Basic Properties of Determinant

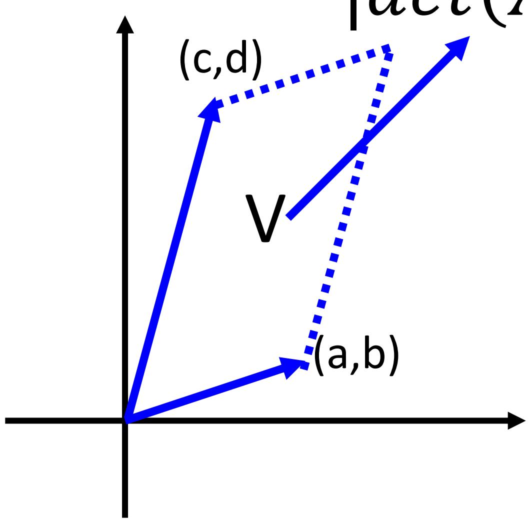
“Volume” in high dimensions

Determinant in High School

- 2 X 2

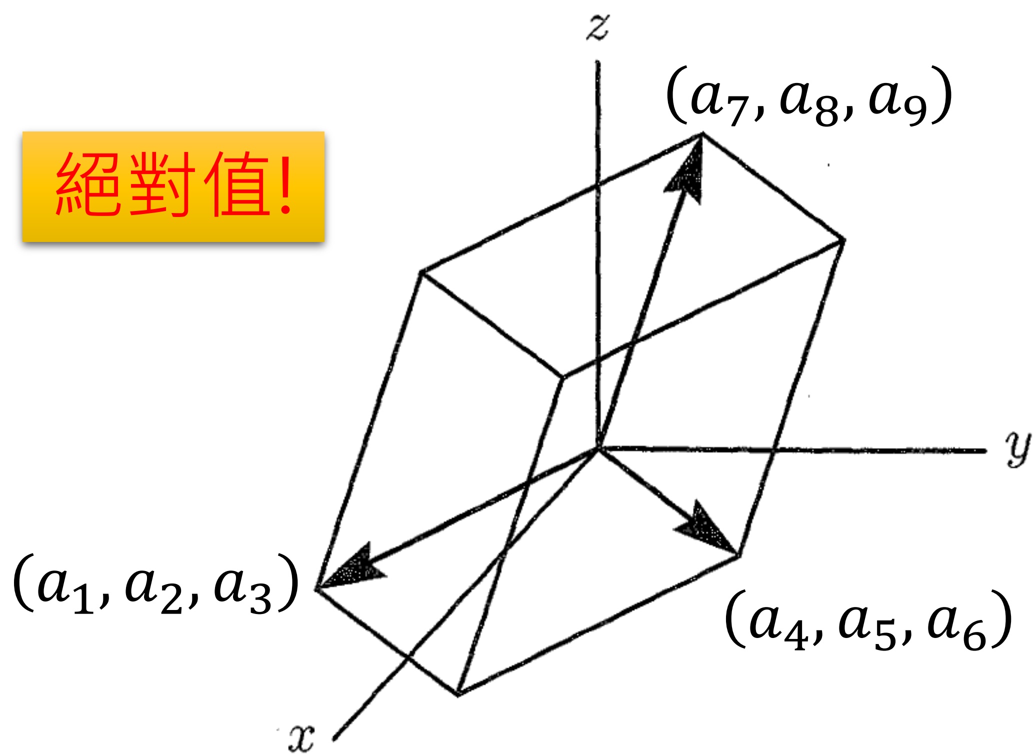
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$|det(A)|$ 絕對值!



- 3 x 3

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$$



Three Basic Properties

- Basic Property 1: $det(I) = 1$
- Basic Property 2: Exchange rows only *reverses the sign* of det (do not change absolute value)
- Basic Property 3: Determinant is "linear" for each row

Area in 2d and Volume in 3d have the above properties

So det is "Volume" in high dimension?

