Basis

Basis

Why nonzero?

- Let V be a nonzero subspace of R^n. A basis B for V is a linearly independent generation set of V.

$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis for R^n.

1. $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is independent
2. $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ generates R^n.

$\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is a basis for R^2

$\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$ $\left\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}-3\\1\end{bmatrix}\right\}$ $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right\}$ …… any two independent vectors form a basis for R^2

Basis

- The pivot columns of a matrix form a basis for its columns space.

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

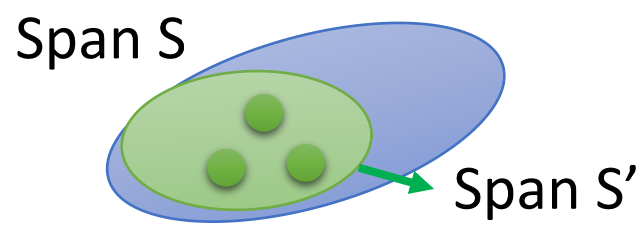
$$\text{Col } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} \right\}$$

Property

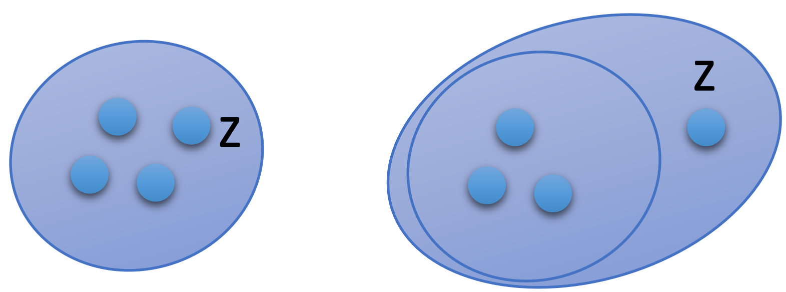
- (a) S is contained in Span S

Basis is always in its subspace

- (b) If a finite set S’ is contained in Span S, then Span S’ is also contained in Span S
  - Because Span S is a subspace



- (c) For any vector z, Span S = Span SU{z} if and only if z belongs to the Span S



