Eigenvalues and Eigenvectors

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Chapter 5

- In chapter 4, we already know how to consider a function from different aspects (coordinate system).
- In chapter 5, we will learn how to find a “good” coordinate system for a function.
- Scope: Chapter 5.1 – 5.4
  - Chapter 5.4 has *

Outline

- What is Eigenvalue and Eigenvector?
  - Eigen (German word): "unique to”
- How to find eigenvectors (given eigenvalues)?
- Check whether a scalar is an eigenvalue
- How to find all eigenvalues?

- Reference: Textbook Chapter 5.1 and 5.2

Definition

Eigenvalues and Eigenvectors

$$A\mathbf{0} = \lambda\mathbf{0}$$

- If $Av = \lambda v$ (v is a vector, λ is a scalar)
  - v is an eigenvector of A excluding zero vector
  - λ is an eigenvalue of A that corresponds to v

A must be square

Eigen value

$$\begin{bmatrix} 5 & 2 & 1 \\ -2 & 1 & -1 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

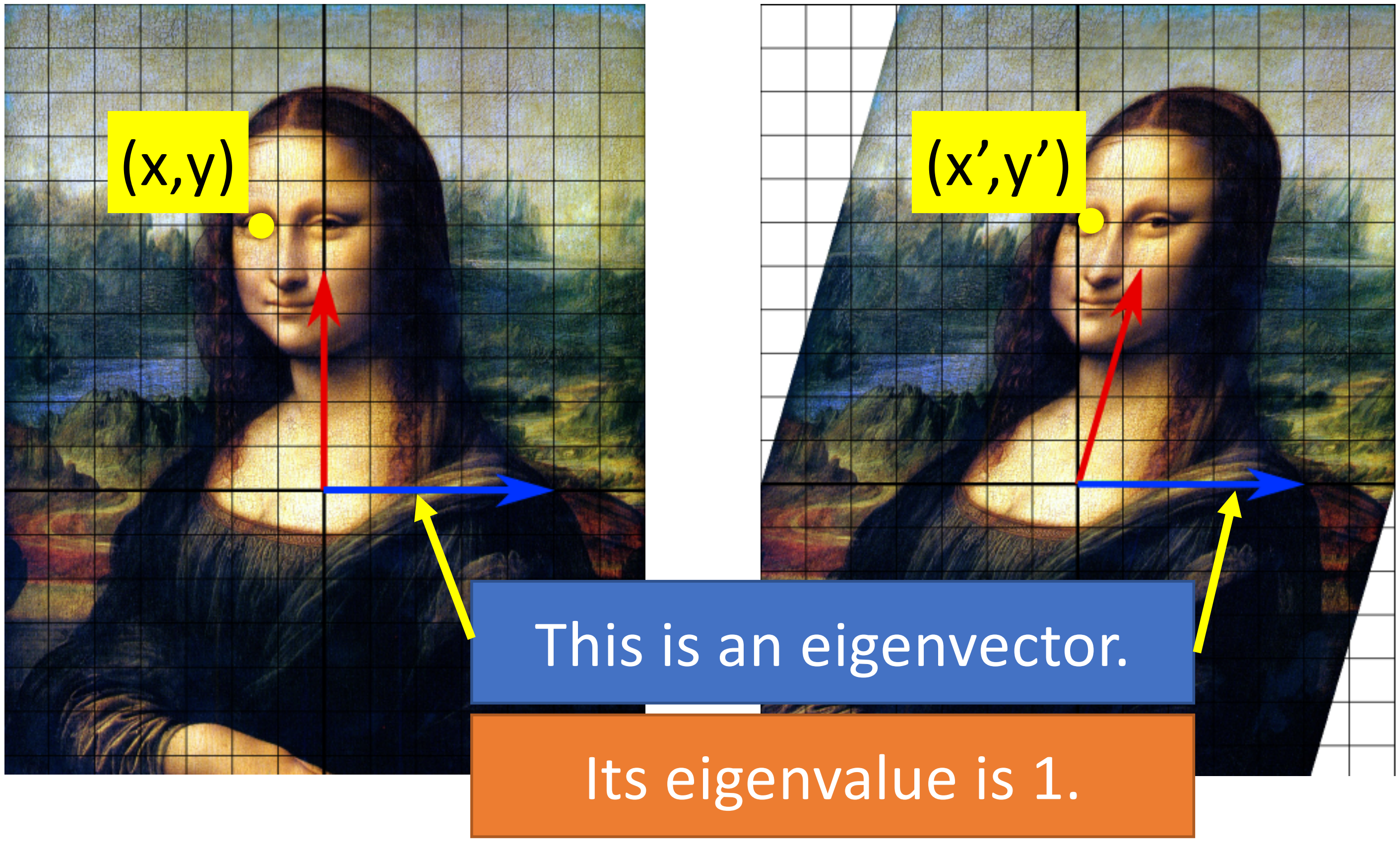
Eigen vector

Eigenvalues and Eigenvectors

- If $Av = \lambda v$ (v is a vector, λ is a scalar)
  - v is an eigenvector of A excluding zero vector
  - λ is an eigenvalue of A that corresponds to v
- T is a ***linear operator.*** If $\mathrm{T}(v) = \lambda v$ (v is a vector, λ is a scalar)
  - v is an eigenvector of T excluding zero vector
  - λ is an eigenvalue of T that corresponds to v

Eigenvalues and Eigenvectors

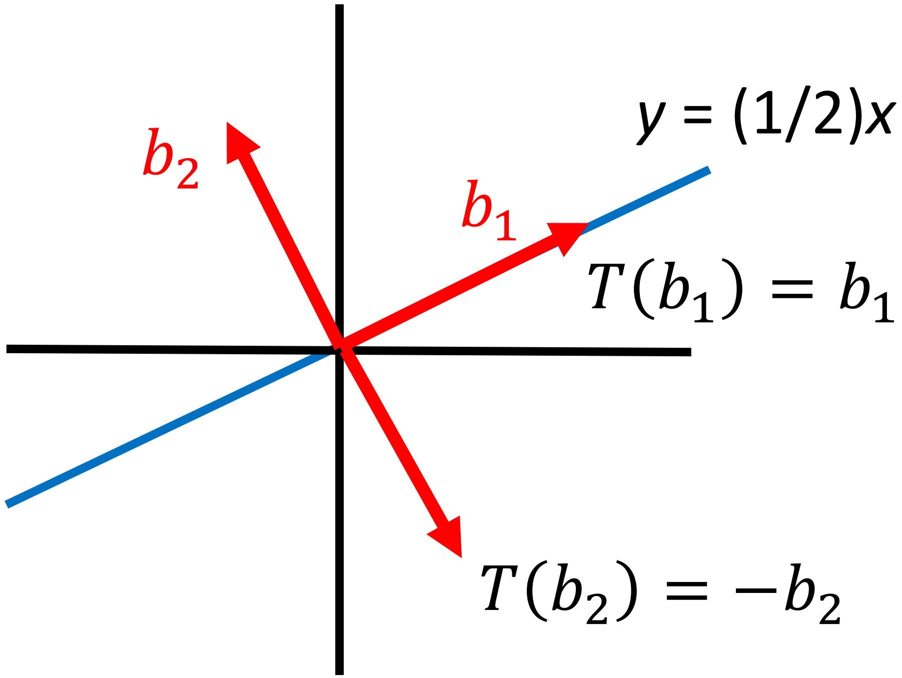
- Example: Shear Transform $\begin{bmatrix} x' \\ y' \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$



Eigenvalues and Eigenvectors

- Example: Reflection

reflection operator T about the line $y = (1/2)x$



$\mathbf{b}_1$ is an eigenvector of T

Its eigenvalue is 1.

