Diagonalization (對角化)

Hung-yi Lee

Review

- If $Av = \lambda v$ (v is a vector, λ is a scalar)
  - v is an eigenvector of A excluding zero vector
  - λ is an eigenvalue of A that corresponds to v
- Eigenvectors corresponding to λ are nonzero solution of $(A - \lambda I_n)\mathbf{v} = \mathbf{0}$

Eigenvectors corresponding to λ

$$= \underline{Null(A - \lambda I_n)} - \{\mathbf{0}\}$$

eigenspace

Eigenspace of λ:

Eigenvectors corresponding to λ + $\{\mathbf{0}\}$

- A scalar t is an eigenvalue of A

$\Leftrightarrow$ $det(A - tI_n) = 0$

Review

- Characteristic polynomial of A is

$det(A - tI_n)$ Factorization

multiplicity

$$= (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2} \dots (t-\lambda_k)^{m_k}(\dots\dots)$$

Eigenvalue: λ_1 λ_2 λ_k

Eigenspace: d_1 d_2 d_k

(dimension) $\leq m_1$ $\leq m_2$ $\leq m_k$

Outline

- An nxn matrix A is called diagonalizable if $A = PDP^{-1}$
  - D: nxn diagonal matrix
  - P: nxn invertible matrix
- Is a matrix A diagonalizable?
  - If yes, find D and P
- Reference: Textbook 5.3

Diagonalizable

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $A^2 = 0$ (?)

- Not all matrices are diagonalizable

If $A = PDP^{-1}$ for some invertible P and diagonal D

$\Rightarrow$ $A^2 = PD^2P^{-1} = 0$ $\Rightarrow$ $D^2 = 0$ $\Rightarrow$ $D = 0$

D is diagonal

$\Rightarrow$ A= 0 

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix} \qquad D^2 = \begin{bmatrix} (d_1)^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (d_n)^2 \end{bmatrix}$$

(d_1 =0, d_n =0; $(d_1)^2$ =0, $(d_n)^2$ =0)

Diagonalizable

$$P = [p_1 \quad \cdots \quad p_n]$$

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

$$= [d_1 e_1 \quad \cdots \quad d_n e_n]$$

- If A is diagonalizable

$$A = PDP^{-1} \Rightarrow AP = PD$$

$$\Rightarrow AP = [\underline{Ap_1} \quad \cdots \quad \underline{Ap_n}]$$

$$\Rightarrow PD = P[d_1 e_1 \quad \cdots \quad d_n e_n]$$

$$= [Pd_1 e_1 \quad \cdots \quad Pd_n e_n]$$

$$= [d_1 P e_1 \quad \cdots \quad d_n P e_n]$$

$$= [\underline{d_1 p_1} \quad \cdots \quad \underline{d_n p_n}] \Rightarrow Ap_i = d_i p_i$$

p_i is an eigenvector of A corresponding to eigenvalue d_i

Diagonalizable

$$P = [p_1 \quad \cdots \quad p_n]$$

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

- If A is diagonalizable

$$A = PDP^{-1}$$

p_i is an eigenvector of A corresponding to eigenvalue d_i

=

There are n eigenvectors that form an invertible matrix

=

There are n independent eigenvectors

=

The eigenvectors of A can form a basis for R^n.

Diagonalizable

$$P = [p_1 \quad \cdots \quad p_n]$$

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

- If A is diagonalizable

$$A = PDP^{-1}$$

p_i is an eigenvector of A corresponding to eigenvalue d_i

How to diagonalize a matrix A?

Step 1: Find *n* independent eigenvectors corresponding if possible, and form an invertible *P*

Step 2: The eigenvalues corresponding to the eigenvectors in P form the diagonal matrix *D.*

Diagonalizable

A set of eigenvectors that correspond to distinct eigenvalues is linear independent.

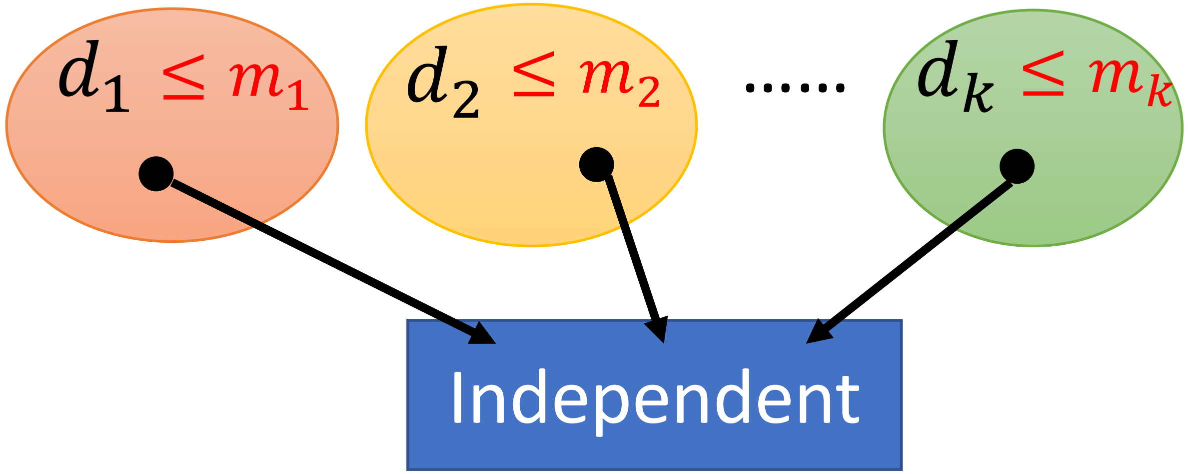
$det(A - tI_n)$ Factorization

$$= (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2}\dots(t-\lambda_k)^{m_k}(\dots\dots)$$

Eigenvalue: λ_1 λ_2 …… λ_k

Eigenspace:

(dimension)



Diagonalizable

A set of eigenvectors that correspond to distinct eigenvalues is linear independent.