Three Basic Properties

- Basic Property 1:
  - $det(I) = 1$

正方形　　面積為 1

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$det(I_2) = 1$$

正立方體　　體積為 1

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$det(I_3) = 1$$

Three Basic Properties

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

- Basic Property 2:
  - Exchanging rows only reverses the sign of det

$$det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 1$$

$$det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1$$

$$det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\right) = -1$$

$$det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = -1$$

$$det\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\right) = 1$$

Three Basic Properties

- Basic Property 2:
  - Exchanging rows only reverses the sign of det

If a matrix A has ***<u>2 equal rows</u>***

$$\Rightarrow \quad det(A) = 0$$

$$A \xrightarrow{\text{exchange the two rows}} A'$$

$$det(A) = K \quad = \quad det(A') = -K$$

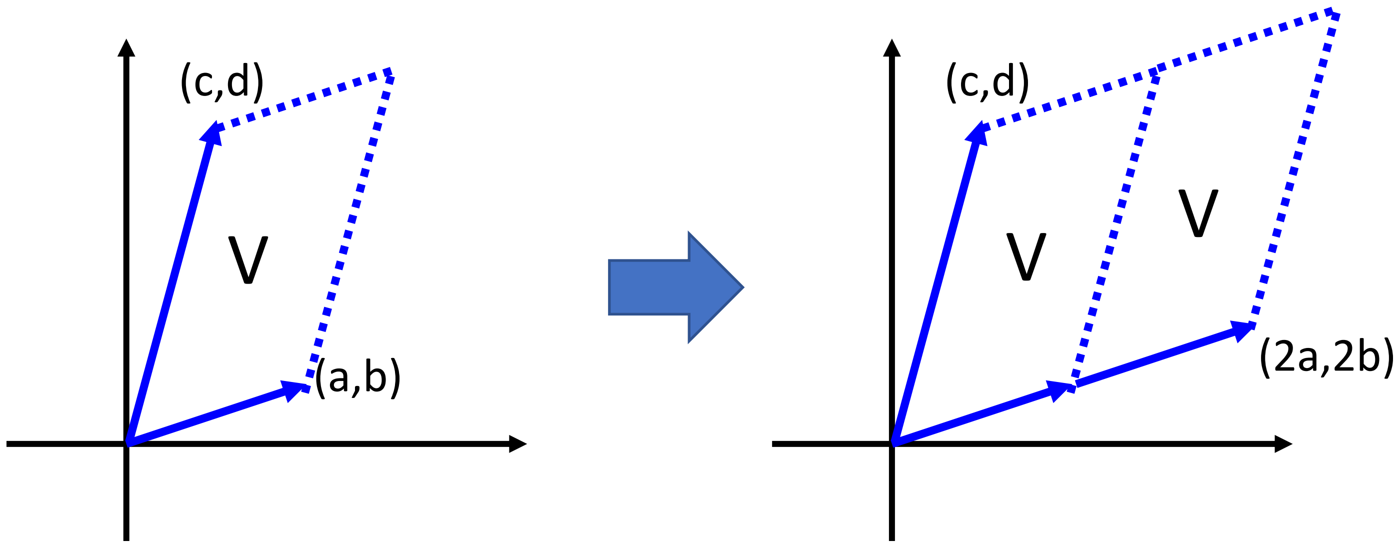
Exchanging the two equal rows yields the same matrix

Three Basic Properties

- Basic Property 3:
  - Determinant is “linear” for each row

3-a

$$det\left(\begin{bmatrix} ta & tb \\ c & d \end{bmatrix}\right) = t\,det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$



Three Basic Properties

- Basic Property 3:
  - Determinant is “linear” for each row

3-a

$$det\left(\begin{bmatrix} ta & tb \\ c & d \end{bmatrix}\right) = t\,det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

Q: find $det(2A)$

$$det\left(2\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = det\left(\begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix}\right)$$

If A is n x n ......

$$= 2det\left(\begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix}\right) = 4det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

A: $det(2A) = 2^n det(A)$

Three Basic Properties

- Basic Property 3:
  - Determinant is "linear" for each row

3-a

$$det\left(\begin{bmatrix} ta & tb \\ c & d \end{bmatrix}\right) = t\,det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

A row of zeros $\Rightarrow$ $det(A) = 0$

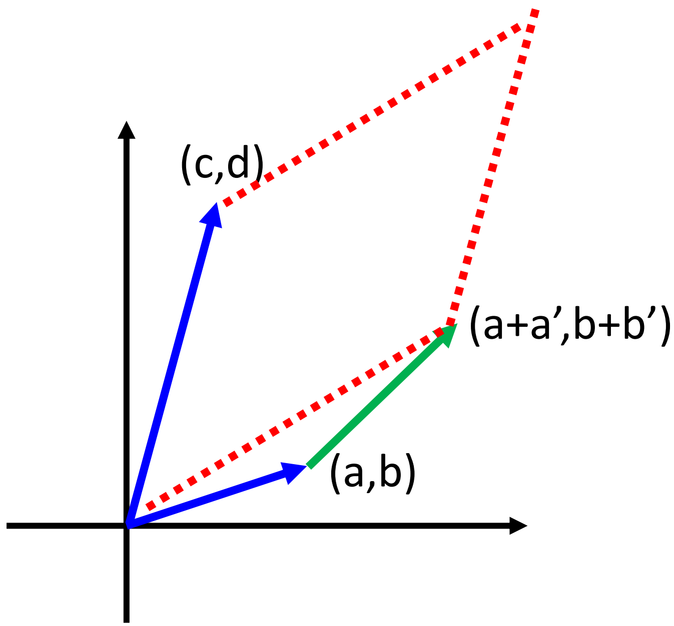
Set $t = 0$!