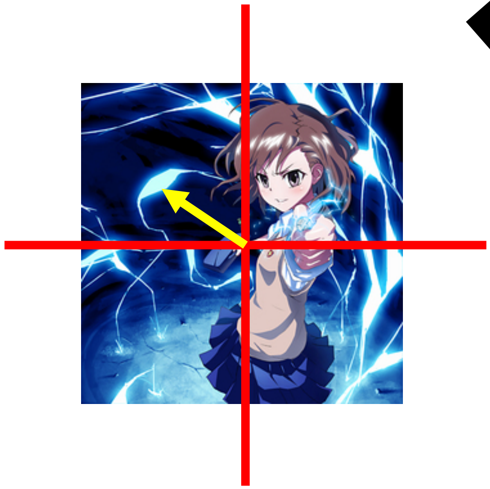
$\mathbf{b}_2$ is an eigenvector of T

Its eigenvalue is -1.

Eigenvalues and Eigenvectors

- Example:

Expansion and Compression

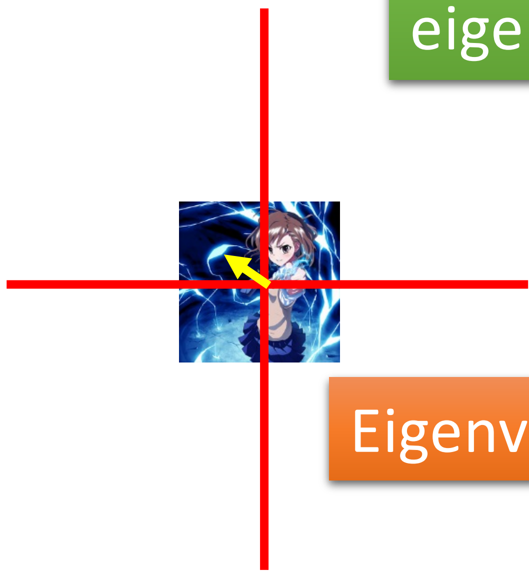


$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Eigenvalue is 2

All vectors are eigenvectors.

$$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$



Eigenvalue is 0.5

Eigenvalues and Eigenvectors

• Example: Rotation

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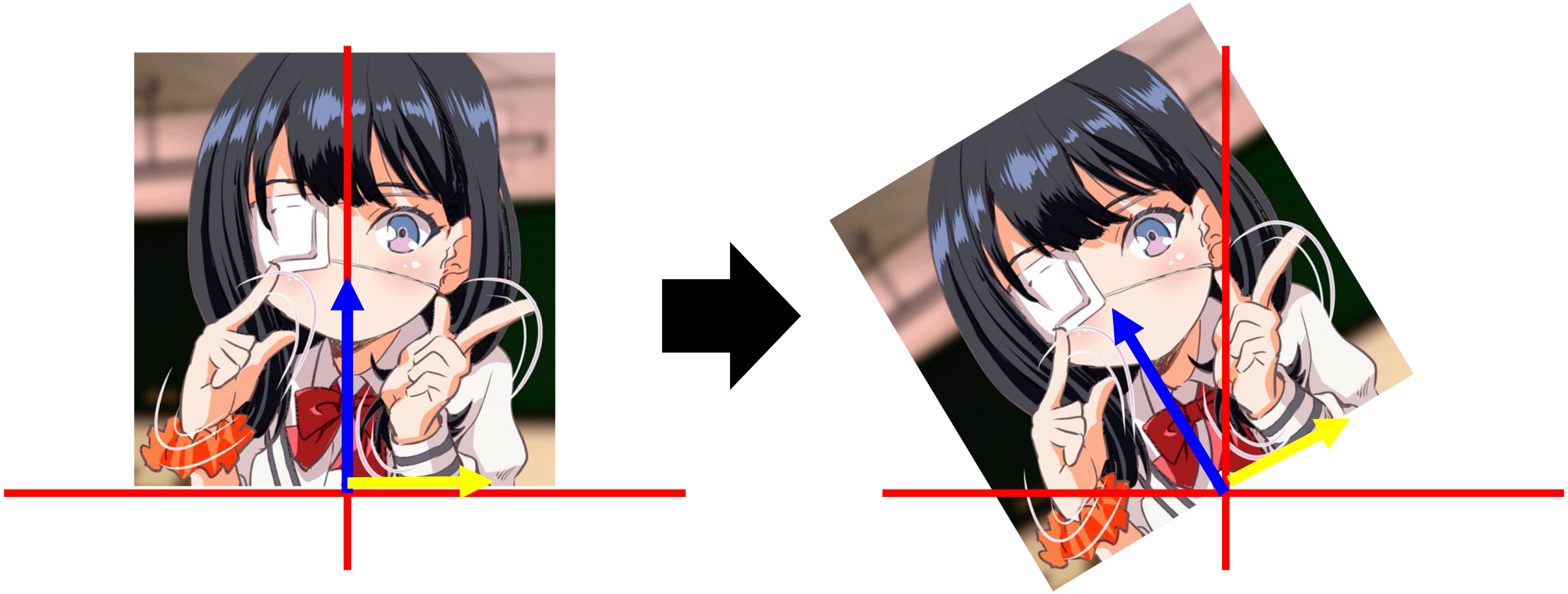


