# Singular Value Decomposition

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# Outline

- Diagonalization can only apply on some square matrices.
- Singular value decomposition (SVD) can apply on any matrix.

- Reference: Chapter 7.7

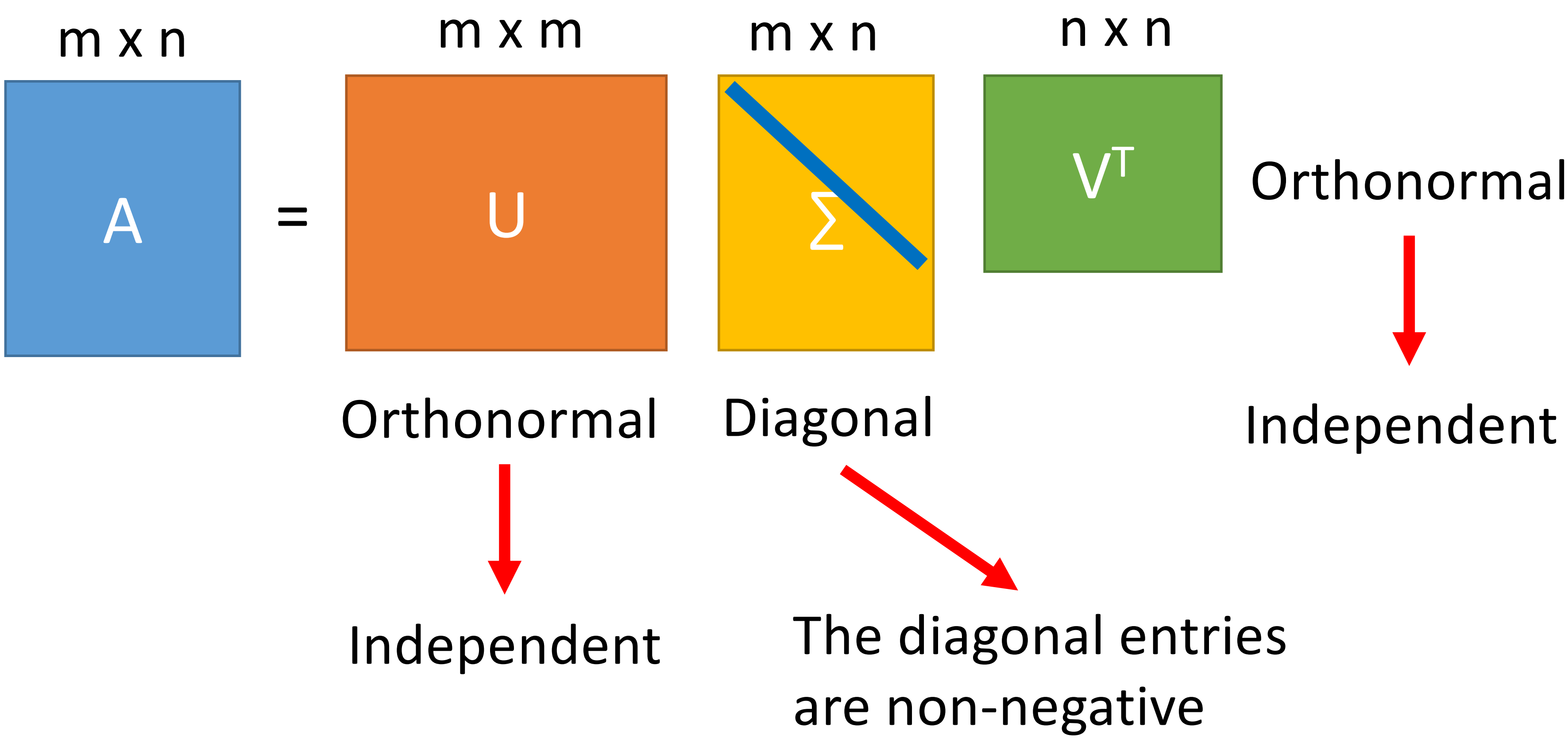
# SVD

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0$$

Singular values

$$\left[\begin{array}{cccc|cccc} \sigma_1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & & \sigma_k & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{array}\right]$$