Eigenvalue: λ_1 λ_2 …… λ_m

Eigenvector: v_1 v_2 …… v_m

Assume dependent → a contradiction

$$\mathbf{v}_k = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{k-1}\mathbf{v}_{k-1}$$

$$A\mathbf{v}_k = c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + \cdots + c_{k-1}A\mathbf{v}_{k-1}$$

$$\lambda_k\mathbf{v}_k = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + \cdots + c_{k-1}\lambda_{k-1}\mathbf{v}_{k-1}$$

(λ_k)

$$-\quad \lambda_k\mathbf{v}_k = c_1\lambda_k\mathbf{v}_1 + c_2\lambda_k\mathbf{v}_2 + \cdots + c_{k-1}\lambda_k\mathbf{v}_{k-1}$$

$$\mathbf{0} = c_1(\lambda_1-\lambda_k)\mathbf{v}_1 + c_2(\lambda_2-\lambda_k)\mathbf{v}_2 + \cdots + c_{k-1}(\lambda_{k-1}-\lambda_k)\mathbf{v}_{k-1}$$

Not $c_1 = c_2 = \cdots = c_{k-1} = 0$ → *Same eigenvalue* → a contradiction

Diagonalizable

$$P = [p_1 \quad \cdots \quad p_n]$$

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

- If A is diagonalizable

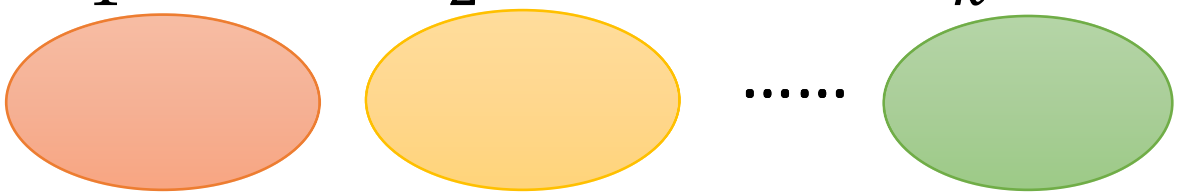
$$A = PDP^{-1}$$

p_i is an eigenvector of A corresponding to eigenvalue d_i

$$det(A - tI_n)$$
$$= (t - \lambda_1)^{m_1}(t - \lambda_2)^{m_2} \dots (t - \lambda_k)^{m_k}(\dots\dots)$$

Eigenvalue: λ_1 λ_2 …… λ_k

Eigenspace: ……



{ Basis for λ_1 Basis for λ_2 Basis for λ_3 }

Independent Eigenvectors

You can't find more!

Diagonalizable - Example

- Diagonalize a given matrix $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

characteristic polynomial is $-(t+1)^2(t-3)$ ⇨ eigenvalues: 3, –1

eigenvalue 3

$$\mathbb{B}_1 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

eigenvalue –1

$$\mathbb{B}_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

$A = PDP^{-1}$, where

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Test for a Diagonalizable Matrix

- An n x n matrix A is diagonalizable if and only if both the following conditions are met.

The characteristic polynomial of *A* factors into a product of linear factors.

$det(A - tI_n)$ Factorization

$= (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2} \ldots (t-\lambda_k)^{m_k}$ ~~(... ...)~~

For each eigenvalue λ of *A*, the multiplicity of λ equals the dimension of the corresponding eigenspace.

Independent Eigenvectors

An n x n matrix A is diagonalizable

=

The eigenvectors of A can form a basis for R^n.

=

$$det(A - tI_n)$$

$$= (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2} \ldots (t-\lambda_k)^{m_k}(\ldots \ldots)$$

Eigenvalue:	λ_1	λ_2	……	λ_k
Eigenspace: (dimension)	$d_1 = m_1$	$d_2 = m_2$	……	$d_k = m_k$