Do any n x n matrix or linear operator have eigenvalues?

How to find eigenvectors (given eigenvalues)

Eigenvalues and Eigenvectors

- An eigenvector of A corresponds to a unique eigenvalue.
- An eigenvalue of A has infinitely many eigenvectors.

Example: $$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \quad v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad u = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

Eigenvalue= -1

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Eigenvalue= -1

Do the eigenvectors correspond to the same eigenvalue form a subspace?

$$Av = \lambda v \qquad A(cv) = \lambda(cv)$$

$$Au = \lambda u \qquad A(u+v) = \lambda(u+v)$$

Eigenspace

- Assume we know λ is the eigenvalue of matrix A
- Eigenvectors corresponding to λ

$$A\mathbf{v} = \lambda\mathbf{v}$$

$$A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0}$$

$$A\mathbf{v} - \lambda I_n\mathbf{v} = \mathbf{0}$$

$$\underbrace{(A - \lambda I_n)}_{\text{matrix}}\mathbf{v} = \mathbf{0}$$

Eigenvectors corresponding to λ are nonzero solution of

$$(A - \lambda I_n)\mathbf{v} = \mathbf{0}$$

Eigenvectors corresponding to λ

$$= \underbrace{Null(A - \lambda I_n)}_{\text{eigenspace}} - \{\mathbf{0}\}$$

Eigenspace of λ:

Eigenvectors corresponding to λ + $\{\mathbf{0}\}$

Check whether a scalar is an eigenvalue

Check Eigenvalues

$Null(A - \lambda I_n)$:
eigenspace of λ

- How to know whether a scalar λ is the eigenvalue of A?

Check the dimension of eigenspace of λ

If the dimension is 0

⇒ Eigenspace only contains {0}

⇒ No eigenvector

⇒ λ is not eigenvalue

Check Eigenvalues

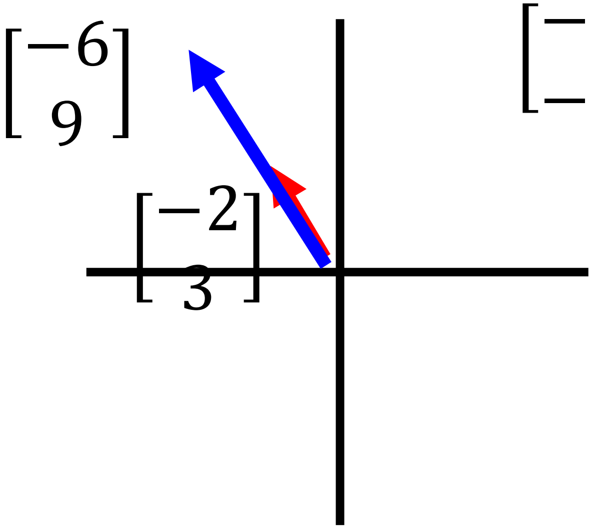
$Null(A - \lambda I_n)$:
eigenspace of λ

