Confirming that a set is a Basis

Intuitive Way

- Definition: A basis B for V is an independent generation set of V.

$$V = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathcal{R}^3 : v_1 - v_2 + 2v_3 = 0 \right\}$$

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Is C a basis of *V*?

Independent? yes

Generation set? difficult

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$ generates V

Another way

Find a basis for V

- Given a subspace V, assume that we already know that dim V = k. Suppose S is a subset of V with k vectors

If S is independent → S is basis

If S is a generation set → S is basis

$$V = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathcal{R}^3 : v_1 - v_2 + 2v_3 = 0 \right\} \quad \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Dim V = 2 (parametric representation)

Is C a basis of V?

C is a subset of V with 2 vectors

Independent? yes

→ C is a basis of V

Another way

Assume that dim V = k. <u>Suppose S is a subset of V with k vectors</u>

If S is independent ⇒ S is basis

By the extension theorem, we can add more vector into S to form a basis.

However, S already have k vectors, so it is already a basis.

If S is a generation set ⇒ S is basis

By the reduction theorem, we can remove some vector from S to form a basis.

However, S already have k vectors, so it is already a basis.

Example

- Is B a basis of *V* ?

$$V = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \in \mathcal{R}^4 : v_1 + v_2 + v_4 = 0 \right\} \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

Independent set in V? yes

Dim V = ? 3 → B is a basis of *V*.

Example

- Is B a basis of V = Span S ?

B is a subset of V with 3 vectors

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\} \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$A = \begin{bmatrix} 1 & -1 & 3 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 2 & -1 & 1 & -1 \end{bmatrix} \Rightarrow R_A = \begin{bmatrix} 1 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad dimA = 3$$

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow R_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad Independent$$

⇒ B is a basis of V.