Determinant

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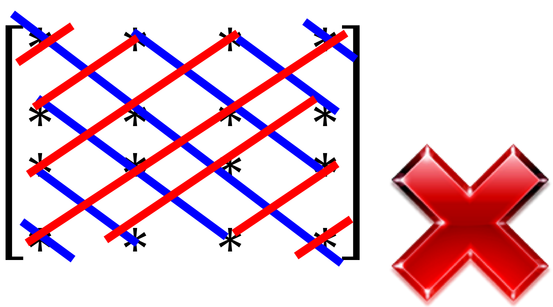
The determinant of a **square matrix** is a **scalar** that provides information about the matrix (e.g. invertible or not)

Determinants in High School

- 2 X 2

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$det(A) = ad - bc$$



- 3 x 3

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$$

$$det(A) =$$

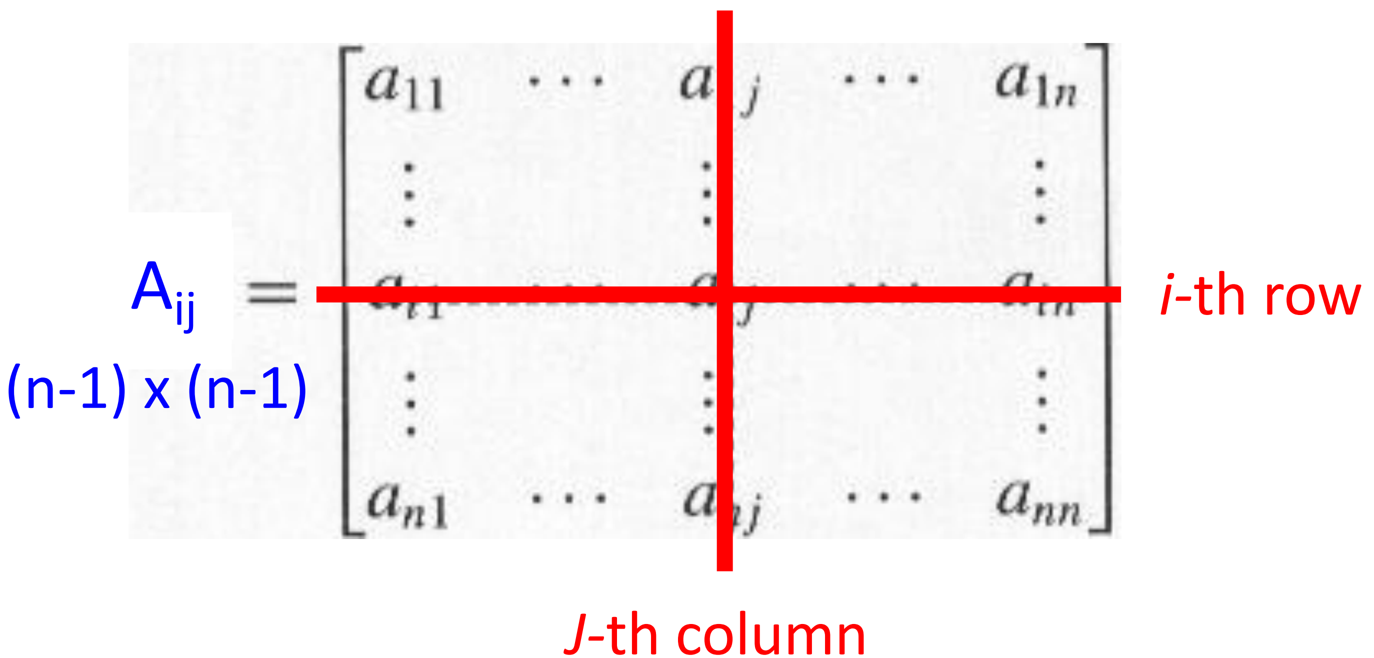
$$a_1 a_5 a_9 + a_2 a_6 a_7 + a_3 a_4 a_8$$