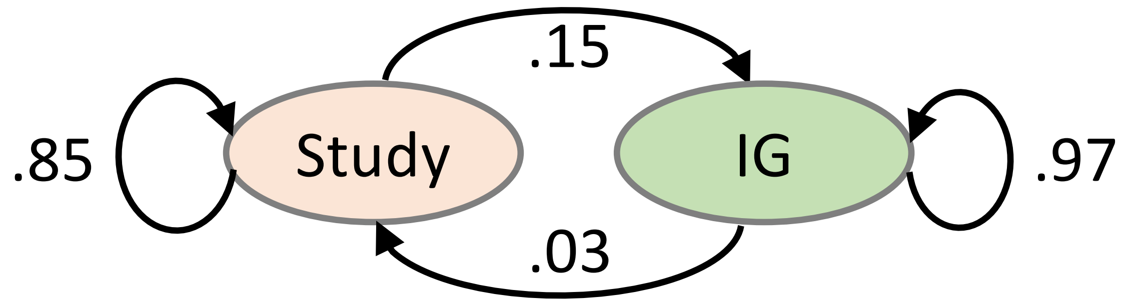
$$d_1 + d_2 + \cdots + d_k = n$$

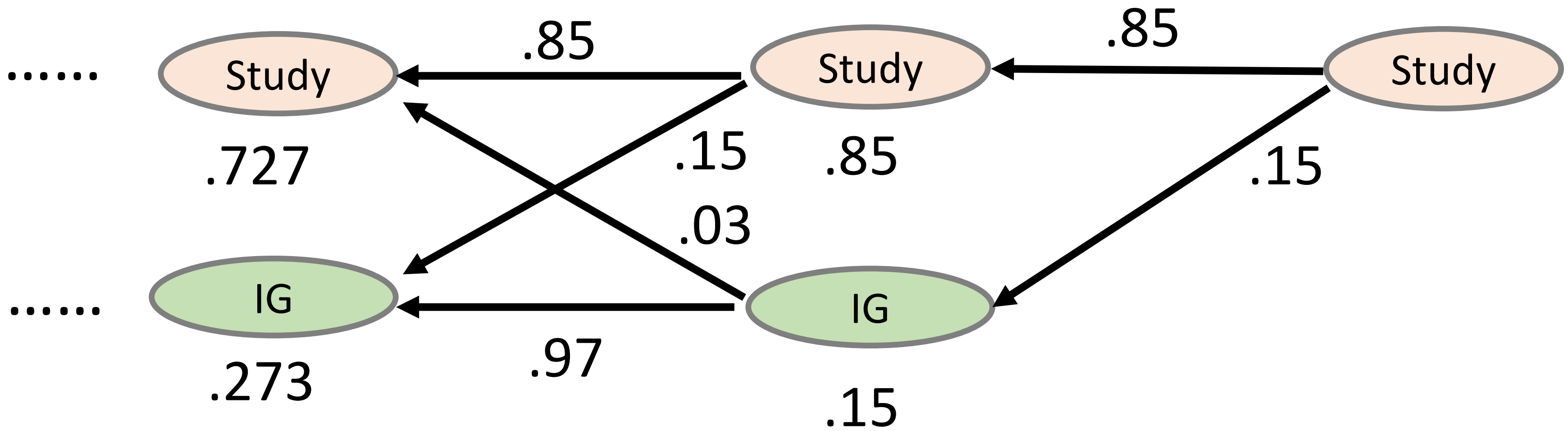
Application of Diagonalization

- If A is diagonalizable,

$$A = PDP^{-1} \Rightarrow A^m = PD^mP^{-1}$$

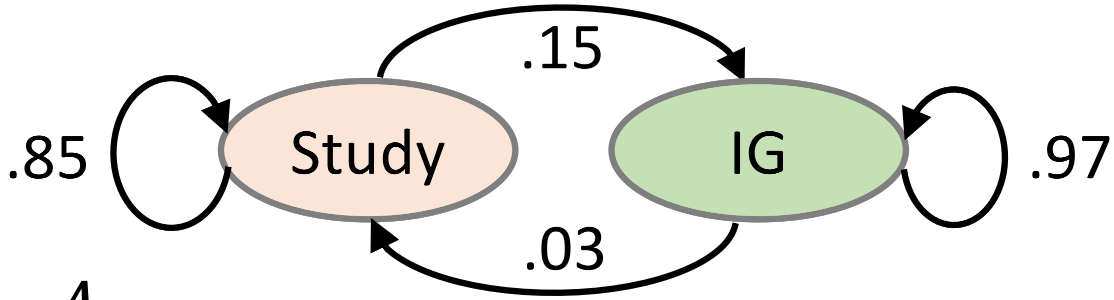
- Example:

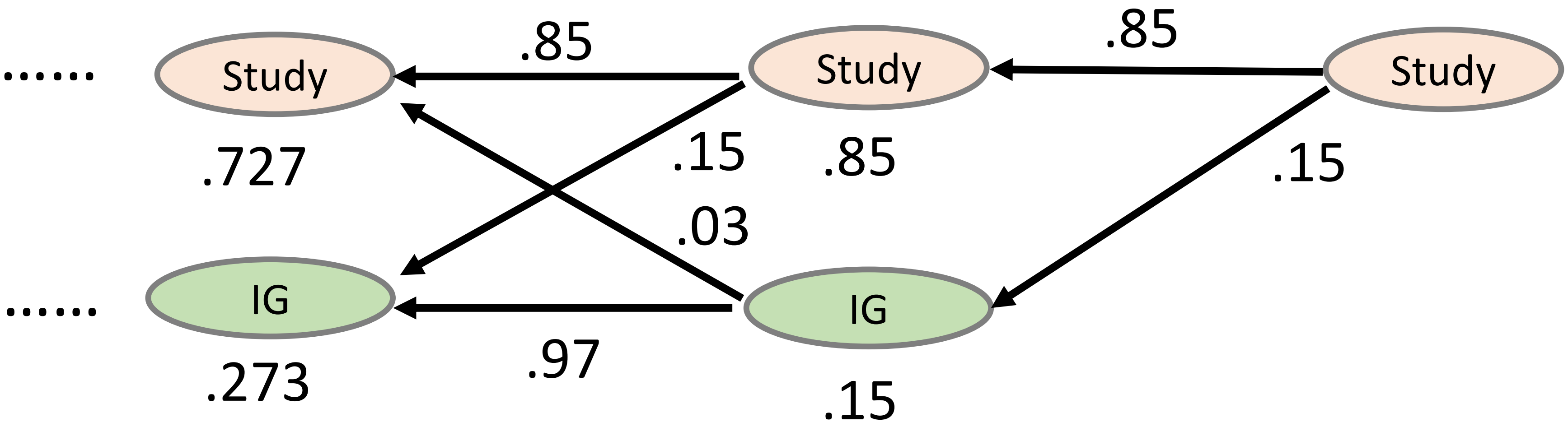




$$\begin{array}{lcc} & \text{From} & \\ & \text{Study} & \text{IG} \\ \text{To} \quad \text{Study} & .85 & .03 \\ \qquad \text{IG} & .15 & .97 \end{array} = A$$

$$A = \begin{bmatrix} .85 & .03 \\ .15 & .97 \end{bmatrix}$$





$$\begin{bmatrix} .727 \\ .273 \end{bmatrix} \quad \begin{bmatrix} .85 & .03 \\ .15 & .97 \end{bmatrix} \quad \begin{bmatrix} .85 \\ .15 \end{bmatrix} \quad \begin{bmatrix} .85 & .03 \\ .15 & .97 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A = PDP^{-1} \Longrightarrow A^m = PD^mP^{-1}$$

Diagonalizable

- Diagonalize a given matrix $A = \begin{bmatrix} .85 & .03 \\ .15 & .97 \end{bmatrix}$

$$\det\,(A - tI_2)$$

$$A - .82I_2 \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$A - I_2 \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -.2 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{p}_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 5 \end{bmatrix}$$

(invertible)

$$D = \begin{bmatrix} .82 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = PDP^{-1} \text{ where } P = \begin{bmatrix} -1 & 1 \\ 1 & 5 \end{bmatrix}, D = \begin{bmatrix} .82 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} A^m &= PD^mP^{-1} \\ &= \begin{bmatrix} -1 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} (.82)^m & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 5 \end{bmatrix}^{-1} \\ &= \frac{1}{6} \begin{bmatrix} 1+5(.82)^m & 1-(.82)^m \\ 5-5(.82)^m & 5+(.82)^m \end{bmatrix} \end{aligned}$$

When $m \to \infty$,

$$A^m = \begin{bmatrix} 1/6 & 1/6 \\ 5/6 & 5/6 \end{bmatrix}$$

The beginning condition does not influence.

$$\begin{bmatrix} 1/6 & 1/6 \\ 5/6 & 5/6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 5/6 \end{bmatrix} \quad \begin{bmatrix} 1/6 & 1/6 \\ 5/6 & 5/6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 5/6 \end{bmatrix}$$

Diagonalization of Linear Operator

- Example 1: $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 8x_1 + 9x_2 \\ -6x_1 - 7x_2 \\ 3x_1 + 3x_2 - x_3 \end{bmatrix}$

The standard matrix is $A = \begin{bmatrix} 8 & 9 & 0 \\ -6 & -7 & 0 \\ 3 & 3 & -1 \end{bmatrix}$

det $\begin{bmatrix} 8-t & 9 & 0 \\ -6 & -7-t & 0 \\ 3 & 3 & -1-t \end{bmatrix}$

$\Rightarrow$ the characteristic polynomial is $-(t+1)^2(t-2)$

Eigenvalue -1:

$B_1 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Eigenvalue 2:

$B_2 = \left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \right\}$

$\Rightarrow B_1 \cup B_2$ is a basis of R^3

Diagonalization of Linear Operator

- Example 2: $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 + x_2 + 2x_3 \\ x_1 - x_2 \\ 0 \end{bmatrix}$

