

(依賴的、不獨立的)

Dependent and Independent

(獨立的、自主的)

Definition

- A set of n vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ is linear *dependent*

Find one



Obtain many

- If there exist scalars x_1, x_2, \dots, x_n , **not all zero**, such that

$$x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \dots + x_n \mathbf{a}_n = \mathbf{0}$$

- A set of n vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ is linear *independent*

$$x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \dots + x_n \mathbf{a}_n = \mathbf{0}$$

$$\text{Only if } x_1 = x_2 = \dots = x_n = 0$$

unique

- A set of n vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ is linear **dependent**
 - If there exist scalars x_1, x_2, \dots, x_n , **not all zero**, such that
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- A set of n vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ is linear **independent**

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n = \mathbf{0}$$

Only if $x_1 = x_2 = \dots = x_n = 0$

$$\left\{ \begin{bmatrix} -4 \\ 12 \\ 6 \end{bmatrix}, \begin{bmatrix} -10 \\ 30 \\ 15 \end{bmatrix} \right\} \text{ Dependent or Independent?}$$

dependent

$$5 \begin{bmatrix} -4 \\ 12 \\ 6 \end{bmatrix} + (-2) \begin{bmatrix} -10 \\ 30 \\ 15 \end{bmatrix} = \mathbf{0}$$

- A set of n vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ is linear **dependent**
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- A set of n vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ is linear **independent**

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n = \mathbf{0}$$

Only if $x_1 = x_2 = \dots = x_n = 0$

$\left\{ \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix} \right\}$ Dependent or Independent?

dependent

$$\underset{\mathbf{1}}{x_1} \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} + \underset{\mathbf{1}}{x_2} \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix} + \underset{-\mathbf{1}}{x_3} \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix} = \mathbf{0}$$

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- A set of n vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ is linear **independent**

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n = \mathbf{0}$$

$$\text{Only if } x_1 = x_2 = \dots = x_n = 0$$

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} \right\} \text{ Dependent or Independent?}$$

dependent

$$x_1 \begin{bmatrix} 3 \\ -1 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} = \mathbf{0}$$

0 Any 0

Any set contains zero vector would be linear dependent

Linear Dependent

(for $n \geq 2$)

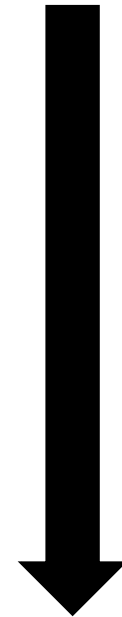
Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$, there exists scalars x_1, x_2, \dots, x_n , that are **not all zero**, such that $x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \dots + x_n \mathbf{a}_n = \mathbf{0}$.

$$x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 \dots + \boxed{x_i \mathbf{a}_i} + \dots + x_n \mathbf{a}_n = \mathbf{0}$$

永遠可以找到某一項有非 0 係數 $x_i \neq 0$

$$x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 \dots + x_n \mathbf{a}_n = -x_i \mathbf{a}_i$$

$$-\left(\frac{x_n}{x_1}\right) \mathbf{a}_1 - \left(\frac{x_n}{x_2}\right) \mathbf{a}_2 \dots - \left(\frac{x_n}{x_i}\right) \mathbf{a}_n = \mathbf{a}_i$$



Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$, if there exists any \mathbf{a}_i that is a linear combination of other vectors

Linear Dependent

(for $n \geq 2$)

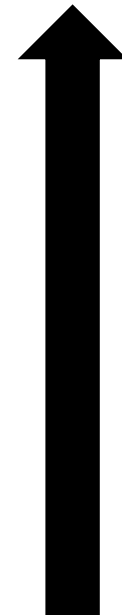
Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$, there exists scalars x_1, x_2, \dots, x_n , that are **not all zero**, such that $x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \dots + x_n \mathbf{a}_n = \mathbf{0}$.

$$\mathbf{a}_i = c_1 \mathbf{a}_1 + c_2 \mathbf{a}_2 \dots + c_n \mathbf{a}_n$$

$$-c_1 \mathbf{a}_1 - c_2 \mathbf{a}_2 \dots + \mathbf{a}_i \dots - c_n \mathbf{a}_n = \mathbf{0}$$

至少這項有非 0 係數

$$x_i \neq 0$$



Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$, if there exists any \mathbf{a}_i that is a linear combination of other vectors

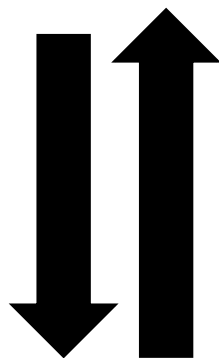
Linear Dependent

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Vector Set 中有人耍廢

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$, there exists scalars x_1, x_2, \dots, x_n , that are **not all zero**, such that $x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \dots + x_n \mathbf{a}_n = \mathbf{0}$.