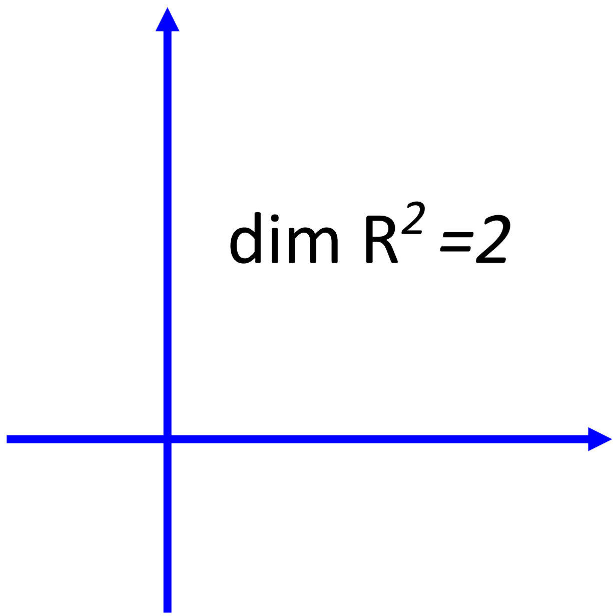
Theorem

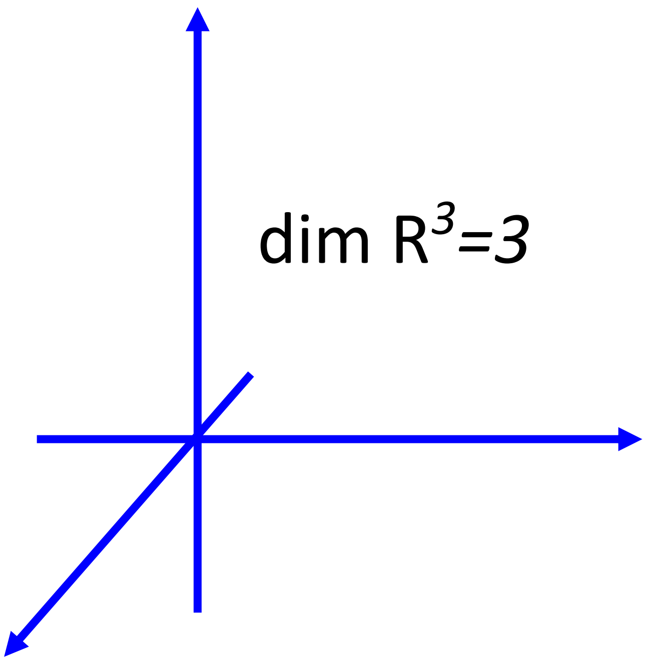
- 1. A basis is the smallest generation set.
- 2. A basis is the largest independent vector set in the subspace.
- 3. Any two bases for a subspace contain the same number of vectors.
  - The number of vectors in a basis for a nonzero subspace V is called dimension of V (dim V).

Theorem 3

Every basis of R^n has n vectors.

- The number of vectors in a basis for a subspace V is called the dimension of V, and is denoted dim V
  - The dimension of zero subspace is 0





Example

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathcal{R}^4 : x_1 - 3x_2 + 5x_3 - 6x_4 = 0 \right\}$$

$x_1 = 3x_2 - 5x_3 + 6x_4$

Find dim V

dim V = 3

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_2 - 5x_3 + 6x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Basis? Independent vector set that generates V

More from Theorems

Any two bases for a subspace contain the same number of vectors.

R^m have a basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$ All bases have m vectors

dim $R^m = m$

A basis is the smallest generation set.

A vector set generates R^m must contain at least *m* vectors.

Because a basis is the smallest generation set

Any other generation set has at least *m* vectors.

A basis is the largest independent set in the subspace.

Any independent vector set in R^m contain at most m vectors.

Independent

All columns are independent

↓

Every column is a pivot column

↓

Every column in RREF(A) is standard vector.

3X4

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

Columns are linea independent

RREF ↓

Cannot be a pivot column

$$\begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}$$

Rank

Matrix A is ***full rank***
if Rank A = min(m,n)

Matrix A is ***rank deficient***
if Rank A < min(m,n)

- Given a mxn matrix A:
  - Rank A $\leq$ min(m, n)
  - Because "the columns of A are independent" is equivalent to "rank A = n"
    - If m < n, the columns of A is dependent.

