Invertible (Proof)
Summary

• Let A be an n x n matrix. A is invertible if and only if
  • The columns of A span \( \mathbb{R}^n \)
  • For every \( b \) in \( \mathbb{R}^n \), the system \( Ax=b \) is consistent
  • The rank of A is n
  • The columns of A are linear independent
  • The only solution to \( Ax=0 \) is the zero vector
  • The nullity of A is zero
  • The reduced row echelon form of A is \( I_n \)
  • A is a product of elementary matrices
  • There exists an n x n matrix B such that \( BA = I_n \)
  • There exists an n x n matrix C such that \( AC = I_n \)
Invertible

• Let $A$ be an $n \times n$ matrix.
  • Onto $\rightarrow$ One-to-one $\rightarrow$ invertible
    • The columns of $A$ span $\mathbb{R}^n$
    • For every $b$ in $\mathbb{R}^n$, the system $Ax=b$ is consistent
    • The rank of $A$ is the number of rows
  • One-to-one $\rightarrow$ Onto $\rightarrow$ invertible
    • The columns of $A$ are linear independent
    • The rank of $A$ is the number of columns
    • The nullity of $A$ is zero
    • The only solution to $Ax=0$ is the zero vector
    • The reduced row echelon form of $A$ is $I_n$

$\text{Rank } A = n$
Is $A$ Invertible?

• Let $A$ be an $n \times n$ matrix. $A$ is invertible if and only if
  • The reduced row echelon form of $A$ is $I_n$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 4 & 8 \end{bmatrix} \quad \xrightarrow{\text{RREF}} \quad I_n \quad \text{Invertible}$$

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \quad \xrightarrow{\text{RREF}} \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{Not Invertible}$$
## Summary

Let $A$ be an $n \times n$ matrix. $A$ is invertible if and only if:

- The columns of $A$ span $\mathbb{R}^n$
- For every $b$ in $\mathbb{R}^n$, the system $Ax = b$ is consistent
- The rank of $A$ is $n$
- The columns of $A$ are linear independent
- The only solution to $Ax = 0$ is the zero vector
- The nullity of $A$ is zero
- The reduced row echelon form of $A$ is $I_n$
- $A$ is a product of elementary matrices
- There exists an $n \times n$ matrix $B$ such that $BA = I_n$
- There exists an $n \times n$ matrix $C$ such that $AC = I_n
A is invertible.

There exists an $n \times n$ matrix $B$ such that $BA = I_n$.

The only solution to $Ax = 0$ is the zero vector.

If $Av = 0$, then ...

$BA = I_n$

$BAv = 0$

$I_nv = v$

$v = 0$
A is invertible.

There exists an $n \times n$ matrix $C$ such that $AC = I_n$.

For every $b$ in $\mathbb{R}^n$, $Ax = b$ is consistent.

For any vector $b$,

$$AC = I_n$$

$$ACb$$

$$I_nb = b$$

$Cb$ is always a solution for $b$. 

For any vector $b$, 

$AC = I_n$
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