(for $n \geq 2$)



\mathbf{a}_i 在耍廢

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$, if there exists any \mathbf{a}_i that is a linear combination of other vectors

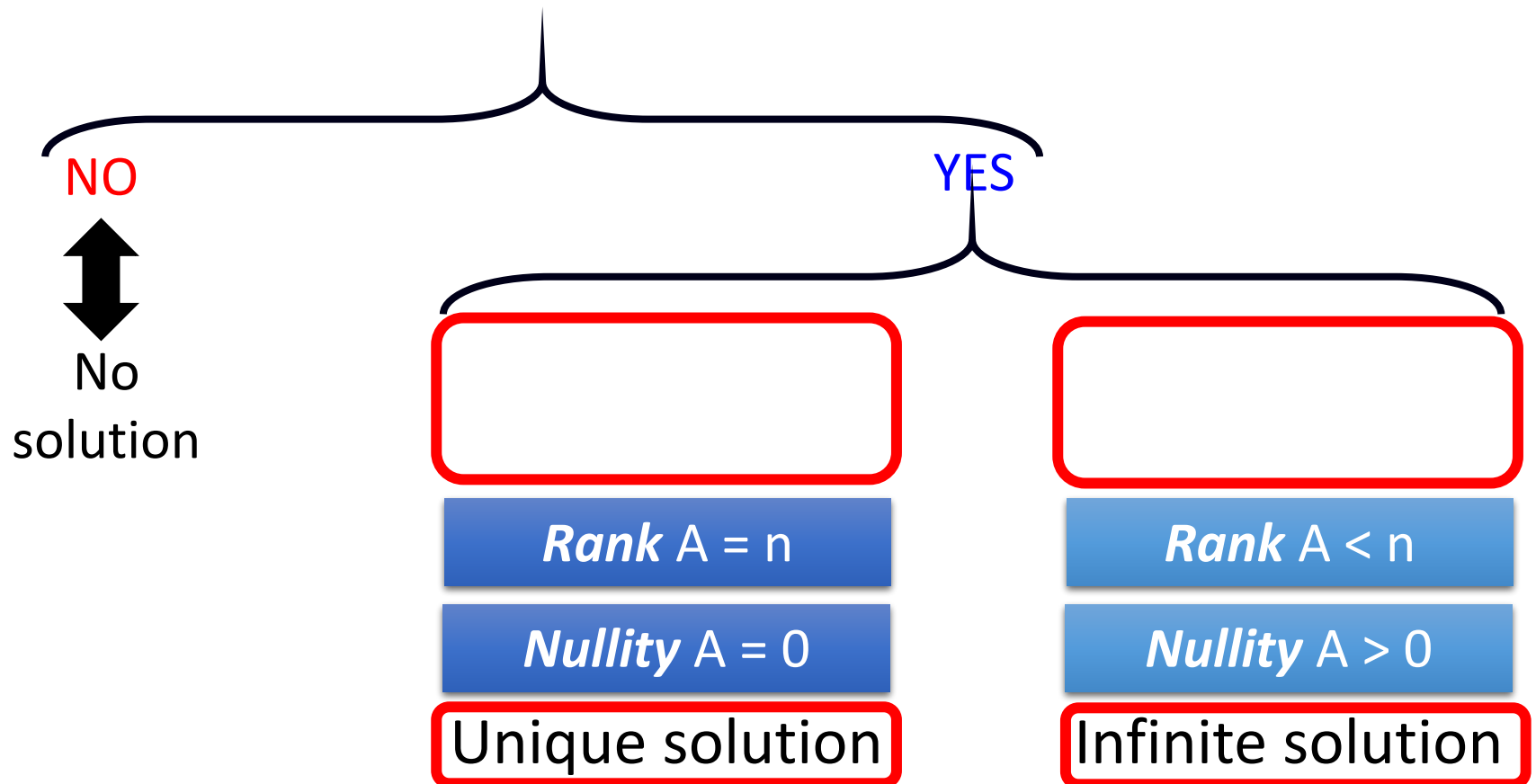
Linear Independent

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Vector Set 中沒有人耍廢

Summary

$$A\mathbf{x} = \mathbf{b}$$



Intuition

Dependent:
Once we have solution, we have
infinite.

- Intuitive link between dependence and the number of solutions

$$\begin{bmatrix} 6 & 1 & 7 \\ 3 & 8 & 11 \\ 3 & 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 22 \\ 12 \end{bmatrix}$$

dependent

$$1 \cdot \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix}$$

$$1 \cdot \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix} = \begin{bmatrix} 14 \\ 22 \\ 12 \end{bmatrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$2 \cdot \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 22 \\ 12 \end{bmatrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

Infinite
Solution

Proof

- Columns of A are **dependent** \rightarrow If $Ax=b$ have solution, it will have **Infinite** solutions

- If $Ax=b$ have **Infinite** solutions \rightarrow Columns of A are **dependent**

Homogeneous linear equations

Proof

$$Ax = \mathbf{0} \text{ (always having } x = \mathbf{0} \text{ as solution)}$$

- Columns of A are **dependent** \rightarrow If $Ax=0$ have solution, it will have **Infinite** solutions

$$A = [a_1 \quad a_2 \quad \cdots \quad a_n] \text{ dependent set}$$

$$Ax = \mathbf{0} \Rightarrow x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = \mathbf{0}$$

there exist scalars x_1, x_2, \cdots, x_n , **not all zero**

- If $Ax=0$ have **Infinite** solutions \rightarrow Columns of A are **dependent**

$$Ax = \mathbf{0} \text{ have non-zero solutions}$$

$$x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = \mathbf{0}$$

x_1, x_2, \cdots, x_n ,
not all zero

Proof

$Ax=0$ have **infinite** solutions

- Columns of A are **dependent** \rightarrow If $Ax=b$ have solution, it will have **Infinite** solutions

We can find non-zero solution u such that $Au = 0$

There exists v such that $Av = b$

$$A(u + v) = b$$

$u + v$ is another solution different to v

- If $Ax=b$ have **Infinite** solutions \rightarrow Columns of A are **dependent**

$$u \neq v \quad \begin{array}{l} Au = b \\ Av = b \end{array}$$

$Ax=0$ have **infinite** solutions

$$A(u - v) = 0$$

Non-zero