Beyond Vectors

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Introduction

- Many things can be considered as “vectors”.
  - E.g. a function can be regarded as a vector
- We can apply the concept we learned on those “vectors”.
  - Linear combination
  - Span
  - Basis
  - Orthogonal ……
- Reference: Chapter 6

Are they vectors?

Are they vectors?

- A matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

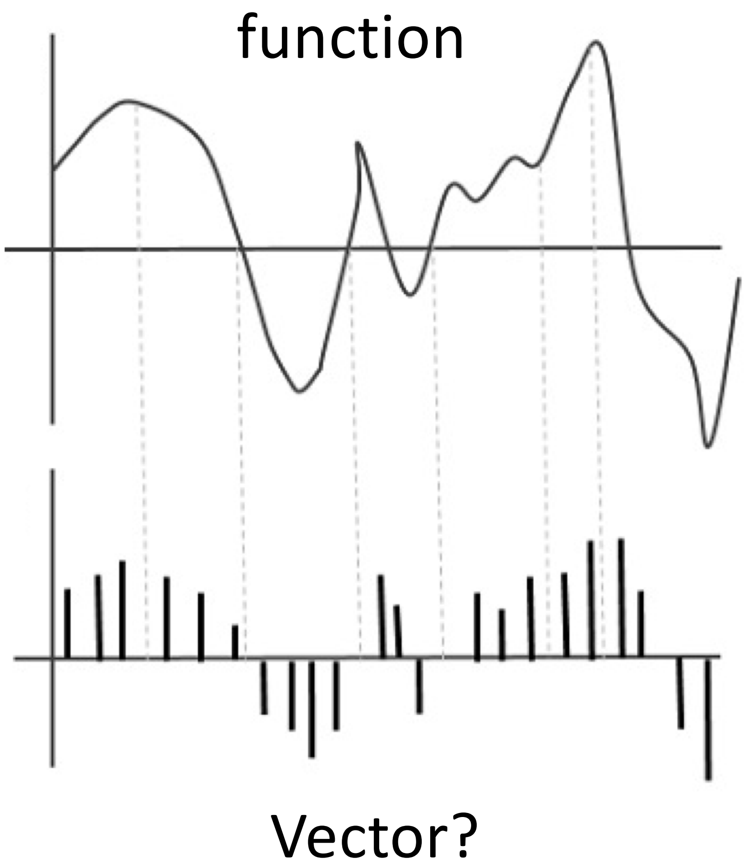
- A linear transform
- A polynomial

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \quad \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Are they vectors?

What is the zero vector?

- Any function is a vector?



Infinite

$$v(t) = e^t \Rightarrow v = \begin{bmatrix} \vdots \\ ? \\ \vdots \end{bmatrix}$$

$$g(t) = t^2 - 1 \Rightarrow g = \begin{bmatrix} \vdots \\ ? \\ \vdots \end{bmatrix}$$