- Example: to check 3 and -2 are eigenvalues of the linear operator T

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -2x_2 \\ -3x_1 + x_2 \end{bmatrix} \quad A = \begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix}$$

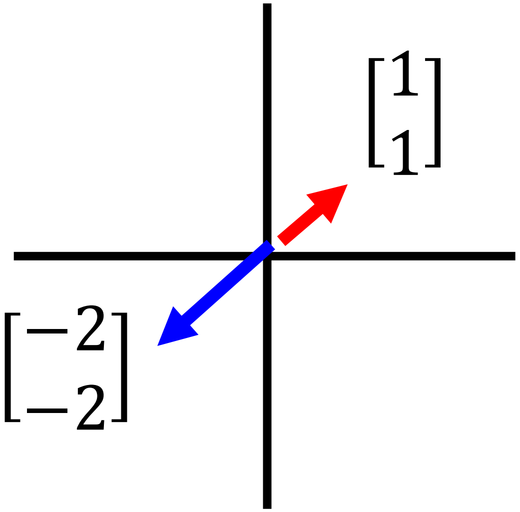
$Null(A - 3I_n)$=?

$$\begin{bmatrix} -3 & -2 \\ -3 & -2 \end{bmatrix}$$



$Null(A + 2I_n)$=?

$$\begin{bmatrix} 2 & -2 \\ -3 & 3 \end{bmatrix}$$



Check Eigenvalues

$Null(A - \lambda I_n)$:
eigenspace of λ

- Example: check that 3 is an eigenvalue of B and find a basis for the corresponding eigenspace

$$B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

find the solution set of $(B - 3I_3)\mathbf{x} = \mathbf{0}$

find the RREF of

$B - 3I_3$

$$= \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_3 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Looking for Eigenvalues

Looking for Eigenvalues

A scalar t is an eigenvalue of A

$\Leftrightarrow$ Existing $v \neq 0$ such that $Av = tv$

$\Leftrightarrow$ Existing $v \neq 0$ such that $Av - tv = 0$

$\Leftrightarrow$ Existing $v \neq 0$ such that $(A - tI_n)v = 0$

$\Leftrightarrow$ $(A - tI_n)v = 0$ has multiple solution

$\Leftrightarrow$ The columns of $(A - tI_n)$ are Dependent

$\Leftrightarrow$ $(A - tI_n)$ is not invertible

$\Leftrightarrow$ $\boxed{det(A - tI_n) = 0}$

Looking for Eigenvalues

- Example 1: Find the eigenvalues of $A = \begin{bmatrix} -4 & -3 \\ 3 & 6 \end{bmatrix}$

A scalar t is an eigenvalue of A $\Longleftrightarrow$ $det(A - tI_n) = 0$

$$A - tI_2 = \begin{bmatrix} -4-t & -3 \\ 3 & 6-t \end{bmatrix}$$

$\det(A - tI_2)$ =0

$\Rightarrow$ t = -3 or 5

The eigenvalues of *A* are -3 or 5.

Looking for Eigenvalues

- Example 1: Find the eigenvalues of $A = \begin{bmatrix} -4 & -3 \\ 3 & 6 \end{bmatrix}$

The eigenvalues of *A* are -3 or 5.

<u>Eigenspace of -3</u>

$$Ax = -3x \Rightarrow (A + 3I)x = 0$$

find the solution

<u>Eigenspace of 5</u>

$$Ax = 5x \Rightarrow (A - 5I)x = 0$$

find the solution

Looking for Eigenvalues

- Example 2: find the eigenvalues of linear operator

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 \\ 2x_1 - x_2 - x_3 \\ -x_3 \end{bmatrix} \xrightarrow[\text{matrix}]{\text{standard}} A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