- Any m x n matrix A



# SVD

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0$$

$$\left[\begin{array}{cccc|cccc} \sigma_1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & \sigma_2 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & & \sigma_k & 0 & 0 & \ldots & 0 \\ \hline 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \end{array}\right]$$

- Any m x n matrix A

| m x n | | m x m | m x n | n x n | |
|---|---|---|---|---|---|
| A | = | U | $\Sigma$ | $V^T$ | Orthonormal |
| $Rank = k$ | | $Rank = m$ | $Rank = k$ | $Rank = n$ | |
| | | Orthonormal | Diagonal | | |

What is the rank of A?

If A is a m x n matrix, and B is a n x k matrix.

$$Rank(AB) \leq min\big(Rank(A), Rank(B)\big)$$

If B is a matrix of rank n, then $Rank(AB) = Rank(A)$

If A is a matrix of rank n, then $Rank(AB) = Rank(B)$

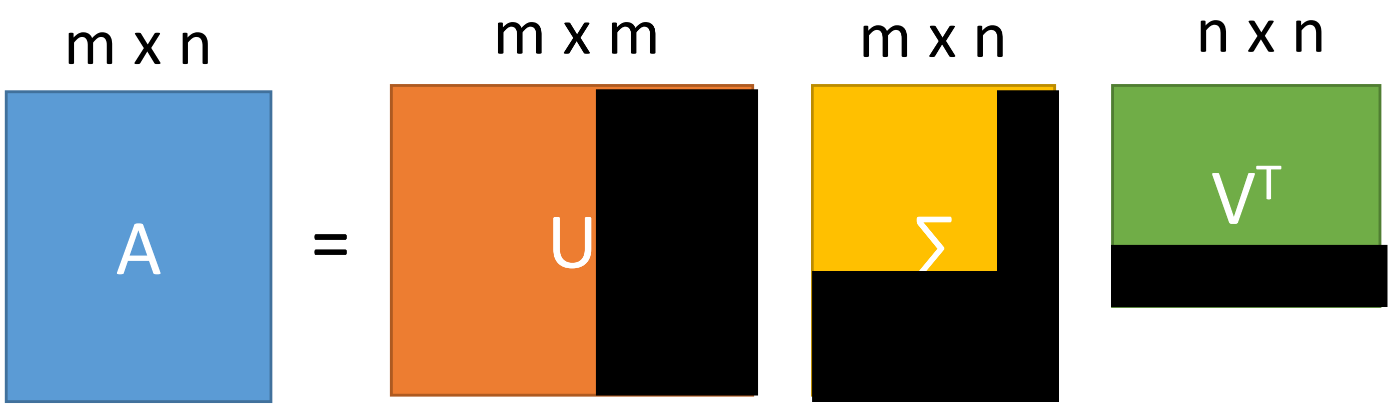
$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0$$

