



How to find RREF

Reduced Row Echelon Form (RREF)

- Gaussian elimination: an algorithm for finding the reduced row echelon form of a matrix.

Original
augmented matrix

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 3 & -6 & -5 & 3 \\ 2 & -1 & 1 & 0 \end{bmatrix}$$

$\rightarrow \dots \rightarrow$

A row echelon form

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 3 & 3 & -6 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$\rightarrow \dots \rightarrow$

The reduced row
echelon form

$$\begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

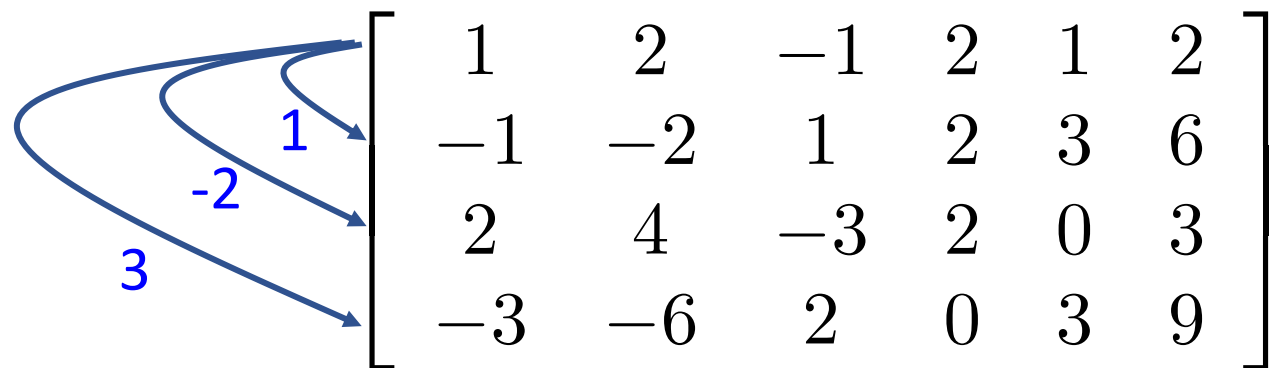
Elementary row
operations

Elementary row
operations

- Because RREF of a matrix is unique, **the order of elementary row operations** is not important.

Example 1

$$\begin{aligned}x_1 + 2x_2 - x_3 + 2x_4 + x_5 &= 2 \\ -x_1 - 2x_2 + x_3 + 2x_4 + 3x_5 &= 6 \\ 2x_1 + 4x_2 - 3x_3 + 2x_4 &= 3 \\ -3x_1 - 6x_2 + 2x_3 + 3x_5 &= 9\end{aligned}$$


$$\left[\begin{array}{cccccc} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{array} \right]$$



$$\left[\begin{array}{cccccc} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & -1 & 6 & 6 & 15 \end{array} \right]$$

Example 1

$$\begin{aligned}x_1 + 2x_2 - x_3 + 2x_4 + x_5 &= 2 \\ -x_1 - 2x_2 + x_3 + 2x_4 + 3x_5 &= 6 \\ 2x_1 + 4x_2 - 3x_3 + 2x_4 &= 3 \\ -3x_1 - 6x_2 + 2x_3 + 3x_5 &= 9\end{aligned}$$


$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & -1 & 6 & 6 & 15 \end{bmatrix}$$



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$$\begin{array}{c} -1 \end{array} \left[\begin{array}{cccccc} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & -1 & 6 & 6 & 15 \end{array} \right]$$



$$\left[\begin{array}{cccccc} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & 0 & 8 & 8 & 16 \end{array} \right]$$

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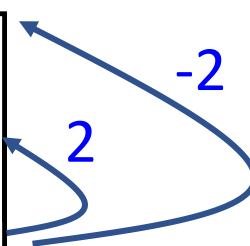
$$\begin{array}{c} -2 \end{array} \begin{array}{c} \curvearrowright \\ \searrow \end{array} \left[\begin{array}{cccccc} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & 0 & 8 & 8 & 16 \end{array} \right]$$



$$\left[\begin{array}{cccccc} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

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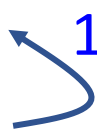
$$\left[\begin{array}{cccccc} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & \cancel{4} & \cancel{1} & \cancel{8} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$




$$\left[\begin{array}{cccccc} 1 & 2 & -1 & 0 & -1 & -2 \\ 0 & 0 & -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

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$$\left[\begin{array}{cccccc} 1 & 2 & -1 & 0 & -1 & -2 \\ 0 & 0 & \cancel{-1} & 1 & 0 & 0 & -3 & \cancel{3} \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$




$$\left[\begin{array}{cccccc} \textcircled{1} & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & \textcircled{1} & 0 & 0 & -3 \\ 0 & 0 & 0 & \textcircled{1} & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$


Example 1

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \left[\begin{array}{cccccc} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \end{array}$$



$$\begin{array}{rcl} x_1 + 2x_2 & + & -x_5 = -5 \\ & x_3 & = -3 \\ & x_4 + x_5 & = 2 \end{array}$$



$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -5 - 2x_2 + x_5 \\ x_2 \\ -3 \\ 2 - x_5 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ 0 \\ -3 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -5 - 2x_2 + x_5 & -8 \\ & x_2 & 1 \\ & -3 \\ & 2 - x_5 & 3 \\ & x_5 & -1 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ 0 \\ -3 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{array}{rcl} x_1 + 2x_2 & + & -x_5 = -5 \\ & x_3 & = -3 \\ & x_4 + x_5 & = 2 \end{array} \quad \Rightarrow \quad \begin{array}{l} x_1 \text{ free} \\ x_2 = -\frac{5}{2} - \frac{1}{2}x_1 + \frac{1}{2}x_5 \\ \cancel{x_1 = -5 - 2x_2 + x_5} \\ \cancel{x_2 \text{ free}} \\ x_3 = -3 \\ x_4 = 2 - x_5 \\ x_5 \text{ free} \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 & -8 \\ -5/2 - 1/2x_1 + 1/2x_5 & 1 \\ -3 \\ 2 - x_5 & 3 \\ x_5 & -1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1/2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -5/2 \\ -3 \\ 2 \\ 0 \end{bmatrix}$$

Example 2

- Find the RREF of

$$R = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & -6 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\times(-2) \quad \begin{bmatrix} R \\ 2R \end{bmatrix} \rightarrow \begin{bmatrix} R \\ \mathbf{0} \end{bmatrix}$$

Example 3

- Find the RREF of $\begin{bmatrix} R & 2R \\ R & -R \end{bmatrix}$

$$R = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} R & 2R \\ R & -R \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} R & 2R \\ \mathbf{0} & -3R \end{bmatrix}$$

$$\begin{bmatrix} R & \mathbf{0} \\ \mathbf{0} & -3R \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} R & \mathbf{0} \\ \mathbf{0} & R \end{bmatrix}$$

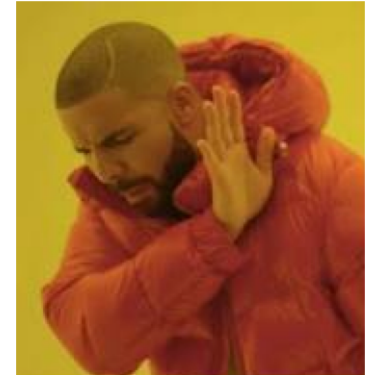
$$\left[\begin{array}{cccc|cccc} 1 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Example 3

This is
RREF!



This is not
RREF!



$$\begin{bmatrix} -1 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$