Basis
Basis

• Let V be a nonzero subspace of $\mathbb{R}^n$. A basis $B$ for $V$ is a linearly independent generation set of $V$.

$\{e_1, e_2, \ldots, e_n\}$ is a basis for $\mathbb{R}^n$.

1. $\{e_1, e_2, \ldots, e_n\}$ is independent
2. $\{e_1, e_2, \ldots, e_n\}$ generates $\mathbb{R}^n$.

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a basis for $\mathbb{R}^2$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

...... any two independent vectors form a basis for $\mathbb{R}^2$
Basis

- The pivot columns of a matrix form a basis for its columns space.

\[
\begin{bmatrix}
1 & 2 & -1 & 2 & 1 & 2 \\
-1 & -2 & 1 & 2 & 3 & 6 \\
2 & 4 & -3 & 2 & 0 & 3 \\
-3 & -6 & 2 & 0 & 3 & 9
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 0 & 0 & -1 & -5 \\
0 & 0 & 1 & 0 & 0 & -3 \\
0 & 0 & 0 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

pivot columns

Col \( \mathbf{A} \) = Span \( \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} \right\} \)
Property

• (a) $S$ is contained in $\text{Span } S$

• (b) If a finite set $S'$ is contained in $\text{Span } S$, then $\text{Span } S'$ is also contained in $\text{Span } S$
  • Because $\text{Span } S$ is a subspace

• (c) For any vector $z$, $\text{Span } S =$ $\text{Span } S \cup \{z\}$ if and only if $z$ belongs to the $\text{Span } S$
Theorem

• 1. A basis is the **smallest** generation set.
• 2. A basis is the **largest** independent vector set in the subspace.
• 3. Any two bases for a subspace **contain the same number of vectors**.
  • The number of vectors in a basis for a nonzero subspace $V$ is called **dimension** of $V$ ($\dim V$).
Theorem 3

• The number of vectors in a basis for a subspace $V$ is called the dimension of $V$, and is denoted $\text{dim } V$
  • The dimension of zero subspace is 0

$\text{dim } \mathbb{R}^2 = 2$

$\text{dim } \mathbb{R}^3 = 3$

Every basis of $\mathbb{R}^n$ has $n$ vectors.
Example

\[ V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : x_1 = 3x_2 - 5x_3 + 6x_4 = 0 \right\} \]

Find \( \dim V \)

\( \dim V = 3 \)

\[ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_2 - 5x_3 + 6x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \]

Basis?  Independent vector set that generates \( V \)
Any two bases for a subspace contain the same number of vectors.

R^m have a basis \{e_1, e_2, \ldots, e_m\}  
All bases have m vectors

A basis is the smallest generation set.

A vector set generates R^m must contain at least m vectors.
Because a basis is the smallest generation set
Any other generation set has at least m vectors.

A basis is the largest independent set in the subspace.

Any independent vector set in R^m contain at most m vectors.
Independent

All columns are independent

Every column is a pivot column

Every column in RREF(A) is standard vector.

Columns are linearly independent

3x4

\[
\begin{bmatrix}
* & * & * \\
* & * & * \\
* & * & * \\
\end{bmatrix}
\]

RREF

\[
\begin{bmatrix}
1 & 0 & 0 & * \\
0 & 1 & 0 & * \\
0 & 0 & 1 & * \\
\end{bmatrix}
\]

Cannot be a pivot column
Rank

- Given a mxn matrix A:
  - \( \text{Rank } A \leq \min(m, n) \)
  - Because “the columns of A are independent” is equivalent to “\( \text{rank } A = n \)”
    - If \( m < n \), the columns of A are dependent.

\[
\begin{bmatrix}
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
\end{bmatrix}
\]

3 X 4
Rank A \( \leq 3 \)

\[
\begin{bmatrix}
* \\
* \\
* \\
\end{bmatrix}, \begin{bmatrix}
* \\
* \\
* \\
\end{bmatrix}, \begin{bmatrix}
* \\
* \\
* \\
\end{bmatrix}, \begin{bmatrix}
* \\
* \\
* \\
\end{bmatrix}
\]

A matrix set has 4 vectors belonging to \( \mathbb{R}^3 \) is dependent

In \( \mathbb{R}^m \), you cannot find more than \( m \) vectors that are independent.
Consistent or not

Span\{a_1, \ldots, a_n\} = \mathbb{R}^m = \text{Rank } A = \text{no. of rows}

m \text{ independent vectors can span } \mathbb{R}^m

More than } m \text{ vectors in } \mathbb{R}^m \text{ must be dependent.}
Theorem 1

A basis is the smallest generation set.

