

More about Rank

# Rank

Maximum number of  
Independent Columns

||

Number of Pivot  
Column

||

Number of Non-zero  
rows

Rank  $R$  = Rank  $A$

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix}$$

Rank = ? **3**

$$R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = ? **3**

# Rank

Maximum number of  
Independent Columns

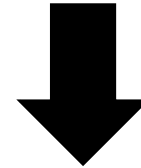
||

Number of Pivot  
Column

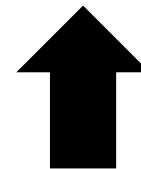
||

Number of Non-zero  
rows

➡ Rank  $A \leq$  Number of columns



Rank  $A \leq \text{Min}(\text{ Number of columns, Number of rows})$



➡ Rank  $A \leq$  Number of rows

# Rank

Matrix A is full rank  
if  $\text{Rank } A = \min(m, n)$

Matrix A is rank deficient  
if  $\text{Rank } A < \min(m, n)$

- Given a  $m \times n$  matrix A:
  - Rank  $A \leq \min(m, n)$
  - Because “the columns of A are independent” is equivalent to “rank  $A = n$ ”
    - If  $m < n$ , the columns of A is dependent.

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

3 X 4

Rank  $A \leq 3$

$$\left\{ \begin{bmatrix} * \\ * \\ * \end{bmatrix}, \begin{bmatrix} * \\ * \\ * \end{bmatrix}, \begin{bmatrix} * \\ * \\ * \end{bmatrix}, \begin{bmatrix} * \\ * \\ * \end{bmatrix} \right\}$$

A matrix set has 4 vectors  
belonging to  $\mathbb{R}^3$  is dependent

In  $\mathbb{R}^m$ , you cannot find more than  $m$  vectors that are independent.

# Independent

All columns are independent



Every column is a pivot column



Every column in RREF(A) is standard vector.

3X4

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

Columns are linearly independent 

RREF



$$\begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}$$

Cannot be a pivot column



這個發現已經提過，現在只是用  
 $\text{Rank } A \leq \min(m, n)$  再說一次

# Basic, Free Variables v.s. Rank

$$Ax = b$$

$$\begin{aligned} x_1 + 2x_2 - x_3 + 2x_4 + 5x_5 &= 2 \\ -x_1 - 2x_2 + x_3 + 2x_4 + 3x_5 &= 6 \\ 2x_1 + 4x_2 - 3x_3 + 2x_4 &= 3 \\ -3x_1 - 6x_2 + 2x_3 + 3x_5 &= 9 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 5 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix}$$

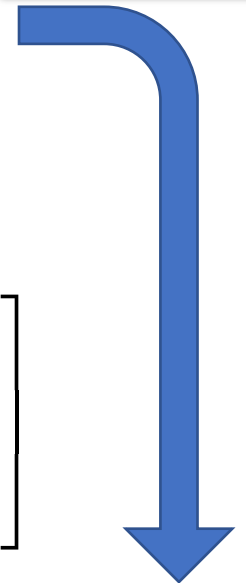
$A$   $b$



$$\begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$RREF(A)$   $b'$

3 useful  
equations



rank = non-zero row = 3 basic variables

nullity = No. column – non-zero row = 2 free variables

$$x_1 = -5 - 2x_2 + x_5$$

$$x_3 = -3$$

$$x_4 = 2 - x_5$$

# Rank

Rank

Maximum number of  
Independent Columns

Number of Pivot  
Column

Number of Non-zero  
rows of RREF

Number of Basic  
Variables

Nullity = no. column - rank

Number of zero rows  
of RREF



Number of Free  
Variables