Solving System of Linear Equations

Equivalent

- Two systems of linear equations are **equivalent** if they have exactly **the same solution set**.

$$\begin{cases} 3x_1 & +x_2 & = 10 \\ x_1 & -3x_2 & = 0 \end{cases}$$

Solution set: $\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}$

$\Longleftrightarrow$ equivalent

$$\begin{cases} x_1 & & = 3 \\ & x_2 & = 1 \end{cases}$$

Solution set: $\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}$

Equivalent

- Applying the following three operations on a system of linear equations will produce an equivalent one.
- 1. Interchange

$$\begin{cases} 3x_1 & +x_2 & = 10 \\ x_1 & -3x_2 & = 0 \end{cases} \Rightarrow \begin{cases} x_1 & -3x_2 & = 0 \\ 3x_1 & +x_2 & = 10 \end{cases}$$

- 2. Scaling (non zero)

$$\begin{cases} 3x_1 & +x_2 & = 10 \\ x_1 & -3x_2 & = 0 \end{cases} \text{X(-3)} \Rightarrow \begin{cases} 3x_1 & +x_2 & = 10 \\ -3x_1 & +9x_2 & = 0 \end{cases}$$

- 3. Row Addition

$$\begin{cases} 3x_1 & +x_2 & = 10 \\ x_1 & -3x_2 & = 0 \end{cases} \text{X(-3)} \Rightarrow \begin{cases} & 10x_2 & = 10 \\ x_1 & -3x_2 & = 0 \end{cases}$$

Solving system of linear equation

- Strategy
  - We know how to transform the given system of linear equations into another equivalent one.
  - We do it again and again until the system of linear equation is very simple
  - Finally, we know the answer at a glance.

$$\begin{cases} x_1 & -3x_2 & = 0 \quad \text{X 3} \\ 3x_1 & +x_2 & = 10 \end{cases} \Rightarrow \begin{cases} x_1 & -3x_2 & = 0 \\ & 10x_2 & = 10 \quad \text{X 1/10} \end{cases}$$

$$\Rightarrow \begin{cases} x_1 & -3x_2 & = 0 \\ & x_2 & = 1 \quad \text{X 3} \end{cases} \Rightarrow \begin{cases} x_1 & & = 3 \\ & x_2 & = 1 \end{cases}$$

Augmented Matrix

- a system of linear equation

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

$\Rightarrow$ $A\mathbf{x} = \mathbf{b}$

$$\underset{m \times n}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

coefficient matrix

Augmented Matrix

- a system of linear equation

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

$$A\mathbf{x} = \mathbf{b}$$

m x (n+1)

m x n   m x 1

$$\left[\; A \mid \mathbf{b} \;\right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

Back to Equivalent

- 1. Interchange

$$\begin{cases} 3x_1 & +x_2 & = 10 \\ x_1 & -3x_2 & = 0 \end{cases} \Rightarrow \begin{cases} x_1 & -3x_2 & = 0 \\ 3x_1 & +x_2 & = 10 \end{cases}$$

- 2. Scaling (non zero)

$$\begin{cases} 3x_1 & +x_2 & = 10 \\ x_1 & -3x_2 & = 0 \;\; \text{X(-3)} \end{cases} \Rightarrow \begin{cases} 3x_1 & +x_2 & = 10 \\ -3x_1 & +9x_2 & = 0 \end{cases}$$

- 3. Row Addition

$$\begin{cases} 3x_1 & +x_2 & = 10 \\ x_1 & -3x_2 & = 0 \;\; \text{X(-3)} \end{cases} \Rightarrow \begin{cases} & 10x_2 & = 10 \\ x_1 & -3x_2 & = 0 \end{cases}$$

Back to Equivalent

elementary row operations

- 1. Interchange — Interchange any two rows of the matrix

$$\begin{bmatrix} 3 & 1 & 10 \\ 1 & -3 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 0 \\ 3 & 1 & 10 \end{bmatrix}$$

- 2. Scaling (non zero) — Multiply every entry of some row by the same nonzero scalar

$$\begin{bmatrix} 3 & 1 & 10 \\ 1 & -3 & 0 \end{bmatrix} \text{X(-3)} \Rightarrow \begin{bmatrix} 3 & 1 & 10 \\ -3 & 9 & 0 \end{bmatrix}$$

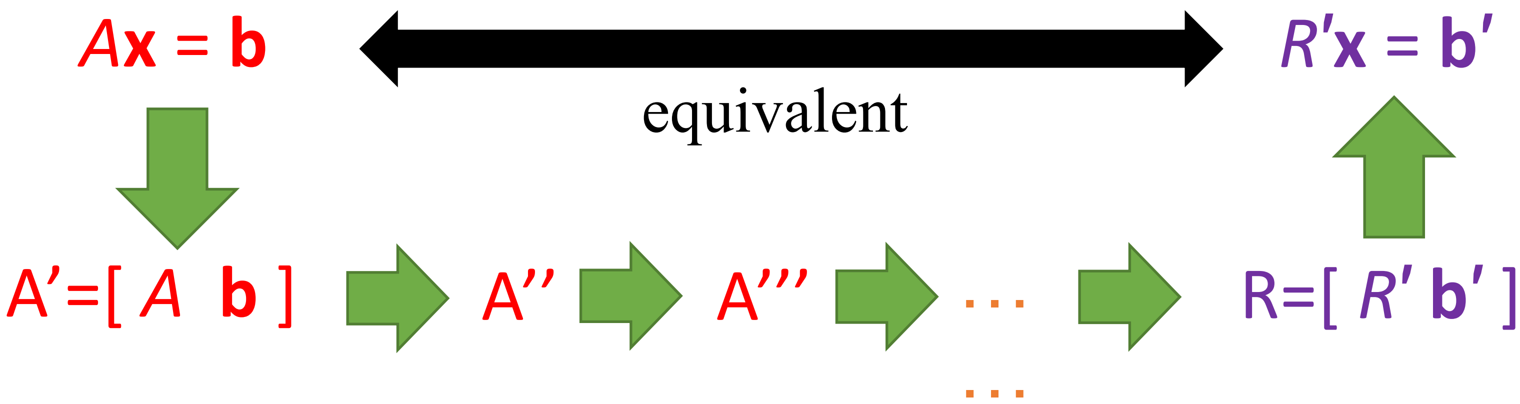
- 3. Row Addition — Add a multiple of one row of the matrix to another row