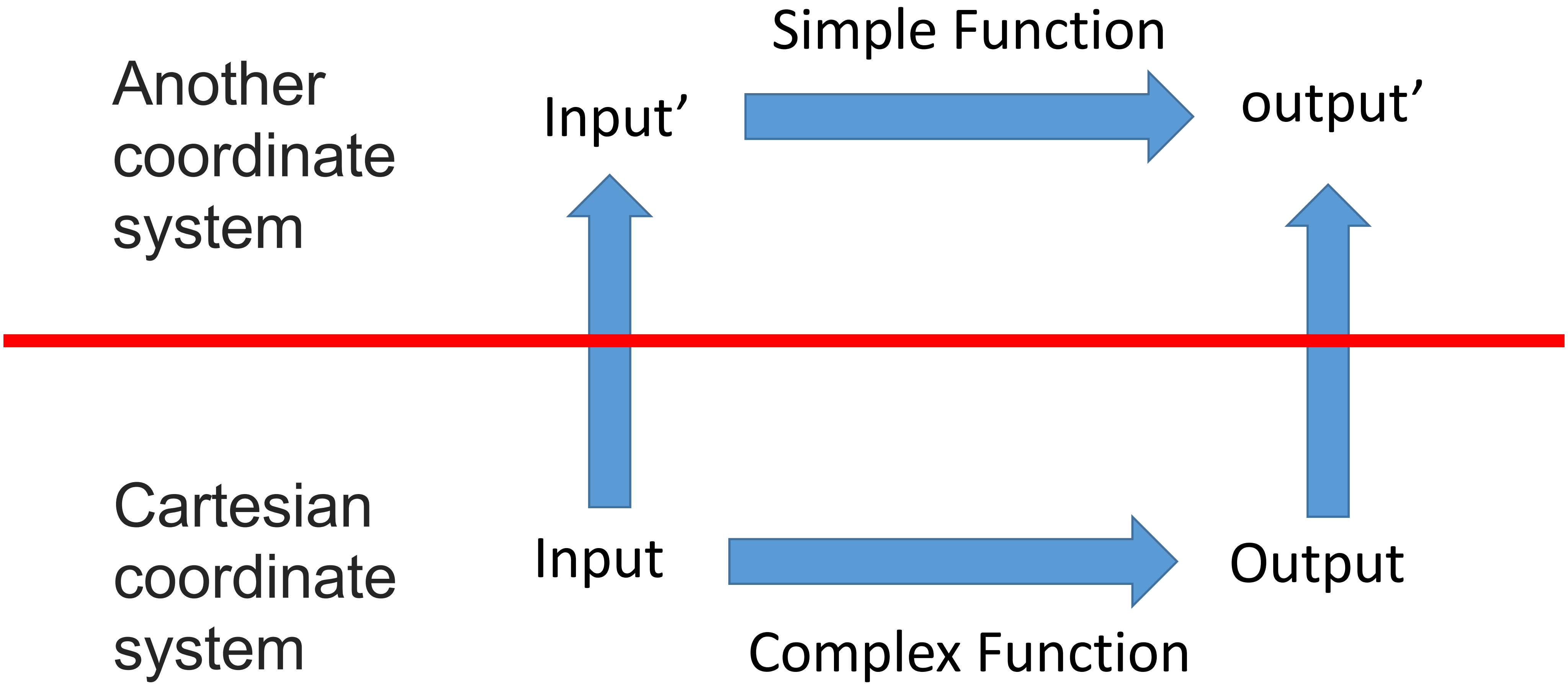
det -t -t -t

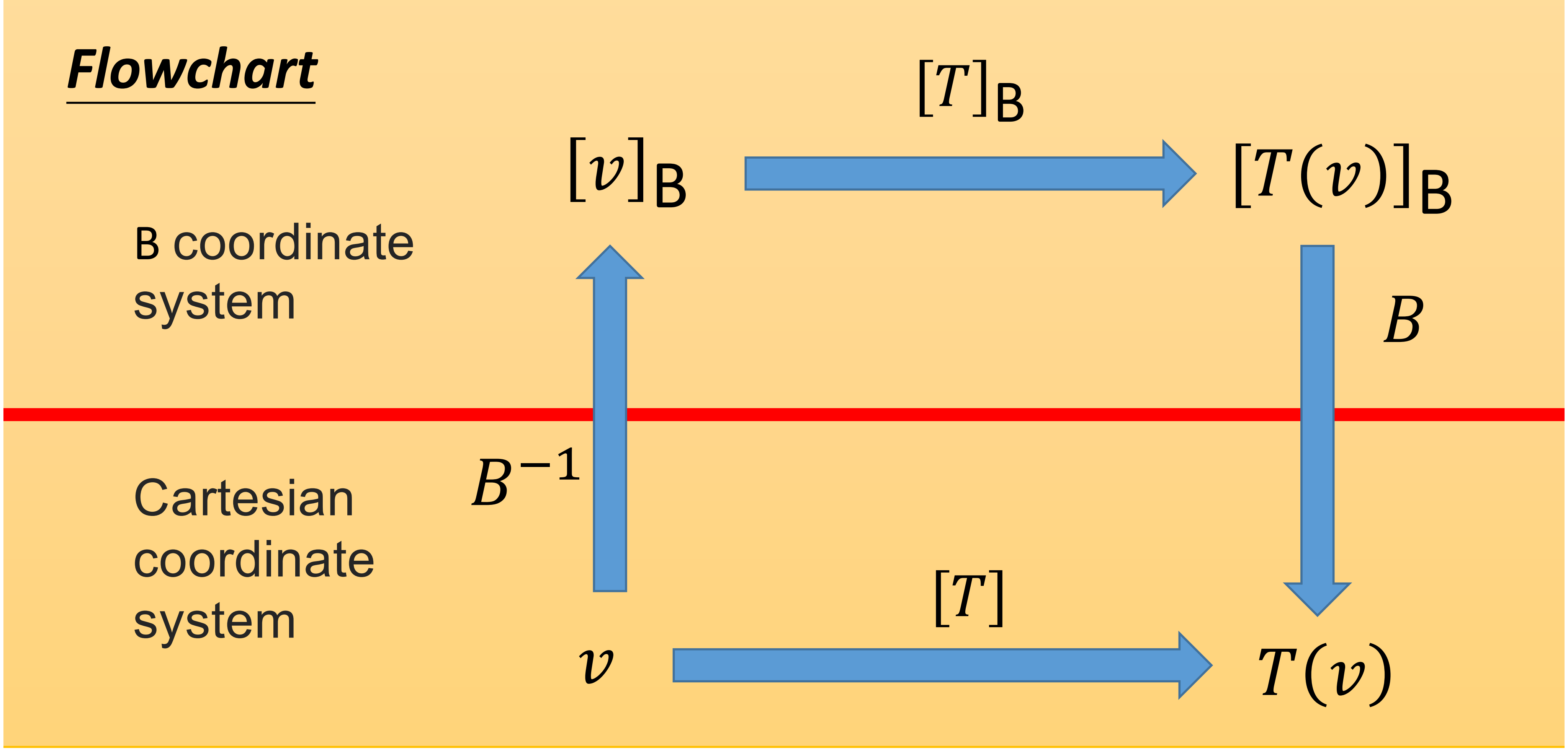
The standard matrix is $A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - x_2 &= 0 \\ x_3 &= 0 \end{aligned} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Review