$$\mathrm{h}(t) = e^t + t^2 - 1$$

$$\Rightarrow v + g$$

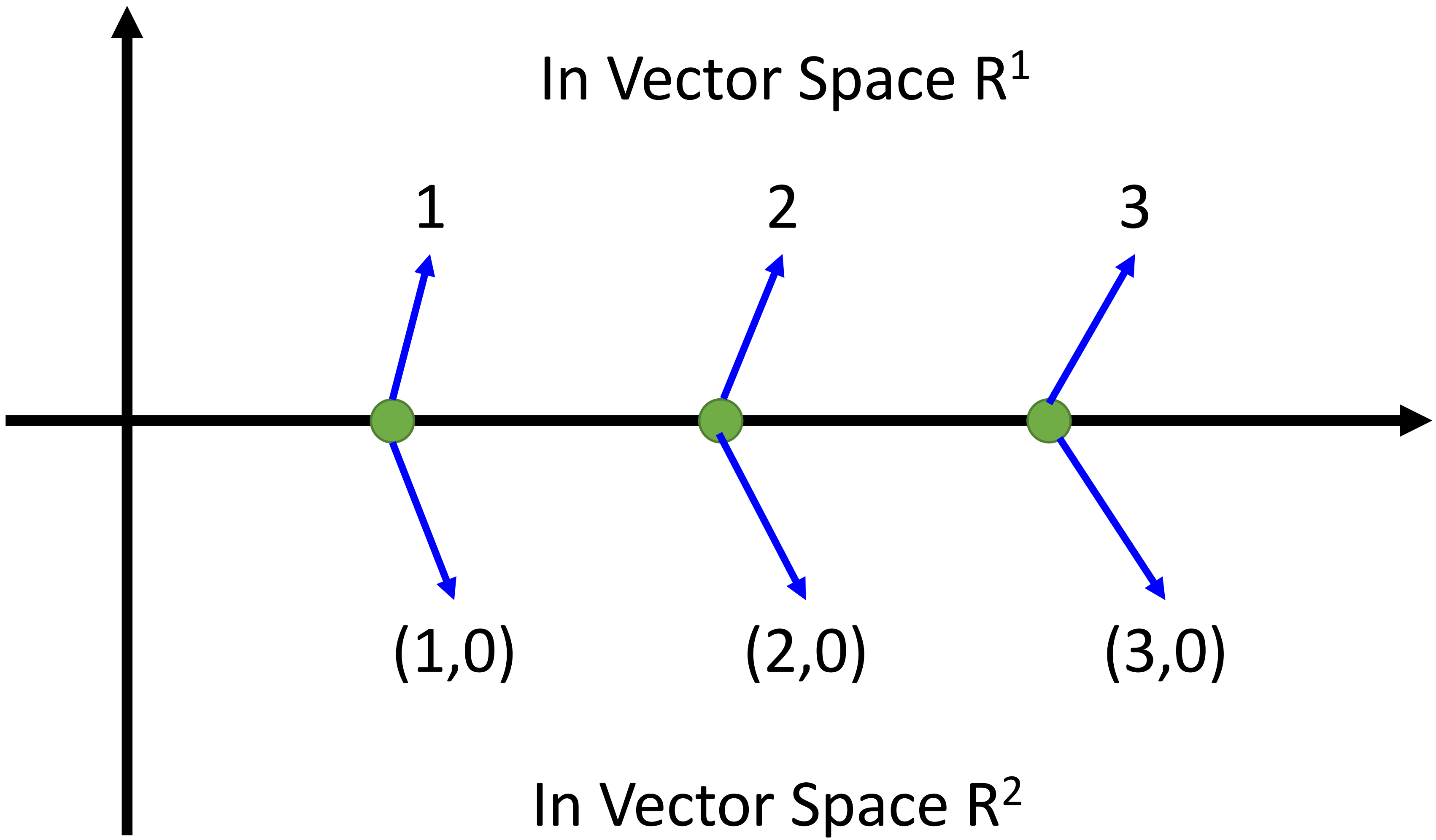
What is a vector?

R^n is a vector space

- If a set of objects V is a **vector space**, then the objects are "vectors".
- Vector space:
  - There are operations called "***addition***" and "***scalar multiplication***".
  - u, v and w are in V, and a and b are scalars. u+v and au are unique elements of V
- The following axioms hold:
  - u + v = v + u, (u +v) + w = u +(v + w)
  - There is a "zero vector" 0 in V such that u + 0 = u

    unique
  - There is –u in V such that u +(-u) = 0
  - 1u = u, (ab)u = a(bu), a(u+v) = au +av, (a+b)u = au +bu

Objects in Different Vector Spaces



Objects in Different Vector Spaces

All the polynomials with degree less than or equal to 2 as a vector space

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \qquad \begin{bmatrix}1\\1\\0\end{bmatrix} \qquad \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$f(t) = 1 \qquad g(t) = t + 1 \qquad h(t) = t^2 + t + 1$$

Vectors with infinite dimensions

All functions as a vector space

Subspaces

Review: Subspace

- A vector set V is called a subspace if it has the following three properties:
- 1. The zero vector **0** belongs to V
- 2. If **u** and **w** belong to V, then **u+w** belongs to V

Closed under (vector) addition

- 3. If **u** belongs to V, and c is a scalar, then c**u** belongs to V

Closed under scalar multiplication

Are they subspaces?