$$\begin{bmatrix} 3 & 1 & 10 \\ 1 & -3 & 0 \end{bmatrix} \text{X(-3)} \Rightarrow \begin{bmatrix} 0 & 10 & 10 \\ 1 & -3 & 0 \end{bmatrix}$$

Solving system of linear equation

A ***complex*** system of linear equations

A ***simple*** system of linear equations



elementary row operations

1. Interchange any two rows of the matrix
2. Multiply every entry of some row by the same nonzero scalar
3. Add a multiple of one row of the matrix to another row

$$\begin{cases} x_1 & -3x_2 & = 0 \\ 3x_1 & +x_2 & = 10 \end{cases}$$ X 3

$$\begin{bmatrix} 1 & -3 & 0 \\ 3 & 1 & 10 \end{bmatrix}$$ X 3

$$\begin{cases} x_1 & -3x_2 & = 0 \\ & 10x_2 & = 10 \end{cases}$$ X 1/10

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 10 & 10 \end{bmatrix}$$ X 1/10

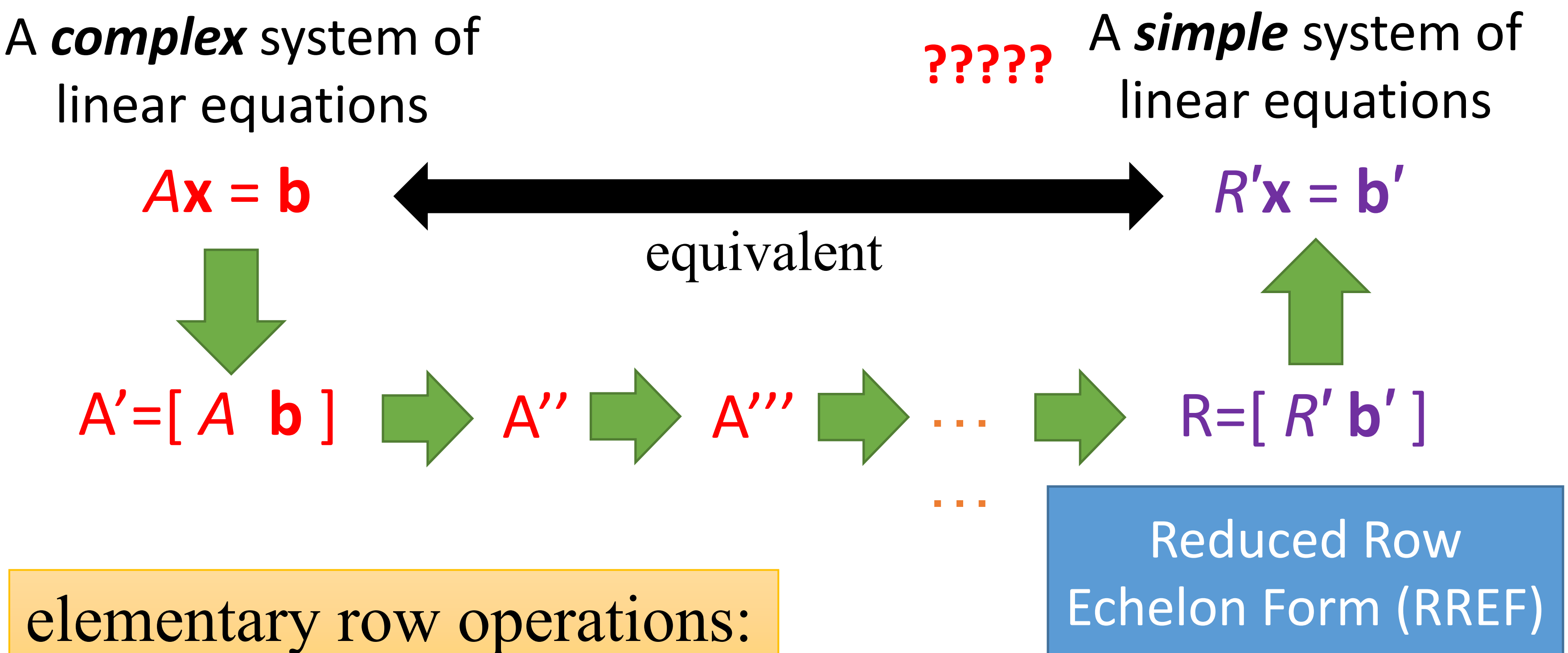
$$\begin{cases} x_1 & -3x_2 & = 0 \\ & x_2 & = 1 \end{cases}$$ X 3

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$ X 3

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{cases} x_1 & & = 3 \\ & x_2 & = 1 \end{cases}$$

Solving system of linear equation



elementary row operations:

1. Interchange any two rows of the matrix
2. Multiply every entry of some row by the same nonzero scalar
3. Add a multiple of one row of the matrix to another row