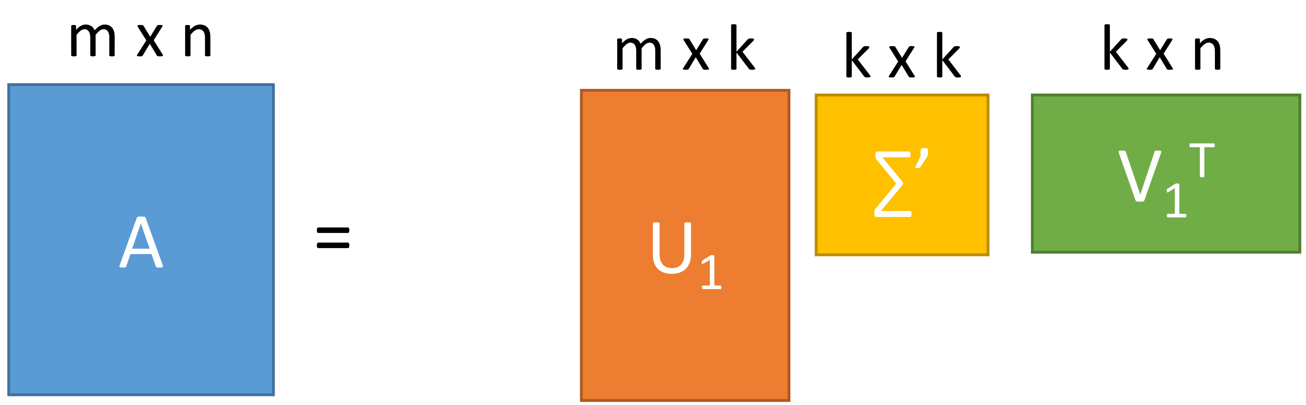
$$\begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & \sigma_k \end{bmatrix}$$

[redacted]

# SVD

- Any m x n matrix A

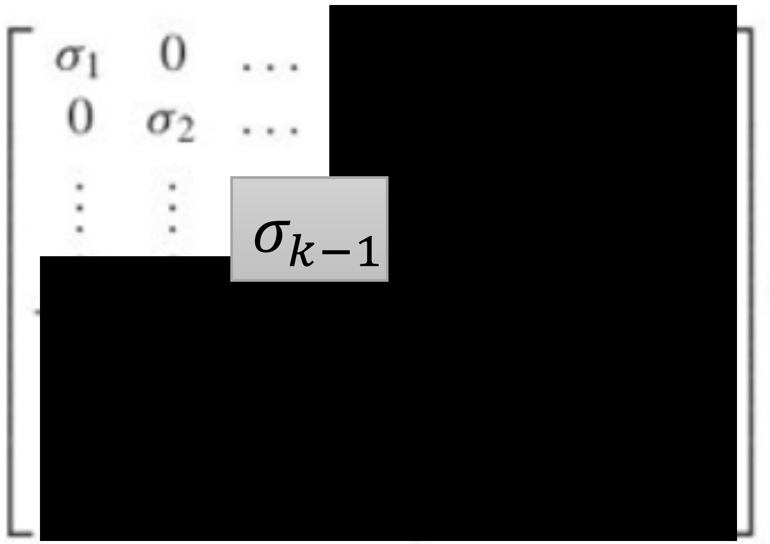


# SVD

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0$

$\sigma_k$ is deleted




- Any m x n matrix A

| m x n | | m x m | m x n | n x n |
|---|---|---|---|---|
| A | = | U [redacted] | Σ [redacted] | $V^T$ [redacted] |

≠

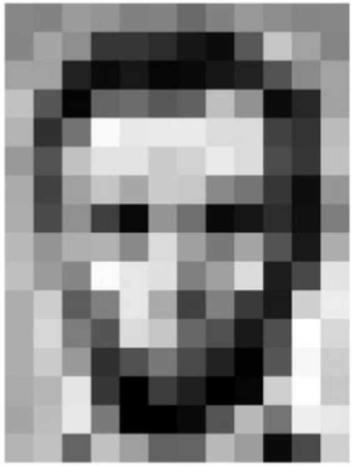
| m x n | | m x (k-1) | (k-1) x (k-1) | (k-1) x n |
|---|---|---|---|---|
| A’ | = | $U_1$ | Σ’ | $V_1^T$ |

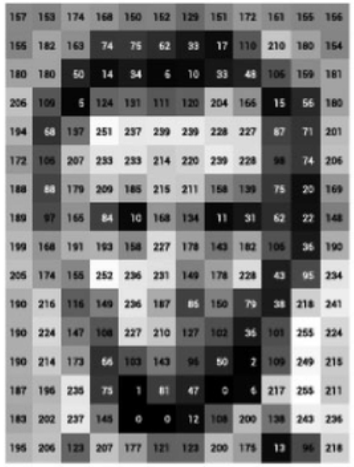
What is the rank of A’?