- All the functions pass 0 at t_0
- All the matrices whose trace equal to zero
- All the matrices of the form

$$\begin{bmatrix} a & a+b \\ b & 0 \end{bmatrix}$$

- All the continuous functions
- All the polynomials with degree n $\quad t^n, -t^n$
- All the polynomials with degree less than or equal to n

P: all polynomials, P_n: all polynomials with degree less than or equal to n

Linear Combination and Span

Linear Combination and Span

- Matrices

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

Linear combination with coefficient a, b, c

$$a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

What is Span S?

All 2x2 matrices whose trace equal to zero

Linear Combination and Span

- Polynomials

$$S = \{1, x, x^2, x^3\}$$

Is $f(x) = 2 + 3x - x^2$ linear combination of the "vectors" in S?

$$f(x) = 2 \cdot 1 + 3 \cdot x + (-1) \cdot x^2$$

$$Span\{1, x, x^2, x^3\} \qquad = P_3$$

$$Span\{1, x, \cdots, x^n, \cdots\} \qquad = P$$

Linear Transformation

Linear transformation

- A mapping (function) T is called linear if for all "vectors" u, v and scalars c:
- Preserving vector addition:

$$T(u+v) = T(u) + T(v)$$

- Preserving vector multiplication: $T(cu) = cT(u)$

Is matrix transpose linear?

Input: m x n matrices, output: n x m matrices

Linear transformation

- Derivative:

linear?

function f
e.g. x^2

Derivative

function f'
e.g. $2x$

- Integral from a to b

linear?

function f
e.g. x^2

Integral

(from a to b)

scalar

$$\int_a^b f(t)dt$$

e.g. $\frac{1}{3}(b^3 - a^3)$

Null Space and Range

- Null Space
  - The null space of T is the set of all “vectors” such that T(v)=0
  - What is the null space of matrix transpose?
- Range
  - The range of T is the set of all images of T.
  - That is, the set of all “vectors” T(v) for all v in the domain
  - What is the range of matrix transpose?

One-to-one and Onto

- U: $M_{m\times n} \to M_{n\times m}$ defined by $U(A) = A^T$.
    - Is U one-to-one? yes
    - Is U onto? yes

- D: $P_3 \to P_3$ defined by $D(f) = f'$
    - Is D one-to-one? no
    - Is D onto? no

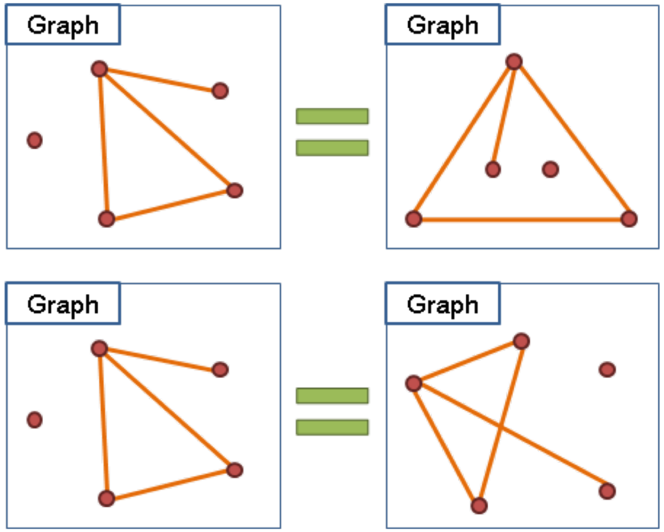
$$x^3 + 2x + 3 \to 3x^2 + 2x$$

$$x^3 + 2x \to 3x^2 + 2x$$

Isomorphism (同構)