A scalar t is an eigenvalue of A $\Longleftrightarrow$ $det(A - tI_n) = 0$

$$A - tI_n = \begin{bmatrix} -1-t & 0 & 0 \\ 2 & -1-t & -1 \\ 0 & 0 & -1-t \end{bmatrix}$$

$$\Rightarrow \quad det(A - tI_n) = (-1-t)^3$$

Looking for Eigenvalues

- Example 3: linear operator on R^2 that rotates a vector by 90°

$$\boxed{\text{A scalar } t \text{ is an eigenvalue of A} \Leftrightarrow det(A - tI_n) = 0}$$

standard matrix of the 90°-rotation: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\det\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - tI_2\right)$$

No eigenvalues, no eigenvectors

Characteristic Polynomial

$\boxed{\text{A scalar } t \text{ is an eigenvalue of A} \Longleftrightarrow det(A - tI_n) = 0}$

A is the standard matrix of linear operator T

$det(A - tI_n)$: Characteristic polynomial of A

linear operator T

$det(A - tI_n) = 0$: Characteristic equation of A

linear operator T

Eigenvalues are the roots of characteristic polynomial or solutions of characteristic equation.

Characteristic Polynomial

- In general, a matrix A and RREF of A have different characteristic polynomials. ⇨ Different Eigenvalues
- Similar matrices have the same characteristic polynomials ⇨ The same Eigenvalues

$$\boxed{B = P^{-1}AP}$$

$$\begin{aligned} det(B - tI) &= det(P^{-1}AP - P^{-1}(tI)P) \\ &= det(P^{-1}(A - tI)P) \\ &= det(P^{-1})det(A - tI)det(P) \\ &= \left(\frac{1}{det(P)}\right)det(A - tI)det(P) = det(A - tI) \end{aligned}$$

Characteristic Polynomial

- Question: What is the order of the characteristic polynomial of an $n \times n$ matrix A?
  - The characteristic polynomial of an $n \times n$ matrix is indeed a polynomial with degree n
  - Consider $\det(A - tI_n)$

$$det \begin{bmatrix} ?-t & \cdots & ? \\ \vdots & \ddots & \vdots \\ ? & \cdots & ?-t \end{bmatrix}$$

- Question: What is the number of eigenvalues of an $n \times n$ matrix A?
  - Fact: An n x n matrix A have less than or equal to n eigenvalues
  - Consider complex roots and multiple roots

Characteristic Polynomial v.s. Eigenspace

- Characteristic polynomial of A is

$det(A - tI_n)$ Factorization

multiplicity

$$= (t - \lambda_1)^{m_1}(t - \lambda_2)^{m_2} \ldots (t - \lambda_k)^{m_k}\underline{(\ldots \ldots)}$$

Eigenvalue:	λ_1	λ_2	λ_k
Eigenspace: (dimension)	d_1 $\leq m_1$	d_2 $\leq m_2$	d_k $\leq m_k$

Characteristic Polynomial

- The eigenvalues of an upper triangular matrix are its diagonal entries.

Characteristic Polynomial:

$$\begin{bmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{bmatrix}$$

$$det \begin{bmatrix} a-t & * & * \\ 0 & b-t & * \\ 0 & 0 & c-t \end{bmatrix}$$

$$= (a-t)(b-t)(c-t)$$

The determinant of an upper triangular matrix is the product of its diagonal entries.

Summary

- If $Av = \lambda v$ (v is a vector, λ is a scalar)
  - v is an eigenvector of A excluding zero vector
  - λ is an eigenvalue of A that corresponds to v
- Eigenvectors corresponding to λ are nonzero solution of $(A - \lambda I_n)\mathbf{v} = \mathbf{0}$

Eigenvectors corresponding to λ

$$= \underbrace{Null(A - \lambda I_n)}_{\text{eigenspace}} - \{\mathbf{0}\}$$

Eigenspace of λ:

Eigenvectors corresponding to λ + $\{\mathbf{0}\}$

- A scalar t is an eigenvalue of A

$$\Leftrightarrow \boxed{det(A - tI_n) = 0}$$