RREF v.s. Span
Consistent or not

• Given $Ax=b$, if the reduced row echelon form of $[A \ b]$ is

$$
\begin{bmatrix}
1 & 0 & 3 & 1 \\
0 & 1 & 2 & -2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

Consistent

b is in the span of the columns of A

• Given $Ax=b$, if the reduced row echelon form of $[A \ b]$ is

$$
\begin{bmatrix}
1 & 0 & 3 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

inconsistent

b is NOT in the span of the columns of A

$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$
Consistent or not

Ax = b is inconsistent (no solution)

The RREF of [A b] is

Only the last column is non-zero

\[
\begin{bmatrix}
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

d \neq 0

Rank A \neq \text{Rank} \ [A \ b]

Need to know b
Ax = b is consistent for every $b$  \[ A: m \times n \]

Every $b$ is in the span of the columns of $A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$

Every $b$ belongs to $\text{Span}\{a_1, \cdots, a_n\}$

$\text{Span}\{a_1, \cdots, a_n\} = R^m$

RREF of $[A \ b]$ cannot have a row whose only non-zero entry is at the last column

RREF of $A$ cannot have zero row

$\text{Rank } A = \text{no. of rows}$
Consistent or not

\[ \text{Span}\{a_1, \ldots, a_n\} = R^m \quad \Rightarrow \quad \text{Rank } A = \text{no. of rows} \]

- \( A: m \times n \)
- More than \( m \) vectors in \( R^m \) must be dependent.

- \( m \) independent vectors can span \( R^m \)
Independent

- All columns are independent
- Every column is a pivot column
- Every column in RREF(A) is standard vector.

3X4

Columns are linear independent

RREF

\[
\begin{bmatrix}
1 & 0 & 0 & * \\
0 & 1 & 0 & * \\
0 & 0 & 1 & *
\end{bmatrix}
\]

Cannot be a pivot column

這個發現已經提過，現在只是從 span 的觀點再說一次
Rank

- Given a mxn matrix A:
  - Rank A ≤ min(m, n)
  - Because “the columns of A are independent” is equivalent to “rank A = n”
    - If m < n, the columns of A is dependent.

\[
\begin{bmatrix}
  * & * & * & * \\
  * & * & * & * \\
  * & * & * & * \\
\end{bmatrix}
\]

3 X 4
Rank A ≤ 3

\[
\begin{bmatrix}
  * \\
  * \\
  * \\
\end{bmatrix}
\] \{ \begin{bmatrix}
  * \\
  * \\
  * \\
\end{bmatrix}, \begin{bmatrix}
  * \\
  * \\
  * \\
\end{bmatrix} \}

A matrix set has 4 vectors belonging to R^3 is dependent

In R^m, you cannot find more than m vectors that are independent.

Matrix A is **full rank** if Rank A = min(m,n)

Matrix A is **rank deficient** if Rank A < min(m,n)

這個發現已經提過，現在只是從span的觀點再說一次
Example

Consider $\mathbb{R}^2$

Does $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$ generate $\mathbb{R}^3$? Yes

$m$ independent vectors can span $\mathbb{R}^m$