

Dimension of Basis

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Col A = Range

- Basis: The pivot columns of A form a basis for Col A.

$$A = \begin{bmatrix} \boxed{1} & 2 & \boxed{-1} & \boxed{2} & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \Rightarrow \text{Col } A = \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} \right\}$$

pivot columns pivot columns

- Dimension:

$$\begin{aligned} \text{Dim (Col } A) &= \text{number of pivot columns} \\ &= \text{rank } A \end{aligned}$$

Rank A (revisit)

Maximum number of Independent Columns

Number of Pivot Columns

Number of Non-zero rows

Number of Basic Variables

Dim (Col A): dimension of column space

Dimension of the range of A

Row A

- Basis: Nonzero rows of RREF(A)

$$A = \begin{bmatrix} 3 & 1 & -2 & 1 & 5 \\ 1 & 0 & 1 & 0 & 1 \\ -5 & -2 & 5 & -5 & -3 \\ -2 & -1 & 3 & 2 & -10 \end{bmatrix} \xrightarrow{\text{RREF}} R = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -5 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row A = Row R

(The elementary row operations do not change the row space.)

a basis of Row R
= a basis of Row A

- Dimension: $\text{Dim}(\text{Row } A) = \text{Number of Nonzero rows} = \text{Rank } A$

Rank A (revisit)

Maximum number of Independent Columns

Number of Pivot Column

Number of Non-zero rows

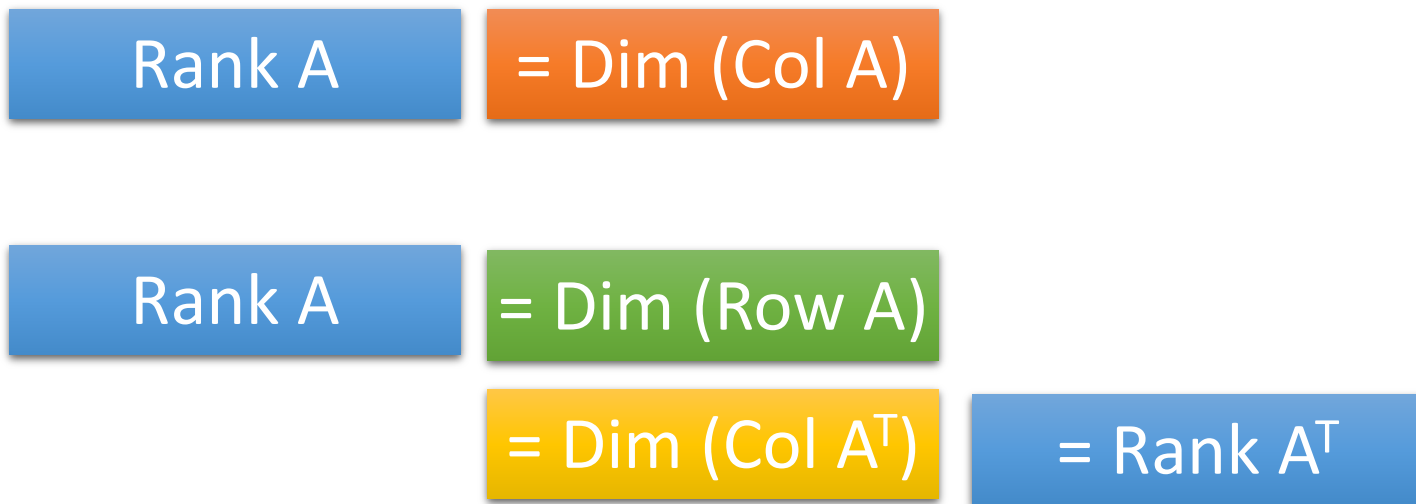
Number of Basic Variables

Dim (Col A): dimension of column space = Dim (Row A)

Dimension of the range of A = Dim (Col A^T)

Rank $A = \text{Rank } A^T$

- Proof




Null A

$$A = \begin{bmatrix} 3 & 1 & -2 & 1 & 5 \\ 1 & 0 & 1 & 0 & 1 \\ -5 & -2 & 5 & -5 & -3 \\ -2 & -1 & 3 & 2 & -10 \end{bmatrix} \quad R = \begin{bmatrix} 10 & 1 & 0 & 1 \\ 0 & 1 & -5 & 0 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Basis:
 - Solving $Ax = 0$
 - Each free variable in the parametric representation of the general solution is multiplied by a vector.
 - The vectors form the basis.

$$\begin{array}{l}
 x_1 + x_3 + x_5 = 0 \\
 x_2 - 5x_3 + 4x_5 = 0 \\
 x_4 - 2x_5 = 0
 \end{array}
 \begin{array}{l}
 \Rightarrow \\
 \Rightarrow \\
 \Rightarrow
 \end{array}
 \begin{array}{l}
 x_1 = -x_3 - x_5 \\
 x_2 = 5x_3 - 4x_5 \\
 x_3 = x_3 \text{ (free)} \\
 x_4 = 2x_5 \\
 x_5 = x_5 \text{ (free)}
 \end{array}
 \begin{array}{l}
 \Rightarrow \\
 \Rightarrow
 \end{array}
 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -4 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$


Basis

Null A

- Basis:
 - Solving $Ax = 0$
 - Each free variable in the parametric representation of the general solution is multiplied by a vector.
 - The vectors form the basis.
- Dimension:

Dim (Null A) = number of free variables

= Nullity A

= $n - \text{Rank A}$

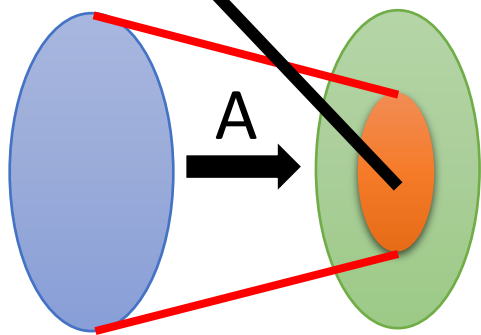
Dimension Theorem

Dim (Col A)
= Rank A

Dim (Null A)
= $n - \text{Rank A}$

If A is $m \times n$
Dim (\mathbb{R}^n) = n

$$\text{Dim of Range} + \text{Dim of Null} = \text{Dim of Domain}$$



range