k-1

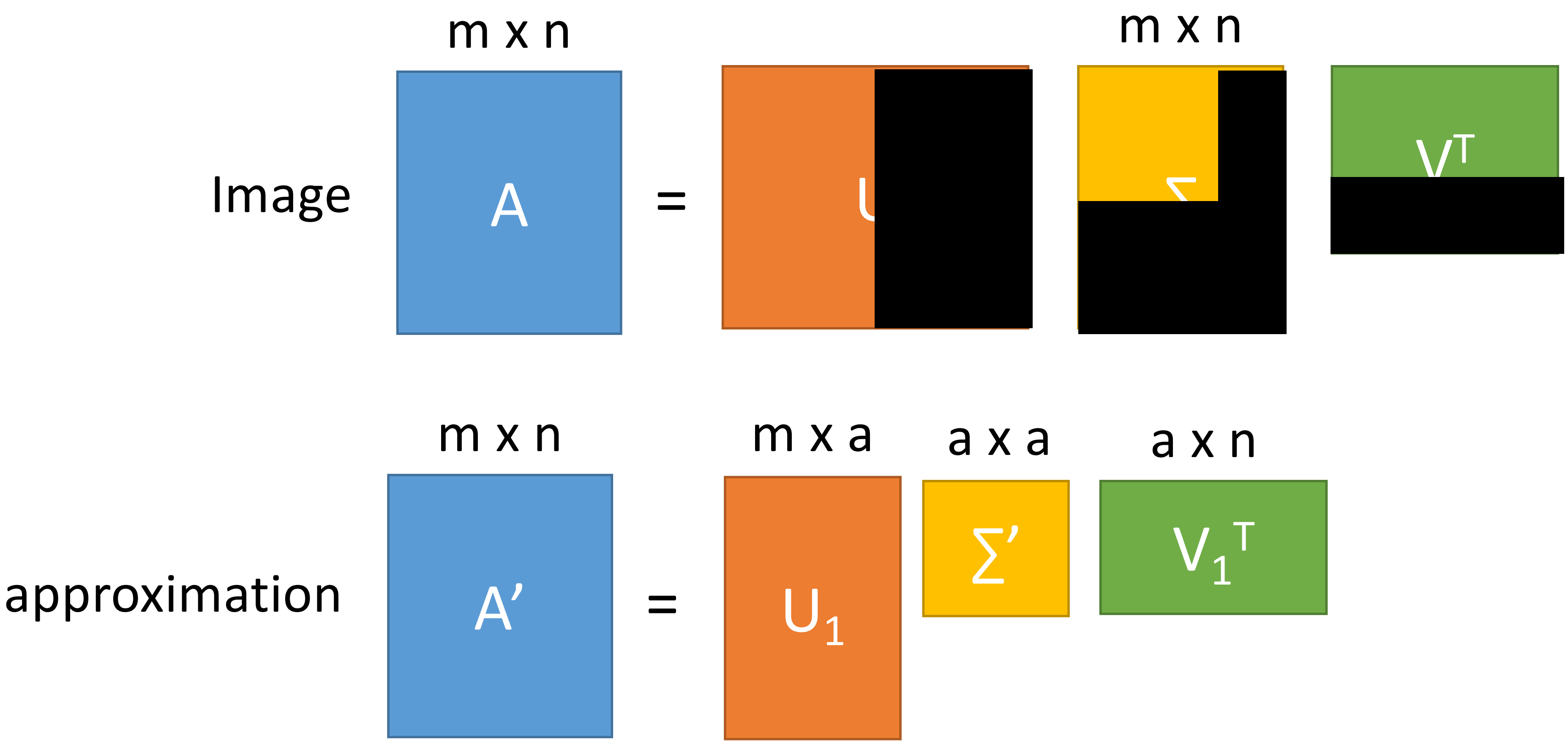
A’ is the rank k-1 matrix minimizing $||A - A'||$

