Column Correspondence Theorem

Column Correspondence Theorem

$$A = [\boldsymbol{a_1} \quad \cdots \quad \boldsymbol{a_n}] \xrightarrow{\text{RREF}} R = [\boldsymbol{r_1} \quad \cdots \quad \boldsymbol{r_n}]$$

If $\boldsymbol{a_j}$ is a linear combination of other columns of A

→

$\boldsymbol{r_j}$ is a linear combination of the ***corresponding*** columns of R with ***the same coefficients***

$\mathbf{a}_5 = -\mathbf{a}_1 + \mathbf{a}_4$

$\mathbf{r}_5 = -\mathbf{r}_1 + \mathbf{r}_4$

$\boldsymbol{a_j}$ is a linear combination of the ***corresponding*** columns of A with ***the same coefficients***

←

If $\boldsymbol{r_j}$ is a linear combination of other columns of R

$\mathbf{a}_3 = 3\mathbf{a}_1 - 2\mathbf{a}_2$

$\mathbf{r}_3 = 3\mathbf{r}_1 - 2\mathbf{r}_2$

Column Correspondence Theorem - Example

$$A = \begin{array}{c} \begin{array}{cccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 \end{array} \\ \begin{bmatrix} \boxed{\begin{matrix} 1 \\ -1 \\ 2 \\ -3 \end{matrix}} & \boxed{\begin{matrix} 2 \\ -2 \\ 4 \\ -6 \end{matrix}} & \begin{matrix} -1 \\ 1 \\ -3 \\ 2 \end{matrix} & \begin{matrix} 2 \\ 2 \\ 2 \\ 0 \end{matrix} & \begin{matrix} 1 \\ 3 \\ 0 \\ 3 \end{matrix} & \begin{matrix} 2 \\ 6 \\ 3 \\ 9 \end{matrix} \end{bmatrix} \end{array} \qquad R = \begin{array}{c} \begin{array}{cccccc} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{r}_4 & \mathbf{r}_5 & \mathbf{r}_6 \end{array} \\ \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 0 \\ 0 \\ 0 \end{matrix}} & \boxed{\begin{matrix} 2 \\ 0 \\ 0 \\ 0 \end{matrix}} & \begin{matrix} 0 \\ 1 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 1 \\ 0 \end{matrix} & \begin{matrix} -1 \\ 0 \\ 1 \\ 0 \end{matrix} & \begin{matrix} -5 \\ -3 \\ 2 \\ 0 \end{matrix} \end{bmatrix} \end{array}$$

$$\mathbf{a}_2 = 2\mathbf{a}_1 \quad \Longleftrightarrow \quad \mathbf{r}_2 = 2\mathbf{r}_1$$

Column Correspondence Theorem - Example

$$A = \begin{array}{c} \begin{array}{cccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 \end{array} \\ \begin{bmatrix} \boxed{1} & 2 & -1 & \boxed{2} & \boxed{1} & 2 \\ \boxed{-1} & -2 & 1 & \boxed{2} & \boxed{3} & 6 \\ \boxed{2} & 4 & -3 & \boxed{2} & \boxed{0} & 3 \\ \boxed{-3} & -6 & 2 & \boxed{0} & \boxed{3} & 9 \end{bmatrix} \end{array} \qquad R = \begin{array}{c} \begin{array}{cccccc} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{r}_4 & \mathbf{r}_5 & \mathbf{r}_6 \end{array} \\ \begin{bmatrix} \boxed{1} & 2 & 0 & \boxed{0} & \boxed{-1} & -5 \\ \boxed{0} & 0 & 1 & \boxed{0} & \boxed{0} & -3 \\ \boxed{0} & 0 & 0 & \boxed{1} & \boxed{1} & 2 \\ \boxed{0} & 0 & 0 & \boxed{0} & \boxed{0} & 0 \end{bmatrix} \end{array}$$

$$\mathbf{a}_2 = 2\mathbf{a}_1 \iff \mathbf{r}_2 = 2\mathbf{r}_1$$

$$\mathbf{a}_5 = -\mathbf{a}_1 + \mathbf{a}_4 \iff \mathbf{r}_5 = -\mathbf{r}_1 + \mathbf{r}_4$$