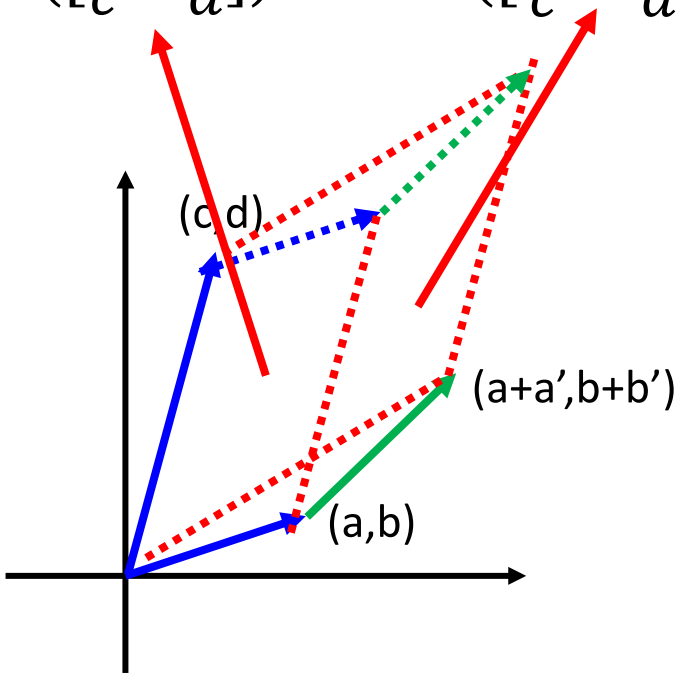
A row of zeros $\Rightarrow$ "volume" is zero

Three Basic Properties

- Basic Property 3:
  - Determinant is "linear" for each row

3-b $$det\left(\begin{bmatrix} a+a' & b+b' \\ c & d \end{bmatrix}\right) = det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + det\left(\begin{bmatrix} a' & b' \\ c & d \end{bmatrix}\right)$$





Three Basic Properties

- Basic Property 3:
  - Determinant is “linear” for each row

Subtract k x row i from row j (elementary row operation)

Determinant doesn’t change

$$det\left(\begin{bmatrix} a & b \\ c-ka & d-kb \end{bmatrix}\right)$$

3-b $$= det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + det\left(\begin{bmatrix} a & b \\ -ka & -kb \end{bmatrix}\right)$$

3-a $$= det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) - k det\left(\begin{bmatrix} a & b \\ a & b \end{bmatrix}\right) = det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

Formulas Again

Formula from Three Properties

1

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

2

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1$$

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} + \det\begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$$ **3-b**

$$= \underbrace{\det\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}}_{\text{3-a}\; = 0} + \underbrace{\det\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}}_{\text{3-a}\; = ad} + \underbrace{\det\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}}_{\text{3-a}\; = -\mathrm{bc}} + \underbrace{\det\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}}_{\text{3-a}\; = 0}$$ **3-b**

$$= ad - bc$$

Finally, we get 3 x 3 x 3 matrices

Most of them have zero determinants

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \det\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \det\begin{bmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \det\begin{bmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= det\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + det\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + det\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= det\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{bmatrix} + det\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{bmatrix} + det\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

3! matrices have non-zero rows

$$= \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$a_{11}a_{22}a_{33} \qquad -a_{11}a_{23}a_{32} \qquad -a_{12}a_{21}a_{33}$$

$$+ \begin{bmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{bmatrix}$$

$$a_{12}a_{23}a_{31} \qquad a_{13}a_{21}a_{32} \qquad -a_{13}a_{22}a_{31}$$

Pick an element at each row,
but they can not be in the same column.

Formula from Three Properties

- Given an n x n matrix A

$$det(A) = sum\ of\ n!\ terms$$

Format of each term: $a_{\underline{1}\underline{\alpha}}a_{\underline{2}\underline{\beta}}a_{\underline{3}\underline{\gamma}}\cdots a_{\underline{n}\underline{\omega}}$

Find an element in each row

permutation of 1,2, …, n

Example

$$det\left(\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\right)$$

$$= det\left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) + det\left(\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right)$$

-1 +1