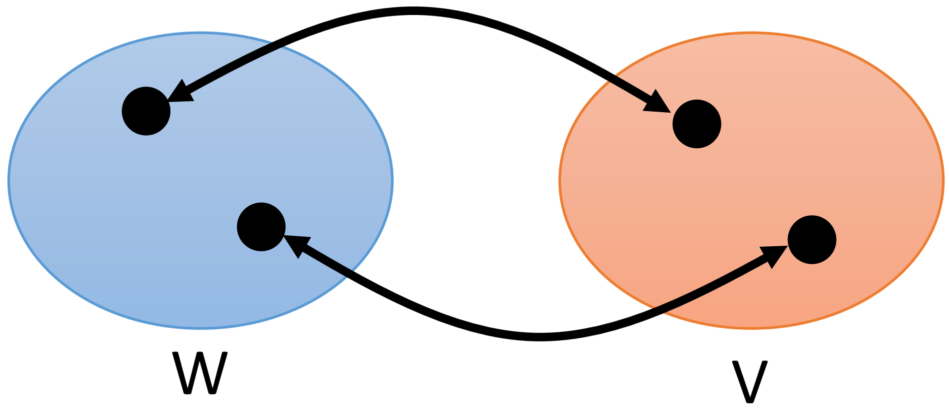
Biology

Graph



Chemistry

Isomorphism



- Let V and W be vector space.
- A linear transformation T: V→W is called an isomorphism if it is one-to-one and onto
  - Invertible linear transform
  - W and V are isomorphic.

Example 1: U: $\mathrm{M}_{m\times n} \rightarrow \mathrm{M}_{n\times m}$ defined by $U(A) = A^T$.

Example 2: T: $\mathrm{P}_2 \rightarrow \mathrm{R}^3$

$$T\left(a + bx + \frac{c}{2}x^2\right) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Basis

A basis for subspace V is a linearly independent generation set of V.

Independent

- Example

  $S = \{x^2 - 3x + 2, 3x^2 – 5x, 2x – 3\}$ is a subset of P_2.

  Is it linearly independent?

$$3(x^2 - 3x + 2) + (-1)(3x^2 - 5x) + 2(2x - 3) = \mathbf{0}$$

No

- Example

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$ is a subset of 2x2 matrices.

Is it linearly independent?

$$a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

implies that $a = b = c = 0$

Yes

Independent

If $\{v_1, v_2, \ldots\ldots, v_k\}$ are independent, and T is an isomorphism, $\{T(v_1), T(v_2), \ldots\ldots, T(v_k)\}$ are independent

- Example

The infinite vector set $\{1, x, x^2, \cdots, x^n, \cdots\}$

Is it linearly independent?

$\Sigma_i c_i x^i = 0$ implies $c_i = 0$ for all i.

Yes

- Example

$S = \{e^t, e^{2t}, e^{3t}\}$ Is it linearly independent?