Column Correspondence Theorem – Reason 1

Elementary row operations 無法改變 column 間的關係

$$a_1 + a_2 = a_3$$

$$A = \begin{bmatrix} 6 & 9 & 15 \\ 8 & 0 & 8 \\ 9 & 2 & 11 \end{bmatrix}$$

Switch rows 1 and 3

$$b_1 + b_2 = b_3$$

$$B = \begin{bmatrix} 9 & 2 & 11 \\ 8 & 0 & 8 \\ 6 & 9 & 15 \end{bmatrix}$$

row 1 x 2

$$c_1 + c_2 = c_3$$

$$C = \begin{bmatrix} 12 & 18 & 30 \\ 8 & 0 & 8 \\ 9 & 2 & 11 \end{bmatrix}$$

row 3 – row 1

$$d_1 + d_2 = d_3$$

$$D = \begin{bmatrix} 6 & 9 & 15 \\ 8 & 0 & 8 \\ 3 & -7 & -4 \end{bmatrix}$$

Column Correspondence Theorem – Reason 2

- ***<u>Basic concept</u>***

RREF

$$\left[\begin{array}{cccc:cc} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

還是 RREF

… 不好說

Column Correspondence Theorem – Reason 2

- ***Basic concept***

Augmented Matrix: $[A \quad b] \xrightarrow{\text{RREF}} [R \quad b']$

Coefficient Matrix: $A \xrightarrow{\text{RREF}} R$

$$\underbrace{\left[\begin{array}{c} \overbrace{\boxed{\begin{array}{ccccc} 1 & 2 & -1 & 2 & 1 \\ -1 & -2 & 1 & 2 & 3 \\ 2 & 4 & -3 & 2 & 0 \\ -3 & -6 & 2 & 0 & 3 \end{array}}}^{\text{A}} \begin{array}{c} 2 \\ 6 \\ 3 \\ 9 \end{array} \end{array}\right]}_{[A \quad b]} \Rightarrow \underbrace{\left[\begin{array}{c} \overbrace{\boxed{\begin{array}{ccccc} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}}}^{\text{R}} \begin{array}{c} -5 \\ -3 \\ 2 \\ 0 \end{array} \end{array}\right]}_{[R \quad b']}$$

Column Correspondence Theorem – Reason 2

- The RREF of matrix A is R

  $Ax = b$ and $Rx = b$ have the same solution set?

- The RREF of augmented matrix $[A \quad b]$ is $[R \quad b']$

  $Ax = b$ and $Rx = b'$ have the same solution set

- The RREF of matrix A is R

  $Ax = 0$ and $Rx = 0$ have the same solution set

If $b = 0$, then $b' = 0$.

Column Correspondence Theorem – Reason 2

- The RREF of matrix A is R, $Ax = 0$ and $Rx = 0$ have the same solution set

$$A = \begin{array}{cccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 \end{array}$$
$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix}$$

$$R = \begin{array}{cccccc} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{r}_4 & \mathbf{r}_5 & \mathbf{r}_6 \end{array}$$
$$R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathbf{a}_2 = 2\mathbf{a}_1$ ⇕ $-2\mathbf{a}_1+\mathbf{a}_2=\mathbf{0}$ ⟺ $Ax = 0$, $x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ⟺ $Rx = 0$, $x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ⟺ $-2\mathbf{r}_1+\mathbf{r}_2=\mathbf{0}$ ⇕ $\mathbf{r}_2 = 2\mathbf{r}_1$

Column Correspondence Theorem – Reason 2

- The RREF of matrix A is R, $Ax = 0$ and $Rx = 0$ have the same solution set

$$A = \begin{array}{c} \begin{array}{cccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 \end{array} \\ \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \end{array} \qquad R = \begin{array}{c} \begin{array}{cccccc} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{r}_4 & \mathbf{r}_5 & \mathbf{r}_6 \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\mathbf{a}_5 = -\mathbf{a}_1 + \mathbf{a}_4 \iff \mathbf{a}_1 - \mathbf{a}_4 + \mathbf{a}_5 = 0 \iff$ $\boxed{Ax = 0}$ $x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \iff$ $\boxed{Rx = 0}$ $x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \iff \mathbf{r}_1 - \mathbf{r}_4 + \mathbf{r}_5 = 0 \iff \mathbf{r}_5 = -\mathbf{r}_1 + \mathbf{r}_4$

How about Rows?

- Are there row correspondence theorem? **NO**

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} \text{—} & \boldsymbol{a_1^T} & \text{—} \\ \text{—} & \boldsymbol{a_2^T} & \text{—} \\ \text{—} & \boldsymbol{a_3^T} & \text{—} \\ \text{—} & \boldsymbol{a_4^T} & \text{—} \end{bmatrix} \qquad R = \begin{bmatrix} \text{—} & \boldsymbol{r_1^T} & \text{—} \\ \text{—} & \boldsymbol{r_2^T} & \text{—} \\ \text{—} & \boldsymbol{r_3^T} & \text{—} \\ \text{—} & \boldsymbol{r_4^T} & \text{—} \end{bmatrix}$$