operations:

1. Interchange any two rows of the matrix
2. Multiply every entry of some row by the same nonzero scalar
3. Add a multiple of one row of the matrix to another row

Reduced Row Echelon Form

- A system of linear equations is easily solvable if its augmented matrix is in ***reduced row echelon form***

Example 1. Unique Solution

$$\begin{array}{cccc} x_1 & x_2 & x_3 & b \\ \end{array}$$
$$\left[\begin{array}{cccc} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

If RREF looks like $[\ I \ \ \mathbf{b}'\ ]$

$$\begin{aligned} x_1 &= -4 \\ x_2 &= -5 \\ x_3 &= \ \ 3 \end{aligned}$$

unique solution

Example 2. Infinite Solution

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & b \end{array} \\ \begin{bmatrix} 1 & -3 & 0 & 2 & 0 & 7 \\ 0 & 0 & 1 & 6 & 0 & 9 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \Rightarrow \begin{aligned} \boxed{x_1} - 3x_2 \quad + 2x_4 \quad &= 7 \\ \boxed{x_3} + 6x_4 \quad &= 9 \\ x_5 &= 2 \\ \cancel{0 = 0} \end{aligned}$$

Free variables

Basic variables

> With free variables, there are infinitely many solutions.

Parametric Representation:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 7 + 3x_2 - 2x_4 \\ \\ 9 - 6x_4 \\ \\ 2 \end{bmatrix}$$

Reduced Row Echelon Form

- ***Example 3. No Solution***

$$\begin{array}{cccc} x_1 & x_2 & x_3 & b \\ \end{array}$$

$$\left[\begin{array}{cccc} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ \boxed{0 \quad 0 \quad 0} & & & \boxed{1} \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{array}{r} x_1 \qquad\qquad -3x_3 = 0 \\ x_2 + 2x_3 = 0 \\ 0x_1 + 0x_2 + 0x_3 = 1 \\ 0x_1 + 0x_2 + 0x_3 = 0 \end{array}$$

inconsistent

When an augmented matrix contains a row in which **the only nonzero entry lies in the last column**

The corresponding system of linear equations has **no solution** (**inconsistent**).