$$- a_3 a_5 a_7 - a_2 a_4 a_9 - a_1 a_6 a_8$$

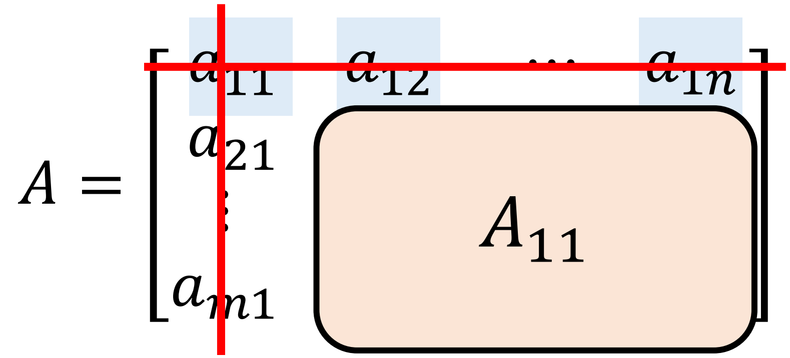
Cofactor Expansion

a_{ij}: scalar
A_{ij}: matrix

- Suppose A is an n x n matrix. A_{ij} is defined as the submatrix of A obtained by removing the i-th row and the j-th column.



Cofactor Expansion



- Pick row 1

$$detA = a_{11}c_{11} + a_{12}c_{12} + \cdots + a_{1n}c_{1n}$$

- Or pick row i

c_{ij}: (i,j)-cofactor

$$detA = a_{i1}c_{i1} + a_{i2}c_{i2} + \cdots + a_{in}c_{in}$$

- Or pick column j

$$detA = a_{1j}c_{1j} + a_{2j}c_{2j} + \cdots + a_{nj}c_{nj}$$

$$c_{ij} = (-1)^{i+j} detA_{ij}$$

$$c_{11} = (-1)^{1+1} detA_{11}$$

Cofactor expansion again …

$$detA = a_{11}c_{11} + \cdots + a_{1n}c_{1n}$$

$$c_{ij} = (-1)^{i+j} detA_{ij}$$

For 1x1 matrix,
det([a]) = a

2 x 2 matrix

$$c_{ij} = (-1)^{i+j} det A_{ij}$$

- Define det([a]) = a

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad det(A) = ad - bc$$

Pick the first row

$$det(A) = ac_{11} + bc_{12} \quad = ad - bc$$

$$c_{11} = (-1)^{1+1} det([d]) \; = d$$

$$c_{12} = (-1)^{1+2} det([c]) \quad = -c$$

3 x 3 matrix

$$c_{ij} = (-1)^{i+j} detA_{ij}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Pick row 2

$$detA = a_{21}c_{21} + a_{22}c_{22} + a_{23}c_{23}$$

4 5 6

$$(-1)^{2+1} detA_{21} \qquad (-1)^{2+2} detA_{22} \qquad (-1)^{2+3} detA_{23}$$

$$A_{21} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad A_{22} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad A_{23} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Example

- Given tridiagonal n x n matrix A

$$A = \begin{bmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix}$$

Find det A when n = 999

$$detA_4$$

$$= a_{11}c_{11} + a_{12}c_{12} + \cancel{a_{13}c_{13}} + \cancel{a_{14}c_{14}}$$

$$\quad 1 \qquad\qquad 1 \qquad\qquad 0 \qquad\qquad 0$$

$$A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$c_{11} = (-1)^2 det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= det(A_3)$$

$$c_{12} = (-1)^3 det \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= -det(A_2)$$

$$= a_{11}c_{11} + a_{12}c_{12} + \cancel{a_{13}c_{13}}$$

$$\quad 1 \qquad\qquad 1 \qquad\qquad 0$$

$$= det(A_2)$$

$$det(A_4) = det(A_3) - det(A_2)$$

$$det(A_n) = det(A_{n-1}) - det(A_{n-2})$$

$$A_n = \begin{bmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix}$$

A_{n-1}

$$B = \begin{bmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix}$$

A_{n-2}

$= 0$

Example

$$det(A_n) = det(A_{n-1}) - det(A_{n-2})$$

$det(A_1) = 1$ $det(A_2) = 0$ $det(A_3) = -1$

$det(A_4) = -1$ $det(A_5) = 0$ $det(A_6) = 1$

$det(A_7) = 1$ $det(A_8) = 0$ … …