(依賴的、不獨立的)

Dependent and Independent

(獨立的、自主的)

Definition

- A set of n vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\}$ is linear ***dependent***

  Find one → Obtain many

  - If there exist scalars $x_1, x_2, \cdots, x_n$, **not all zero**, such that

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \mathbf{0}$$

- A set of n vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\}$ is linear ***independent***

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \mathbf{0}$$

Only if $x_1 = x_2 = \cdots = x_n = 0$ unique

- A set of n vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\}$ is linear ***dependent***
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- A set of n vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\}$ is linear ***independent***

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \boldsymbol{0}$$

Only if $x_1 = x_2 = \cdots = x_n = 0$

$$\left\{\begin{bmatrix}-4\\12\\6\end{bmatrix}, \begin{bmatrix}-10\\30\\15\end{bmatrix}\right\}$$

Dependent or Independent?

dependent

$$x_1\begin{bmatrix}-4\\12\\6\end{bmatrix} + x_2\begin{bmatrix}-10\\30\\15\end{bmatrix} = \boldsymbol{0}$$

5 -2

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- A set of n vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\}$ is linear ***independent***

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \mathbf{0}$$

Only if $x_1 = x_2 = \cdots = x_n = 0$

$$\left\{\begin{bmatrix}6\\3\\3\end{bmatrix}, \begin{bmatrix}1\\8\\3\end{bmatrix}, \begin{bmatrix}7\\11\\6\end{bmatrix}\right\}$$

Dependent or Independent?

dependent

$$x_1\begin{bmatrix}6\\3\\3\end{bmatrix} + x_2\begin{bmatrix}1\\8\\3\end{bmatrix} + x_3\begin{bmatrix}7\\11\\6\end{bmatrix} = \mathbf{0}$$

1 1 -1

- A set of n vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\}$ is linear ***dependent***
  - If there exist scalars $x_1, x_2, \cdots, x_n$, **not all zero**, such that $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \mathbf{0}$
- A set of n vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\}$ is linear ***independent***

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \mathbf{0}$$

Only if $x_1 = x_2 = \cdots = x_n = 0$

$$\left\{\begin{bmatrix} 3 \\ -1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}\right\}$$

Dependent or Independent?

dependent

$$x_1 \begin{bmatrix} 3 \\ -1 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} = \mathbf{0}$$

0 Any **0**

Any set contains zero vector would be linear dependent

Linear Dependent

(for $n \geq 2$)

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, there exists scalars $x_1, x_2, \ldots, x_n$, that are **not all zero**, such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 +$ □ $+ x_n\mathbf{a}_n = \mathbf{0}$.

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 \ldots + \boxed{x_i\boldsymbol{a}_i} + \cdots + x_n\boldsymbol{a}_n = \mathbf{0}$$

永遠可以找到某一項有非 0係數 $x_i \neq 0$

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 \ldots + x_n\boldsymbol{a}_n = -x_i\boldsymbol{a}_i$$

$$-\left(\frac{x_n}{x_1}\right)\boldsymbol{a}_1 - \left(\frac{x_n}{x_2}\right)\boldsymbol{a}_2 \ldots - \left(\frac{x_n}{x_i}\right)\boldsymbol{a}_n = \boldsymbol{a}_i$$

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, if there exists any $\mathbf{a}_i$ that is a linear combination of other vectors

Linear Dependent

(for $n \geq 2$)

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, there exists scalars $x_1, x_2, \ldots, x_n$, that are **not all zero**, such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$.

$$\boldsymbol{a_i} = c_1\boldsymbol{a_1} + c_2\boldsymbol{a_2} \ldots + c_n\boldsymbol{a_n}$$

$$-c_1\boldsymbol{a_1} - c_2\boldsymbol{a_2} \ldots + \boxed{\boldsymbol{a_i}} \ldots - c_n\boldsymbol{a_n} = \boldsymbol{0}$$

至少這項有非0係數

$x_i \neq 0$

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, if there exists any $\mathbf{a}_i$ that is a linear combination of other vectors

Linear Dependent = Vector Set 中有人要廢

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, there exists scalars $x_1, x_2, \ldots, x_n$, that are **not all zero**, such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \square + x_n\mathbf{a}_n = \mathbf{0}$.

(for $n \geq 2$)

$\mathbf{a}_i$ 在要廢

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, if there exists any $\mathbf{a}_i$ that is a linear combination of other vectors

Linear Independent = Vector Set 中沒有人要廢

Summary

$$A\mathbf{x} = \mathbf{b}$$

NO

No solution

YES

Rank A = n

Nullity A = 0

Unique solution

Rank A < n

Nullity A > 0

Infinite solution

Intuition

Dependent:
Once we have solution, we have infinite.

- Intuitive link between dependence and the number of solutions

dependent

$$1 \cdot \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 1 & 7 \\ 3 & 8 & 11 \\ 3 & 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 22 \\ 12 \end{bmatrix}$$

$$1 \cdot \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix} = \begin{bmatrix} 14 \\ 22 \\ 12 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$2 \cdot \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 22 \\ 12 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

Infinite Solution

Proof

- Columns of A are **dependent** ⟶ If Ax=b have solution, it will have **Infinite** solutions

- If Ax=b have **Infinite** solutions ⟶ Columns of A are **dependent**

Homogeneous linear equations

Proof

$Ax = \mathbf{0}$ (always having $x = \mathbf{0}$ as solution)

- Columns of A are **dependent** ⟶ If Ax=**0** have solution, it will have **Infinite** solutions

$A = [a_1 \quad a_2 \quad \cdots \quad a_n]$ dependent set

$$Ax = \mathbf{0} \Rightarrow x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \mathbf{0}$$

there exist scalars $x_1, x_2, \cdots, x_n$, **not all zero**

- If Ax=**0** have **Infinite** solutions ⟶ Columns of A are **dependent**

$Ax = \mathbf{0}$ have **non-zero** solutions

$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \mathbf{0}$

$x_1, x_2, \cdots, x_n$, **not all zero**

Proof

Ax=**0** have **infinite** solutions

- Columns of A are **dependent** ⟶ If Ax=b have solution, it will have **Infinite** solutions

We can find non-zero solution **u** such that $A\boldsymbol{u} = \mathbf{0}$

There exists **v** such that $A\boldsymbol{v} = \mathbf{b}$

$A(\boldsymbol{u} + \boldsymbol{v}) = \mathrm{b}$

$\boldsymbol{u} + \boldsymbol{v}$ is another solution different to **v**

- If Ax=b have **Infinite** solutions ⟶ Columns of A are **dependent**

$u \neq v$

$Au = \mathrm{b}$

$Av = \mathrm{b}$

Ax=**0** have **infinite** solutions

$A(u - v) = \mathbf{0}$

Non-zero