Another coordinate system
Input’
Simple Function
output’
Cartesian coordinate system
Input
Complex Function
Output



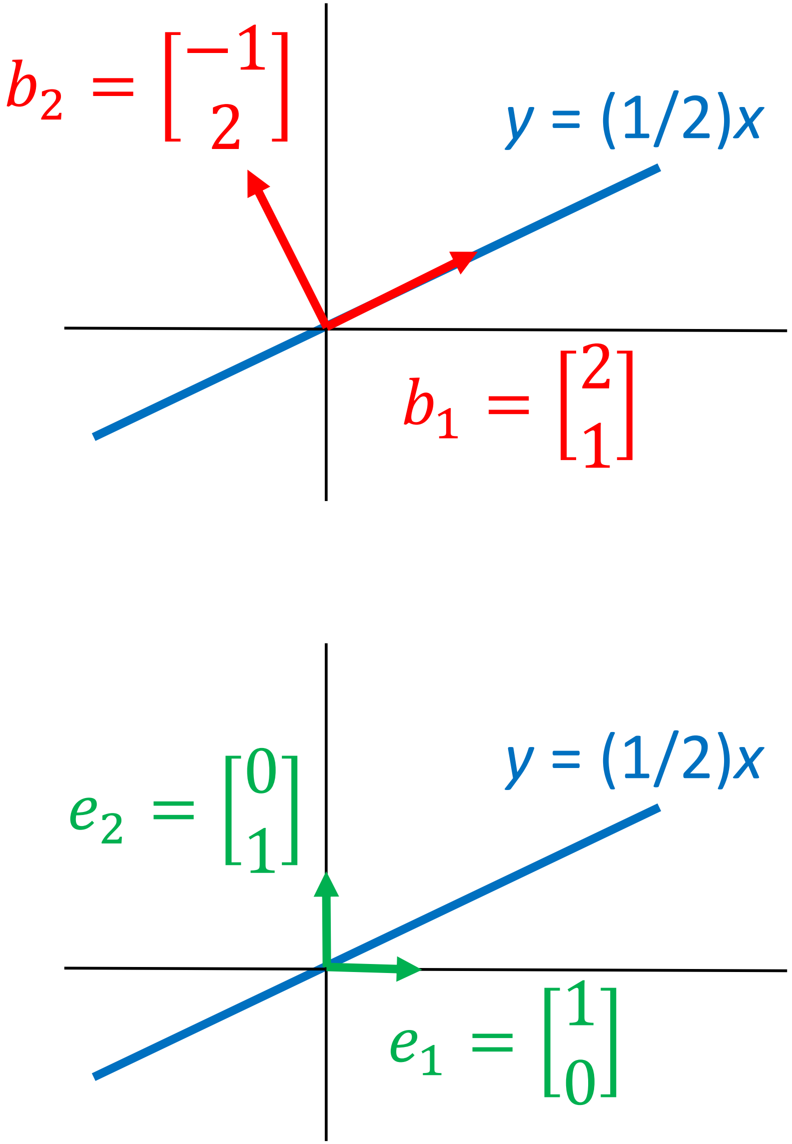
$$[T] = B[T]_B B^{-1}$$

similar

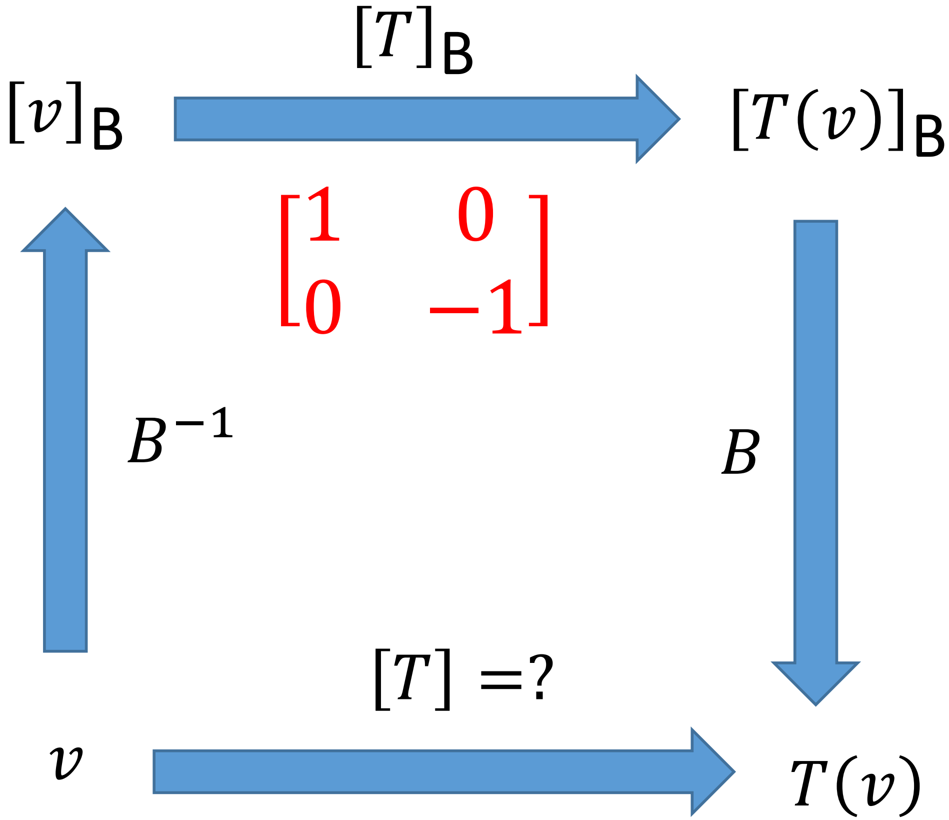