Yes

$$ae^t + be^{2t} + ce^{3t} = 0 \qquad a + b + c = 0$$

$$ae^t + 2be^{2t} + 3ce^{3t} = 0 \qquad a + 2b + 3c = 0$$

$$ae^t + 4be^{2t} + 9ce^{3t} = 0 \qquad a + 4b + 9c = 0$$

Basis

- Example

For the subspace of all 2 x 2 matrices,

The basis is

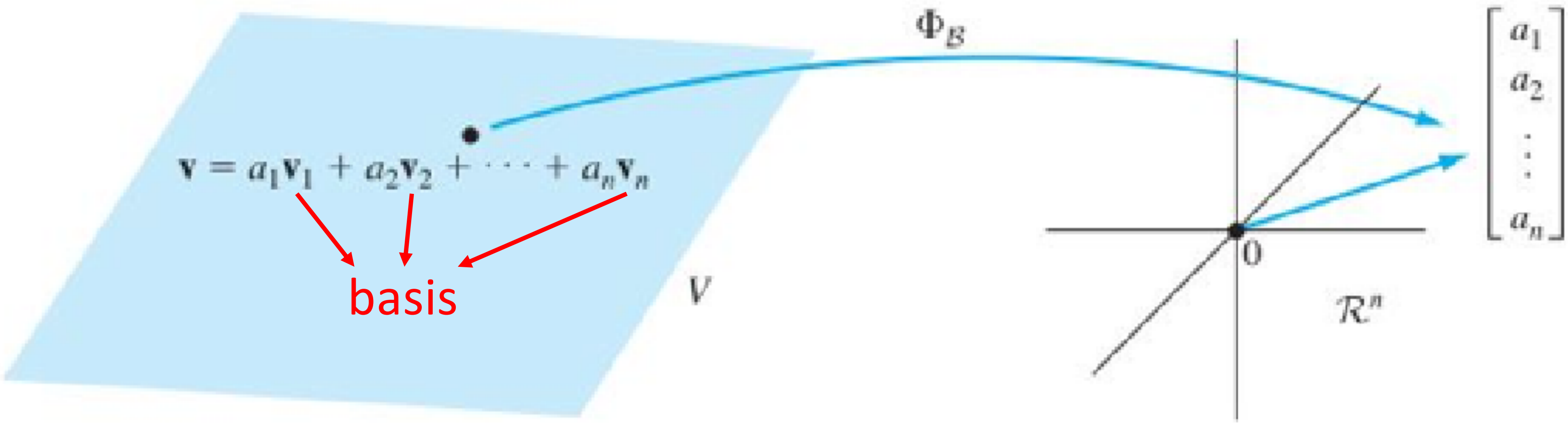
$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$ Dim = 4

- Example

$S = \{1, x, x^2, \cdots, x^n, \cdots\}$ is a basis of P. Dim = inf

Vector Representation of Object

- Coordinate Transformation



P_n: Basis: $\{1, x, x^2, \cdots, x^n\}$

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \Rightarrow \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Matrix Representation of Linear Operator

- Example:
  - D (derivative): $P_2 \rightarrow P_2$

Represent it as a matrix

$$\begin{tikzcd}
\begin{bmatrix}2\\-3\\5\end{bmatrix} \quad \text{vector} \arrow[r, "\text{Multiply a matrix}"'] & \text{vector} \quad \begin{bmatrix}-3\\10\\0\end{bmatrix} \\
2-3x+5x^2 \quad \text{polynomial} \arrow[u] \arrow[r, "\text{Derivative}"'] & \text{polynomial} \quad -3+10x \arrow[u]
\end{tikzcd}$$

Matrix Representation of Linear Operator

- Example:
  - D (derivative): $P_2 \rightarrow P_2$

Represent it as a matrix

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\1\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\1\end{bmatrix}$$

vector → vector

Multiply a matrix

$$\begin{bmatrix}0 & 1 & 0\\0 & 0 & 2\\0 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}0\\0\\0\end{bmatrix} \quad \begin{bmatrix}1\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\2\\0\end{bmatrix}$$

1

x

x^2

polynomial → polynomial

Derivative

0

1

$2x$

Matrix Representation of Linear Operator

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

- Example:
  - D (derivative): $P_2 \rightarrow P_2$

Represent it as a matrix

Not invertible

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} \ \text{vector} \arrow[r, "\text{Multiply a matrix}"'] & \text{vector} \ \begin{bmatrix} -4 \\ 6 \\ 0 \end{bmatrix} \\
5 - 4x + 3x^2 \ \text{polynomial} \arrow[u] \arrow[r, "\text{Derivative}"'] & \text{polynomial} \ -4 + 6x \arrow[u]
\end{tikzcd}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Matrix Representation of Linear Operator

