RREF v.s. Span

Consistent or not

- Given Ax=b, if the reduced row echelon form of [A b] is

$$\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Consistent

b is in the span of the columns of A

- Given Ax=b, if the reduced row echelon form of [A b] is

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$$

inconsistent

b is NOT in the span of the columns of A

Consistent or not

Ax =b is inconsistent (no solution)

The RREF of [A b] is

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad d \neq 0$$

Only the last column is non-zero

Rank A $\neq$ Rank [A b]

3 4

Need to know b

Ax =b is consistent for ***every*** b $A: m \times n$

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Every b is in the span of the columns of A= $[a_1 \quad \cdots \quad a_n]$

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Every b belongs to $Span\{a_1, \quad \cdots \quad , a_n\}$

||

$Span\{a_1, \quad \cdots \quad , a_n\} = R^m$

||

RREF of [A b] cannot have a row whose only non-zero entry is at the last column

||

RREF of A cannot have zero row 沒有任何破綻

||

Rank A = no. of rows

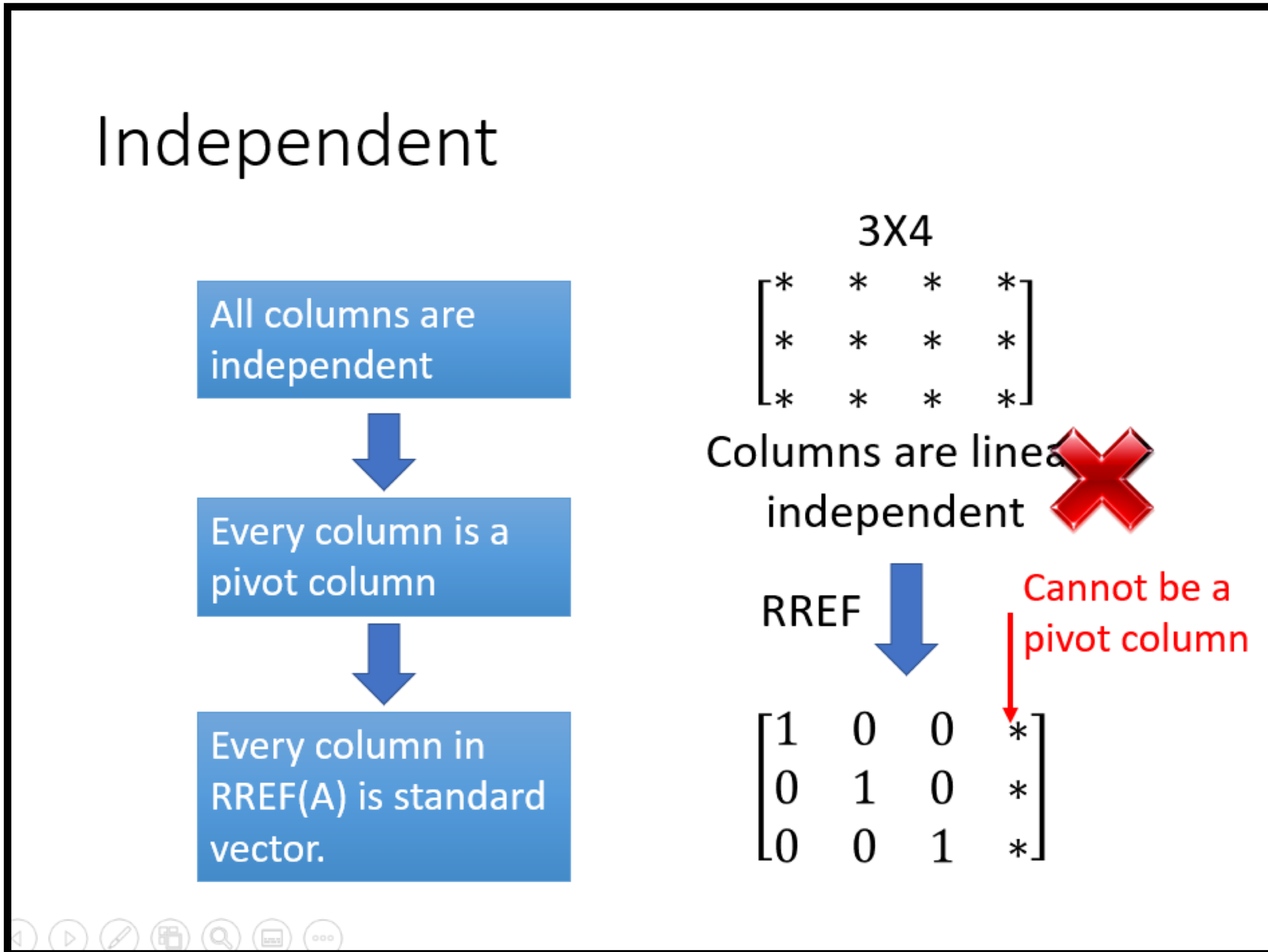
Consistent or not

$A: m \times n$