If there is a generation set $S$ for subspace $V$, 

The size of basis for $V$ is smaller than or equal to $S$.

Reduction Theorem

There is a basis containing in any generation set $S$.

$S$ can be reduced to a basis for $V$ by removing some vectors.
Theorem 1 – Reduction Theorem

Suppose $S = \{u_1, u_2, \ldots, u_k\}$ is a generation set of subspace $V$.

Let $A = \begin{bmatrix} u_1 & u_2 & \cdots & u_k \end{bmatrix}$.

Subspace $V = \text{Span } S$ Let $A = \begin{bmatrix} u_1 & u_2 & \cdots & u_k \end{bmatrix}$.

$= \text{Col } A$

The basis of $\text{Col } A$ is the pivot columns of $A$. Subset of $S$
Theorem 1 – Reduction Theorem

Subspace $V = \text{Span} S = \text{Col} A = \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}$

Smallest generation set

Generation set

$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \right\}$

$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \Rightarrow \text{RREF} \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
Theorem 2

A basis is the largest independent set in the subspace.

If the size of basis is \( k \), then you cannot find more than \( k \) independent vectors in the subspace.

**Extension Theorem**

Given an independent vector set \( S \) in the space, \( S \) can be extended to a basis by adding more vectors.
Theorem 2 – Extension Theorem

Independent set: 我不是一個 basis 就是正在成為一個 basis

There is a subspace V
Given a independent vector set S (elements of S are in V)
\[
\begin{align*}
&\text{If Span } S = V, \text{ then } S \text{ is a basis} \\
&\text{If Span } S \neq V, \text{ find } v_1 \text{ in } V, \text{ but not in Span } S \\
&S = S \cup \{v_1\} \text{ is still an independent set} \\
&\text{If Span } S = V, \text{ then } S \text{ is a basis} \\
&\text{If Span } S \neq V, \text{ find } v_2 \text{ in } V, \text{ but not in Span } S \\
&S = S \cup \{v_2\} \text{ is still an independent set}
\end{align*}
\]

…… You will find the basis in the end.
Theorem 3

• Any two bases of a subspace \(V\) contain the same number of vectors

Suppose \(\{u_1, u_2, \ldots, u_k\}\) and \(\{w_1, w_2, \ldots, w_p\}\) are two bases of \(V\).

Let \(A = [u_1 \ u_2 \ \cdots \ u_k]\) and \(B = [w_1 \ w_2 \ \cdots \ w_p]\).

Since \(\{u_1, u_2, \ldots, u_k\}\) spans \(V\), \(\exists \ c_i \in \mathbb{R}^k\) s.t. \(Ac_i = w_i\) for all \(i\)

\[\Rightarrow A[c_1 \ c_2 \ \cdots \ c_p] = [w_1 \ w_2 \ \cdots \ w_p] \Rightarrow AC = B\]

Now \(Cx = 0\) for some \(x \in \mathbb{R}^p\) \(\Rightarrow ACx = Bx = 0\)

\(B\) is independent vector set \(\Rightarrow x = 0 \Rightarrow c_1 \ c_2 \ \cdots \ c_p\) are independent

\(c_i \in \mathbb{R}^k \Rightarrow p \leq k\)

Reversing the roles of the two bases one has \(k \leq p \Rightarrow p = k\).
Theorem 4.9 (P258)

• If $V$ and $W$ are subspaces of $\mathbb{R}^n$ with $V$ contained in $W$, then $\dim V \leq \dim W$

• If $\dim V = \dim W$, $V=W$

• Proof:
  $B_V$ is a basis of $V$, $V$ in $W$, $B_V$ in $W$

  $B_V$ is an independent set in $W$

  By extension theorem, $B_V$ is in the basis of $W$  \( \dim V \leq \dim W \)

  If $\dim V = \dim W = k$

  $B_V$ is a linear independent set in $W$, with $k$ elements

  It is also the span of $W$
$\mathbb{R}^3$ is the only 3-dim subspace of itself

The 2-dim subspace with basis \{u,v\}

The 0-dim subspace

The 1-dim subspace with basis \{u\}
Concluding Remarks

• 1. A basis is the smallest generation set.
• 2. A basis is the largest independent vector set in the subspace.
• 3. Any two bases for a subspace contain the same number of vectors.
  • The number of vectors in a basis for a nonzero subspace V is called dimension of V (dim V).
Concluding Remarks

- **Generation set**
- **Same size**
- **Independent vector set**
- **Basis**

敦煌 ... 主要是使用雕（通过减除材料来造型）及塑（通过叠加材料来造型）的方式 ...... (from wiki)