# Application

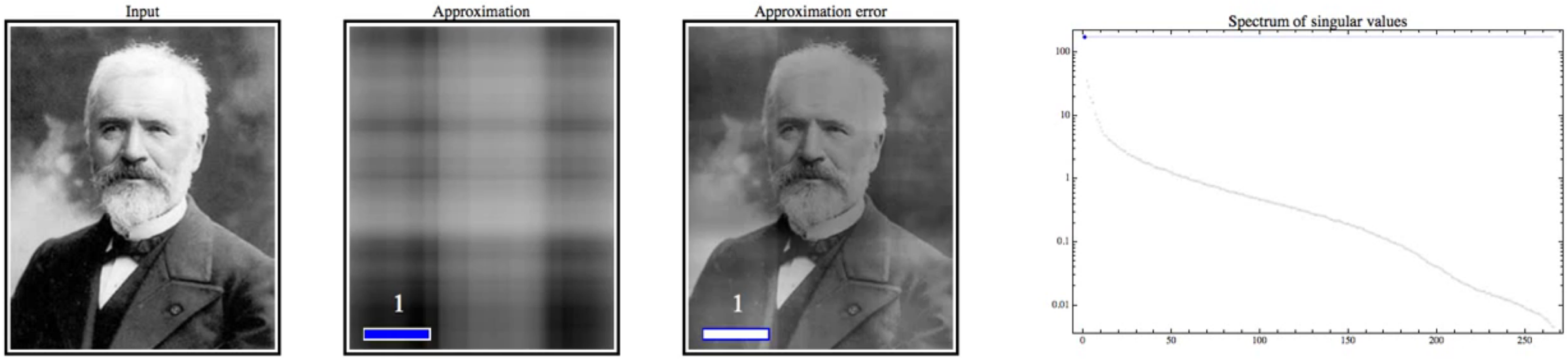




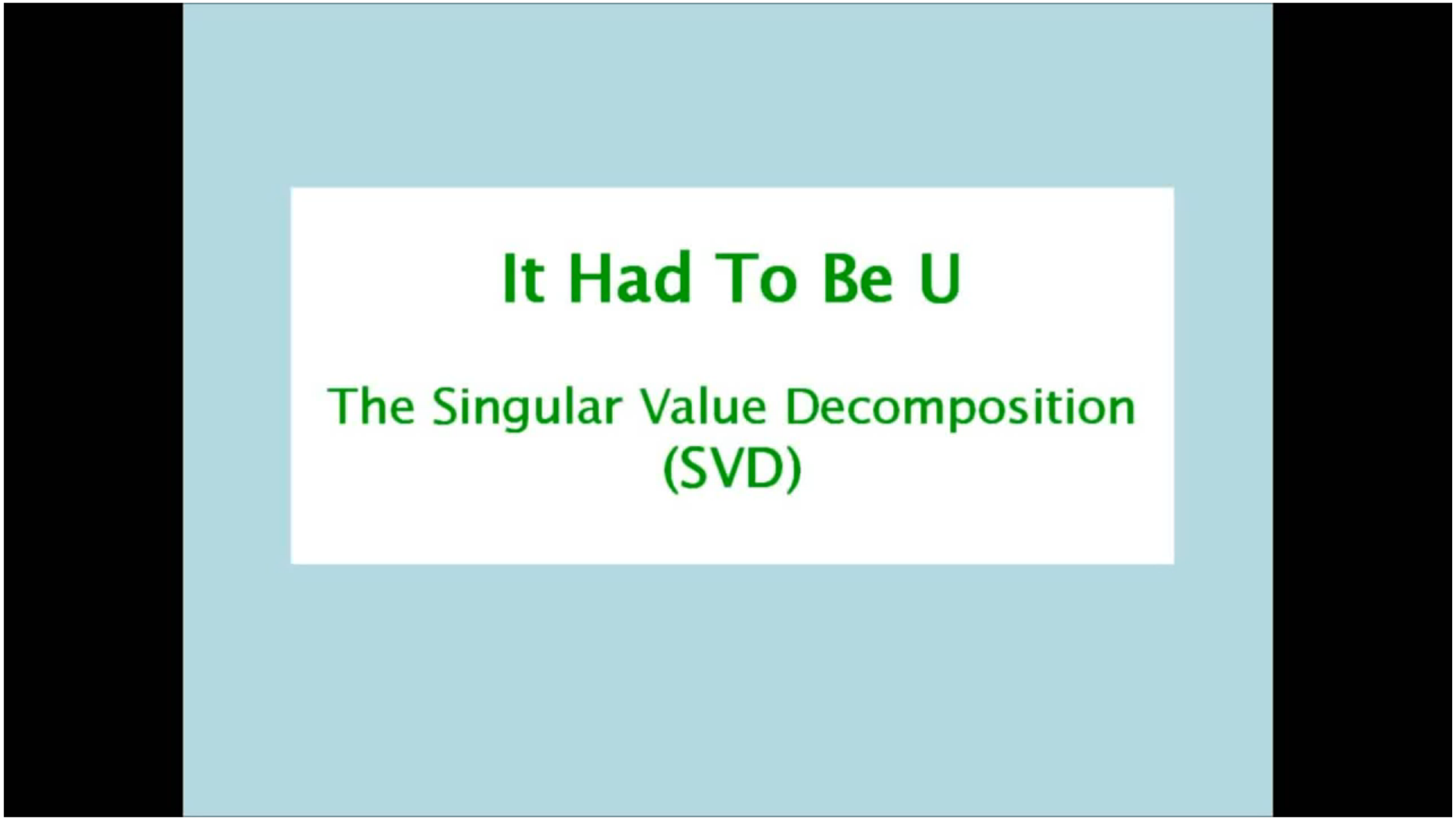
| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 157 | 153 | 174 | 168 | 150 | 152 | 129 | 151 | 172 | 161 | 155 | 156 |
| 155 | 182 | 163 | 74 | 75 | 62 | 33 | 17 | 110 | 210 | 180 | 154 |
| 180 | 180 | 50 | 14 | 34 | 6 | 10 | 33 | 48 | 106 | 159 | 181 |
| 206 | 109 | 5 | 124 | 131 | 111 | 120 | 204 | 166 | 15 | 56 | 180 |
| 194 | 68 | 137 | 251 | 237 | 239 | 239 | 228 | 227 | 87 | 71 | 201 |
| 172 | 105 | 207 | 233 | 233 | 214 | 220 | 239 | 228 | 98 | 74 | 206 |
| 188 | 88 | 179 | 209 | 185 | 215 | 211 | 158 | 139 | 75 | 20 | 169 |
| 189 | 97 | 165 | 84 | 10 | 168 | 134 | 11 | 31 | 62 | 22 | 148 |
| 199 | 168 | 191 | 193 | 158 | 227 | 178 | 143 | 182 | 106 | 36 | 190 |
| 205 | 174 | 155 | 252 | 236 | 231 | 149 | 178 | 228 | 43 | 95 | 234 |
| 190 | 216 | 116 | 149 | 236 | 187 | 86 | 150 | 79 | 38 | 218 | 241 |
| 190 | 224 | 147 | 108 | 227 | 210 | 127 | 102 | 36 | 101 | 255 | 224 |
| 190 | 214 | 173 | 66 | 103 | 143 | 96 | 50 | 2 | 109 | 249 | 215 |
| 187 | 196 | 235 | 75 | 1 | 81 | 47 | 0 | 6 | 217 | 255 | 211 |
| 183 | 202 | 237 | 145 | 0 | 0 | 12 | 108 | 200 | 138 | 243 | 236 |
| 195 | 206 | 123 | 207 | 177 | 121 | 123 | 200 | 175 | 13 | 96 | 218 |



# Low rank approximation using the singular value decomposition




https://www.youtube.com/watch?v=pAiVb7gWUrM https://www.youtube.com/watch?v=fKVRSbFKnEw



$A$ is an $m \times n$ matrix of rank $k$.