$$[T]_B = B^{-1}[T]B$$

similar

- Example: reflection operator *T* about the line $y = (1/2)x$



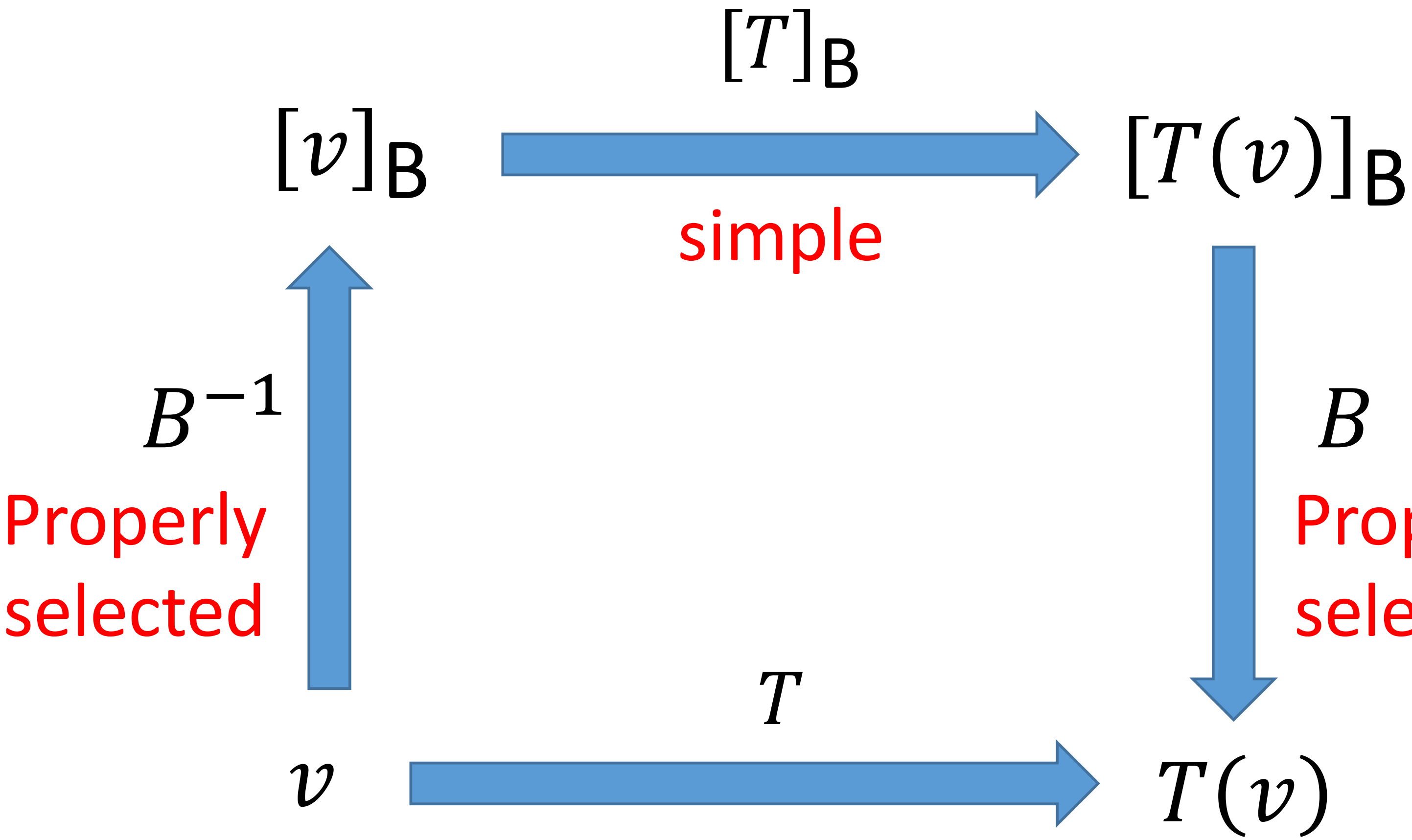
$$B^{-1} = \begin{bmatrix} 0.4 & 0.2 \\ -0.2 & 0.4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$