$Span\{a_1, \cdots, a_n\} = R^m$ = Rank A = no. of rows

m independent vectors can span R^m

More than m vectors in R^m must be dependent.



$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}$$

這個發現已經提過，現在只是從 span 的觀點再說一次

Rank

Matrix A is ***full rank***
if Rank A = min(m,n)

Matrix A is ***rank deficient***
if Rank A < min(m,n)

- Given a mxn matrix A:
  - Rank A $\leq$ min(m, n)
  - Because "the columns of A are independent" is equivalent to "rank A = n"
    - If m < n, the columns of A is dependent.

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

3 X 4

Rank A $\leq$ 3

$$\left\{ \begin{bmatrix} * \\ * \\ * \end{bmatrix}, \begin{bmatrix} * \\ * \\ * \end{bmatrix}, \begin{bmatrix} * \\ * \\ * \end{bmatrix}, \begin{bmatrix} * \\ * \\ * \end{bmatrix} \right\}$$

A matrix set has 4 vectors belonging to R^3 is dependent

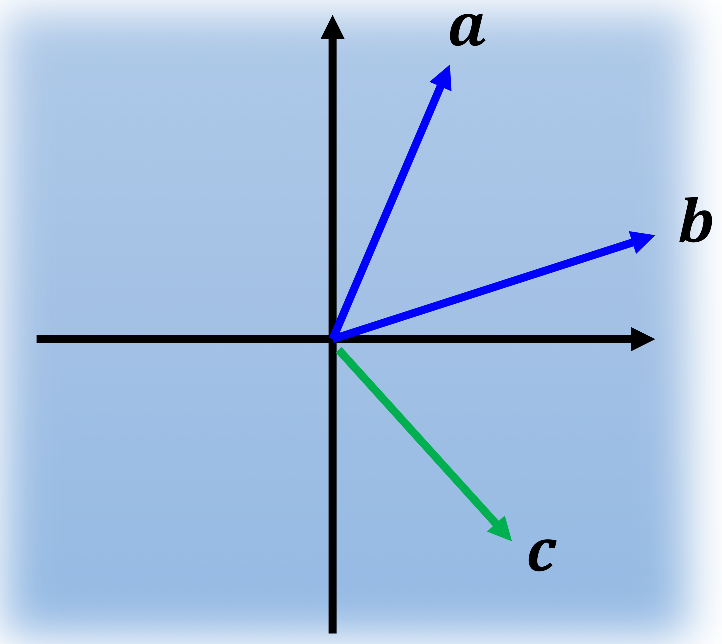
In R^m, you cannot find more than m vectors that are independent.

這個發現已經提過，現在只是從 span 的觀點再說一次

Example

m independent vectors can span R^m

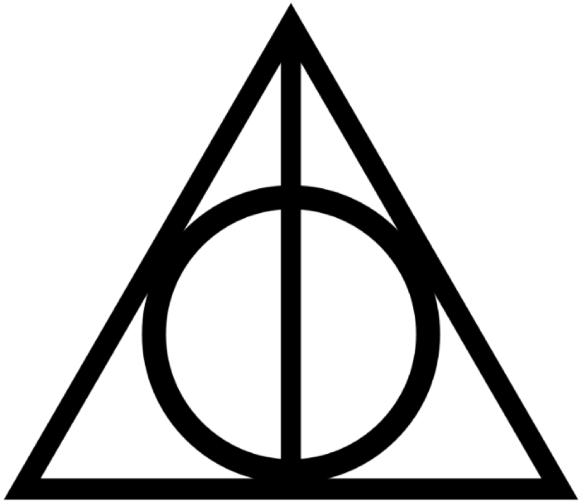
Consider R^2



$$\text{Does } \mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\} \text{ generate } \mathcal{R}^3?$$

yes

independent




三人行，
必有我师焉。
孔子