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

3 X 4

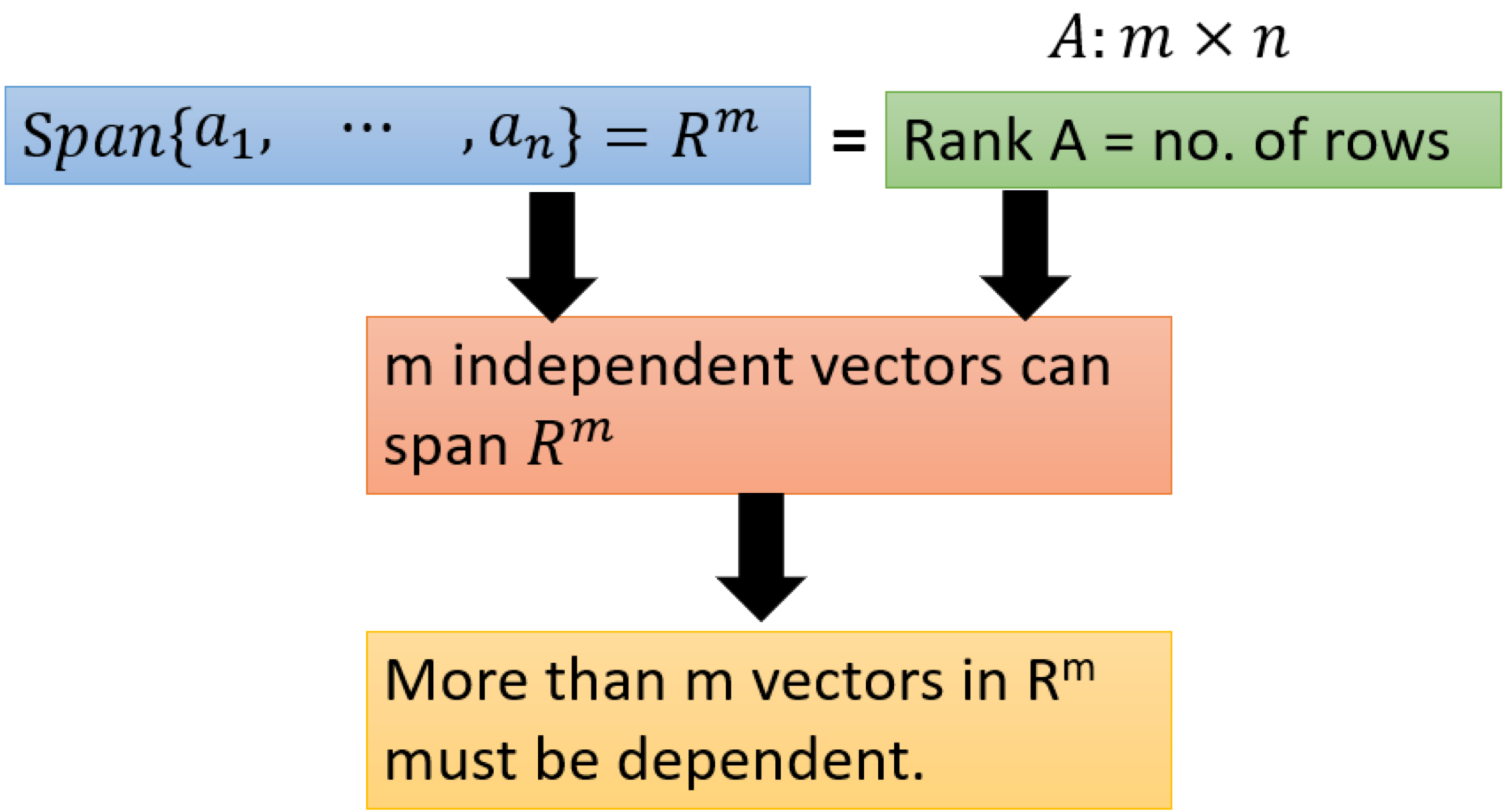
Rank A $\leq$ 3

$$\left\{ \begin{bmatrix} * \\ * \\ * \end{bmatrix}, \begin{bmatrix} * \\ * \\ * \end{bmatrix}, \begin{bmatrix} * \\ * \\ * \end{bmatrix}, \begin{bmatrix} * \\ * \\ * \end{bmatrix} \right\}$$

A matrix set has 4 vectors belonging to R^3 is dependent

In R^m, you cannot find more than m vectors that are independent.

Consistent or not



Theorem 1

A basis is the smallest generation set.

If there is a generation set S for subspace V,

The size of basis for V is smaller than or equal to S.

Reduction Theorem

There is a basis containing in any generation set S.

S can be reduced to a basis for V by removing some vectors.

Theorem 1 – Reduction Theorem

所有的 generation set 心中都有一個 basis

S can be reduced to a basis for V by removing some vectors.

Suppose S = {$\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$} is a generation set of subspace V

Subspace $V = Span\ S$ Let $A = [\ \mathbf{u}_1\ \ \mathbf{u}_2\ \ \cdots\ \ \mathbf{u}_k\ ]$.

$= Col\ A$

→ The basis of Col A is the pivot columns of A Subset of S

Theorem 1 – Reduction Theorem

所有的 generation set 心中都有一個 basis

Subspace $V = Span\, S = Col\, A$ = Span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} \right\}$

Smallest generation set

$$S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 3 \\ 9 \end{bmatrix} \right\}$$

Generation set

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem 2

A basis is the largest independent set in the subspace.

If the size of basis is k, then you cannot find more than k ***independent*** vectors in the subspace.