- Example:
  - D (derivative): Function set F → Function set F
  - Basis of F is $\{e^t \cos t, e^t \sin t\}$

$\begin{bmatrix}1\\0\end{bmatrix}$ $\begin{bmatrix}0\\1\end{bmatrix}$

vector

invertible

vector

$\begin{bmatrix}1\\-1\end{bmatrix}$ $\begin{bmatrix}1\\1\end{bmatrix}$

Multiply a matrix

$$\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}$$

$e^t \cos t$

$e^t \sin t$

Function in F

Derivative

$\boxed{(1)e^t \cos t + (-1)e^t \sin t}$

Function in F

$\boxed{(1)e^t \cos t + (1)e^t \sin t}$

Matrix Representation of Linear Operator

Basis of F is
$\{e^t \cos t, e^t \sin t\}$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\text{vector} \arrow[r, shift right, "\text{Multiply a matrix}"', "\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}"'] & \text{vector} \arrow[l, shift right, "\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}"'] \\
\text{Function in F} \arrow[u] \arrow[r, shift right, "\text{Derivative}"'] & \text{Function in F} \arrow[u] \arrow[l, shift right, "\text{Antiderivative}"']
\end{tikzcd}$$

$\begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\boxed{(-\frac{1}{2})e^t \cos t + (\frac{1}{2})e^t \sin t}$

$e^t \sin t$

Eigenvalue and Eigenvector

$T(\boldsymbol{v}) = \lambda \boldsymbol{v}$, $\boldsymbol{v} \neq \mathbf{0}$, $\boldsymbol{v}$ is eigenvector, λ is eigenvalue

Eigenvalue and Eigenvector

- Consider derivative (linear transformation, input & output are functions)

Is e^{at} an "eigenvector"? ae^{at} What is the "eigenvalue"? a

Every scalar is an eigenvalue of derivative.

- Consider Transpose (also linear transformation, input & output are functions)

Is $\lambda = 1$ an eigenvalue?

Symmetric matrices form the eigenspace

Is $\lambda = -1$ an eigenvalue?

Skew-symmetric matrices form the eigenspace.

Symmetric:

$$A^T = A$$

Skew-symmetric:

$$A^T = -A$$

Consider Transpose of 2x2 matrices

$$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

vector

vector

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$$

What are the eigenvalues?

$$\begin{bmatrix}1&0\\0&0\end{bmatrix}$$

2x2 matrices

transpose

2x2 matrices

$$\begin{bmatrix}1&0\\0&0\end{bmatrix}$$

Eigenvalue and Eigenvector

- ***Consider Transpose of 2x2 matrices***

Matrix representation of transpose

$$det \begin{bmatrix} 1-t & 0 & 0 & 0 \\ 0 & 0-t & 1 & 0 \\ 0 & 1 & 0-t & 0 \\ 0 & 0 & 0 & 1-t \end{bmatrix}$$

Characteristic polynomial

$$(t-1)^3(t+1)$$

$$\lambda = 1$$

Symmetric matrices

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Dim=3

$$\lambda = -1$$

Skew-symmetric matrices

$$\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$$

Dim=1

Inner Product

Inner Product

Norm (length): $\|v\| = \sqrt{\langle v, v \rangle}$

Orthogonal: Inner product is zero

$\langle u, v \rangle$

"vector" v $\rightarrow$ **Inner Product** $\rightarrow$ scalar

"vector" u $\rightarrow$

For any vectors u, v and w, and any scalar a, the following axioms hold:

1. $\langle u, u \rangle > 0$ if $u \neq 0$
2. $\langle u, v \rangle = \langle v, u \rangle$
3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
4. $\langle au, v \rangle = a \langle u, v \rangle$

Dot product is a special case of **inner product** $c(u \cdot v)\ \ c > 0$

Can you define other inner product for normal vectors?