$$[T] = B[T]_B B^{-1}$$

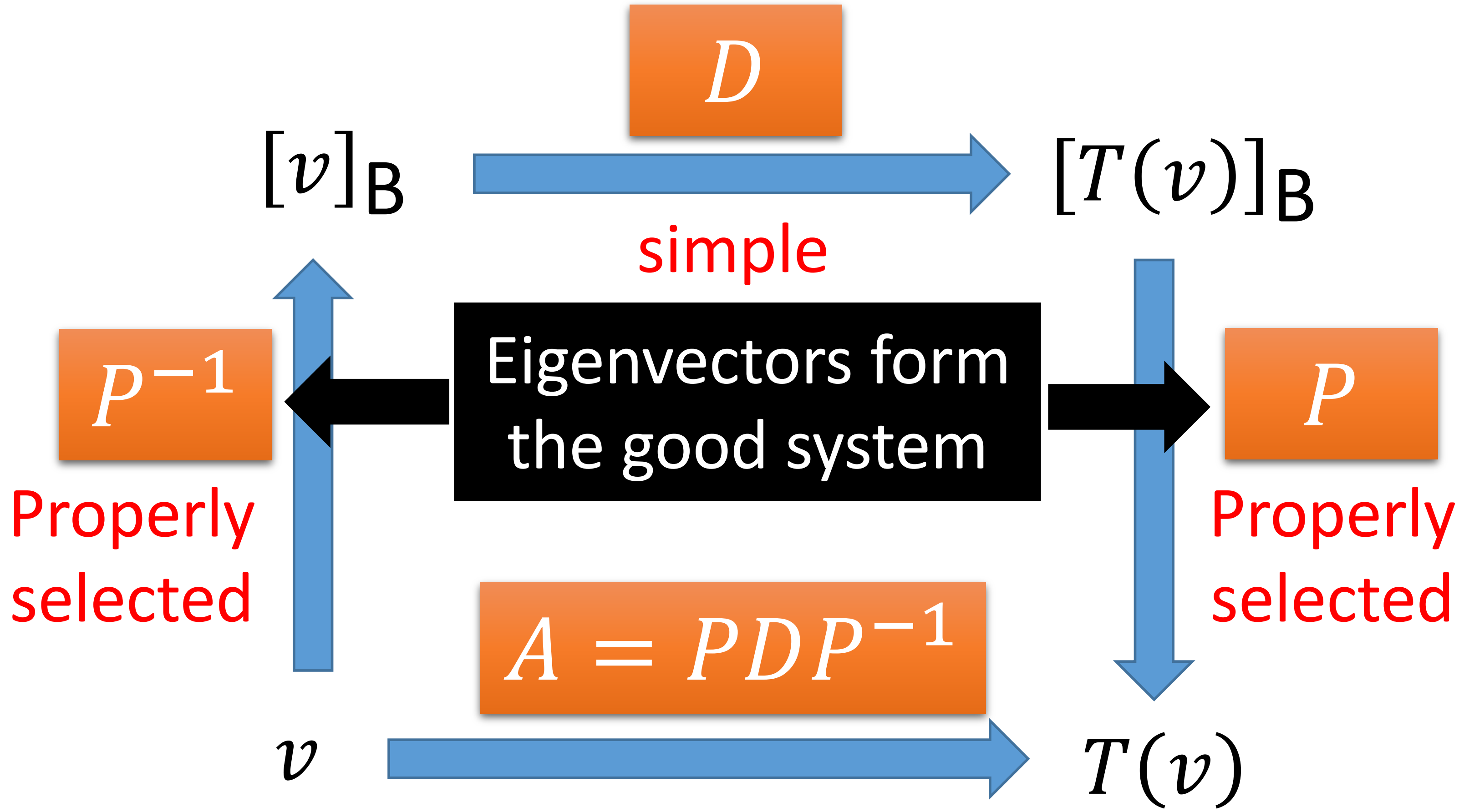
Diagonalization of Linear Operator

- Reference: Chapter 5.4



Diagonalization of Linear Operator

- If a linear operator T is diagonalizable



Diagonalization of Linear Operator

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 8x_1 + 9x_2 \\ -6x_1 - 7x_2 \\ 3x_1 + 3x_2 - x_3 \end{bmatrix}$$

-1: $B_1 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

2: $B_2 = \left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \right\}$

$[v]_B$ → $[T]_B$ → $[T(v)]_B$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 3 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} -1 & 0 & 3 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 9 & 0 \\ -6 & -7 & 0 \\ 3 & 3 & -1 \end{bmatrix}$$

v → $T(v)$