Extension Theorem

Given an independent vector set S in the space

S can be extended to a basis by adding more vectors

Theorem 2 – Extension Theorem

Independent set:
我不是一個 basis 就是正在成為一個 basis

There is a subspace V

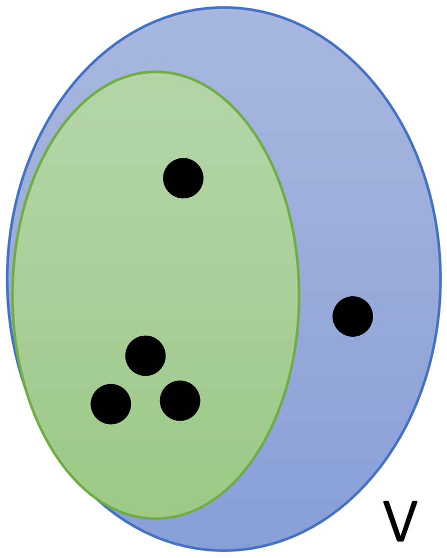
Given a independent vector set S (elements of S are in V)

- If Span S = V, then S is a basis
- If Span S ≠ V, find v_1 in V, but not in Span S

  S = S ∪ $\{v_1\}$ is still an independent set

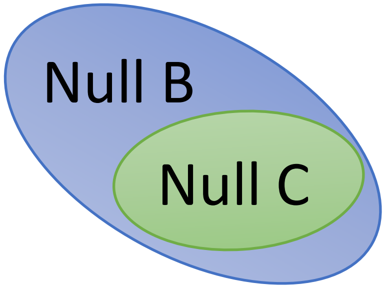
- If Span S = V, then S is a basis
- If Span S ≠ V, find v_2 in V, but not in Span S

  S = S ∪ $\{v_2\}$ is still an independent set



…… You will find the basis in the end.

Theorem 3



- Any two bases of a subspace V contain the same number of vectors

Suppose $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_p\}$ are two bases of V.

Let $A = [\mathbf{u}_1\ \mathbf{u}_2 \cdots \mathbf{u}_k]$ and $B = [\mathbf{w}_1\ \mathbf{w}_2 \cdots \mathbf{w}_p]$.

Since $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ spans V, $\exists\ \mathbf{c}_i \in \mathrm{R}^k$ s.t. $A\mathbf{c}_i = \mathbf{w}_i$ for all i

$\Rightarrow A[\mathbf{c}_1\ \mathbf{c}_2 \cdots \mathbf{c}_p] = [\mathbf{w}_1\ \mathbf{w}_2 \cdots \mathbf{w}_p] \Rightarrow AC = B$

Now $C\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \in \mathrm{R}^p$ $\Rightarrow AC\mathbf{x} = B\mathbf{x} = \mathbf{0}$

B is independent vector set $\Rightarrow x = \mathbf{0} \Rightarrow \mathbf{c}_1\ \mathbf{c}_2 \cdots \mathbf{c}_p$ are independent

$\mathbf{c}_i \in \mathrm{R}^k \Rightarrow p \le k$

Reversing the roles of the two bases one has $k \le p \Rightarrow p = k$.

Theorem 4.9 (P258)

- If V and W are subspaces of R^n with V contained in W, then dim V $\leq$ dim W
- If dim V = dim W, V=W
- Proof:

B_V is a basis of V, V in W, B_V in W

⇒ B_V is an independent set in W

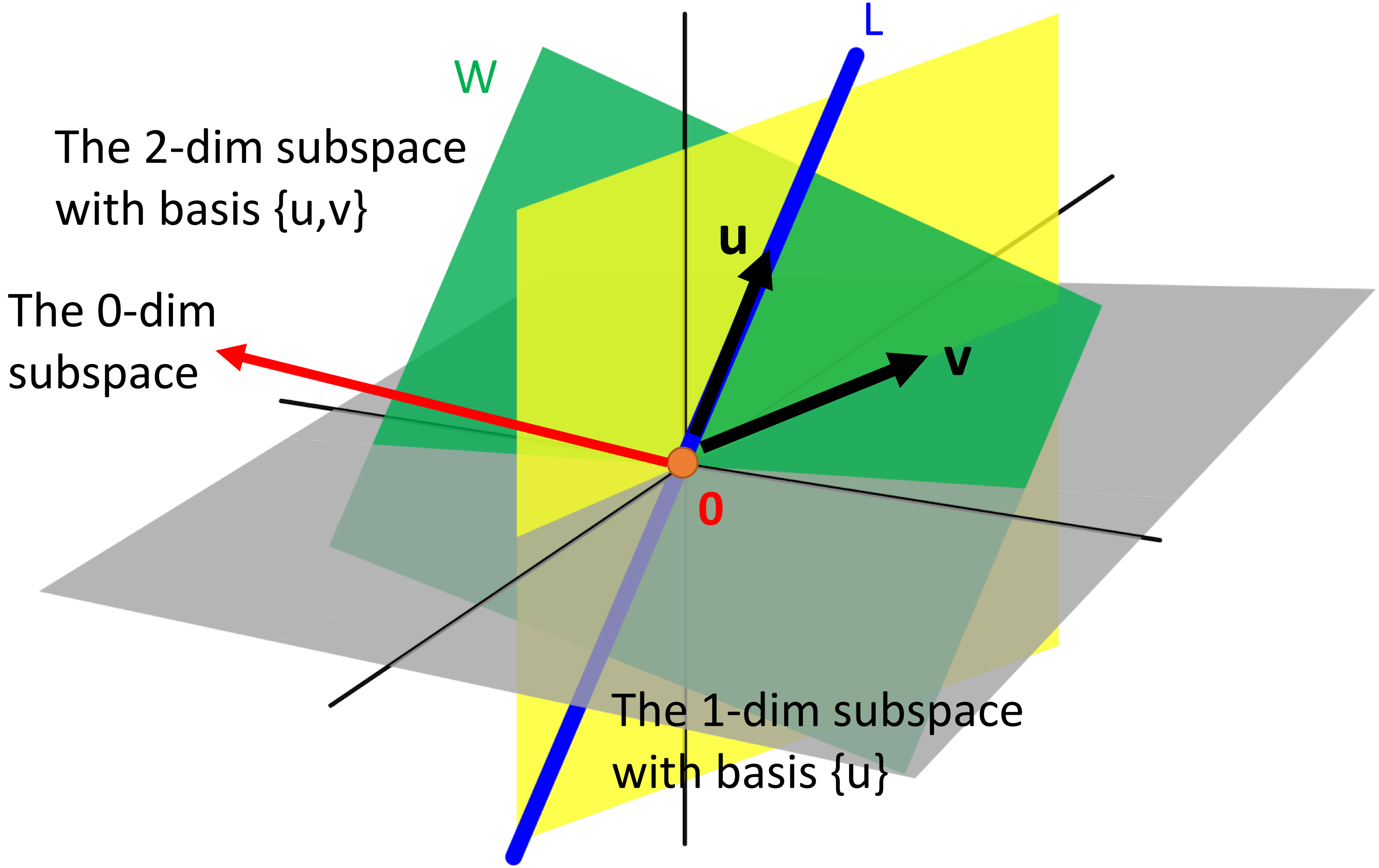
By extension theorem, B_V is in the basis of W ⇒ dim V $\leq$ dim W

If dim V = dim W =k

B_V is a linear independent set in W, with k elements

⇒ It is also the span of W

R^3 is the only 3-dim subspace of itself



Concluding Remarks

- 1. A basis is the smallest generation set.
- 2. A basis is the largest independent vector set in the subspace.
- 3. Any two bases for a subspace contain the same number of vectors.
  - The number of vectors in a basis for a nonzero subspace V is called dimension of V (dim V).

Concluding Remarks

Generation set

刪去

Same size

Basis

疊加

Independent vector set

雕塑 ... 主要是使用雕（通過減除材料來造型）及塑（通過疊加材料來造型）的方式 ...... (from wiki)