Inner Product

- Inner Product of Matrix

Frobenius inner product

$$\langle A, B\rangle = trace(AB^T) = trace(BA^T)$$

$$\left\langle \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \right\rangle = 1 \cdot 5 + 2 \cdot 6 + 3 \cdot 7 + 4 \cdot 8 = 70$$

Element-wise multiplication

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \|A\| = \sqrt{1^2 + 2^2 + 3^2 + 4^2}$$

Inner Product

1. $\langle u, u \rangle > 0$ if $u \neq 0$
2. $\langle u, v \rangle = \langle v, u \rangle$
3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
4. $\langle au, v \rangle = a\langle u, v \rangle$

- Inner product for any function with input [-1, 1]

$$\langle g, h \rangle = \int_{-1}^{1} g(x)h(x)dx$$

$$= \int_{-1}^{1} xdx = 0$$

Is $g(x) = 1$ and $h(x) = x$ orthogonal?

yes

$$\langle g, h \rangle = \sum_{i=-10}^{10} g\left(\frac{i}{10}\right) h\left(\frac{i}{10}\right)$$

Can it be inner product for general functions?

no

$$u\left(\frac{i}{10}\right) = 0, otherwise \neq 0 \qquad \langle u, u \rangle = 0, but\ u \neq 0$$

Orthogonal/Orthonormal Basis

- Let u be any vector, and w is the orthogonal projection of u on subspace W.
- Let $S = \{v_1, v_2, \cdots, v_k\}$ be an orthogonal basis of W.

$$w = \underset{\substack{\big\downarrow \\ \dfrac{u \cdot v_1}{\|v_1\|^2}}}{c_1} v_1 + \underset{\substack{\big\downarrow \\ \dfrac{u \cdot v_2}{\|v_2\|^2}}}{c_2} v_2 + \cdots + \underset{\substack{\big\downarrow \\ \dfrac{u \cdot v_k}{\|v_k\|^2}}}{c_k} v_k$$

- Let $S = \{v_1, v_2, \cdots, v_k\}$ be an orthonormal basis of W.

$$w = \underset{\substack{\big\downarrow \\ u \cdot v_1}}{c_1} v_1 + \underset{\substack{\big\downarrow \\ u \cdot v_2}}{c_2} v_2 + \cdots + \underset{\substack{\big\downarrow \\ u \cdot v_k}}{c_k} v_k$$

Orthogonal Basis

Let $\{u_1, u_2, \cdots, u_k\}$ be a basis of a subspace V. How to transform $\{u_1, u_2, \cdots, u_k\}$ into an orthogonal basis $\{v_1, v_2, \cdots, v_k\}$?

Gram-Schmidt Process

Then $\{v_1, v_2, \cdots, v_k\}$ is an orthogonal basis for W

After normalization, you can get orthonormal basis.

Orthogonal/Orthonormal Basis

- Find orthogonal/orthonormal basis for P_2
  - Define an inner product of P_2 by

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(t)g(t)\,dt$$

  - Find a basis $\{1, x, x^2\}$ $\xrightarrow{}$ v_1, v_2, v_3

u_1, u_2, u_3

$$\mathbf{v}_1 = \mathbf{u}_1$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 \quad = \underline{x^2 - \frac{1}{3}}$$

Orthogonal/Orthonormal Basis

- Find orthogonal/orthonormal basis for P_2
  - Define an inner product of P_2 by