There exist: Orthonormal basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ for $R^n$

Orthonormal basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_m\}$ for $R^m$

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0$

$$A\boldsymbol{v}_i = \begin{cases} \sigma_i \boldsymbol{u}_i, & if\ 1 \leq i \leq k \\ \boldsymbol{0}, & i > k \end{cases}$$

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$$A[\boldsymbol{v}_1 \quad \dots \quad \boldsymbol{v}_n] = [\sigma_1 \boldsymbol{u}_1 \quad \dots \quad \boldsymbol{0}]$$

$V$

m x n

m x m

$$= [\boldsymbol{u}_1 \quad \dots \quad \boldsymbol{u}_m] \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$U$ $\quad$ $\Sigma$

$AV = U\Sigma \qquad A = U\Sigma V^T$

$A$ is an $m \times n$ matrix of rank $k$.

$A^T A \rightarrow n \times n$ matrix

$A^T A$ is symmetric

$A^T A$ is positive semidefinite

$$\boldsymbol{v}^T A^T A \boldsymbol{v} = (A\boldsymbol{v})^T A\boldsymbol{v}$$

$$= A\boldsymbol{v} \cdot A\boldsymbol{v} = \|A\boldsymbol{v}\|^2 \geq 0$$

Matrix B is positive semidefinite

If $\boldsymbol{v}^T B \boldsymbol{v} \geq 0$ For every $\boldsymbol{v}$

Non-negative eigenvalues:

$$\boldsymbol{v}^T B \boldsymbol{v} = \boldsymbol{v}^T (\lambda \boldsymbol{v})$$

$$= \lambda \|\boldsymbol{v}\|^2 \geq 0 \rightarrow \lambda \geq 0$$

$$Rank(A^T A) = Rank(A)$$

$$Nullity(A^T A) = Nullity(A)$$

$$Null(A^T A) = Null(A)$$

$$A^T A \boldsymbol{v} = \mathbf{0}$$

$$\boldsymbol{v}^T A^T A \boldsymbol{v} = \mathbf{0}$$

$$\|A\boldsymbol{v}\|^2 = \mathbf{0}$$

$$A\boldsymbol{v} = \mathbf{0}$$

$A$ is an $m \times n$ matrix of rank $k$.

There exist: Orthonormal basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ for $R^n$

Orthonormal basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_m\}$ for $R^m$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0$$

$$A\boldsymbol{v}_i = \begin{cases} \sigma_i \boldsymbol{u}_i, & if\ 1 \leq i \leq k \\ \mathbf{0}, & i > k \end{cases}$$

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$A^T A$ → $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1}, \cdots, \boldsymbol{v}_n\}$ are eigenvectors

symmetric
semidefinite
Rank $k$

$$\underbrace{\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_k}_{> 0} \quad \underbrace{\lambda_{k+1} \quad \cdots \quad \lambda_n}_{= 0}$$

$$\sigma_i = \sqrt{\lambda_i}$$

$$u_i = \frac{A\boldsymbol{v}_i}{\sigma_i} \qquad u_i \cdot u_j = \frac{A\boldsymbol{v}_i}{\sigma_i} \cdot \frac{A\boldsymbol{v}_j}{\sigma_j} = \frac{1}{\sigma_i \sigma_j} (A\boldsymbol{v}_i)^T A\boldsymbol{v}_j = \frac{1}{\sigma_i \sigma_j} \boldsymbol{v}_i{}^T \overbrace{A^T A\boldsymbol{v}_j}^{\lambda_j \boldsymbol{v}_j}$$

$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_k\}$ are orthonormal → Orthonormal basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_m\}$

*Thank You for Your Attention*

https://www.youtube.com/watch?v=R9UoFyqJca8