Cramer’s Rule

Formula for A-1

$$C = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}$$

- $A^{-1} = \frac{1}{det(A)} C^T$
  - $det(A)$: scalar
  - C: cofactors of A (C has the same size as A, so does C^T)
  - C^T is adjugate of A (adj A, 伴隨矩陣)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$det(A)$$

$$= ad - bc$$

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$C^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1}$$

$$= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Formula for A^{-1}

$$A^{-1} = \frac{1}{det(A)} C^T$$

- $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, A^{-1} = ?$

$$det(A) = aei + bfg + cdh - ceg - bdi - afh$$

$$C = \begin{bmatrix} +\begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & +\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & +\begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ +\begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & +\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

Formula for A^{-1}

$$A^{-1} = \frac{1}{det(A)} C^T$$

- Proof: $AC^T = det(A) I_n$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} c_{11} & \cdots & c_{n1} \\ \vdots & \ddots & \vdots \\ c_{1n} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} det(A) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & det(A) \end{bmatrix}$$

transpose

Diagonal: By definition of determinants

Not Diagonal:

(Exercise 82, P221)

$$AC^T = det(A)I_n$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} c_{11} & \cdots & c_{n1} \\ \vdots & \ddots & \vdots \\ c_{1n} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} det(A) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & det(A) \end{bmatrix}$$

$$det \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = a_{11}c_{11} + a_{12}c_{12} + \cdots + a_{1n}c_{1n}$$

$$det \begin{bmatrix} a_{n1} & \cdots & a_{nn} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = a_{n1}c_{11} + a_{n2}c_{12} + \cdots + a_{nn}c_{1n}$$

$$= 0$$

Cramer's Rule

$$\begin{bmatrix} c_{11} & \cdots & c_{n1} \\ \vdots & \ddots & \vdots \\ c_{1n} & \cdots & c_{nn} \end{bmatrix}$$

$$A^{-1} = \frac{1}{det(A)} C^T \qquad Ax = b \qquad x = A^{-1}b = \frac{1}{det(A)} C^T b$$

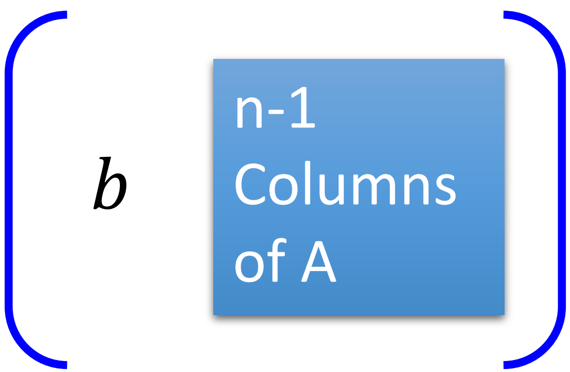
$$x_1 = \frac{1}{det(A)} (c_{11}b_1 + c_{21}b_2 + \cdots + c_{n1}b_n)$$

$$det(B_1)$$

$$x_1 = \frac{det(B_1)}{det(A)}$$

B_1 = with column 1 replaced by b

$$x_2 = \frac{det(B_2)}{det(A)}$$



$$\vdots$$

B_j = with column j replaced by b

$$x_j = \frac{det(B_j)}{det(A)}$$