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(t)g(t)\,dt$$

  - Get an orthogonal basis $\{1, x, x^2-1/3\}$

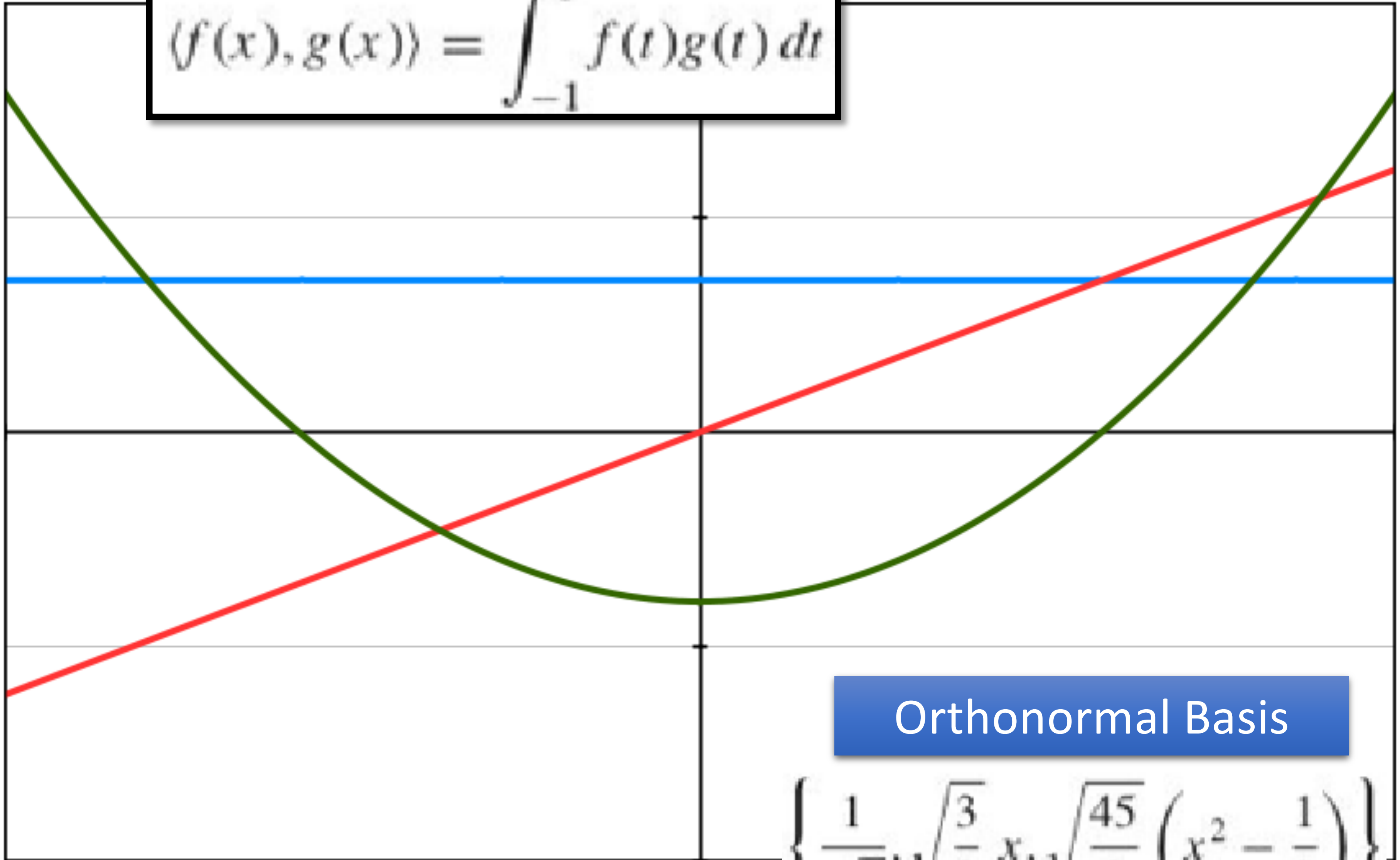
$$\|\mathbf{v}_1\| = \sqrt{\int_{-1}^{1} 1^2\,dx} = \sqrt{2} \qquad \|\mathbf{v}_2\| = \sqrt{\int_{-1}^{1} x^2\,dx} = \sqrt{\frac{2}{3}}$$

$$\|\mathbf{v}_3\| = \sqrt{\int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx} = \sqrt{\frac{8}{45}}$$

Orthonormal Basis

$$\left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\,x, \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) \right\}$$

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(t)g(t)\,dt$$



Orthonormal Basis

$$\left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\,x, \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) \right\}$$