Matrix Multiplication

Four aspects for multiplication

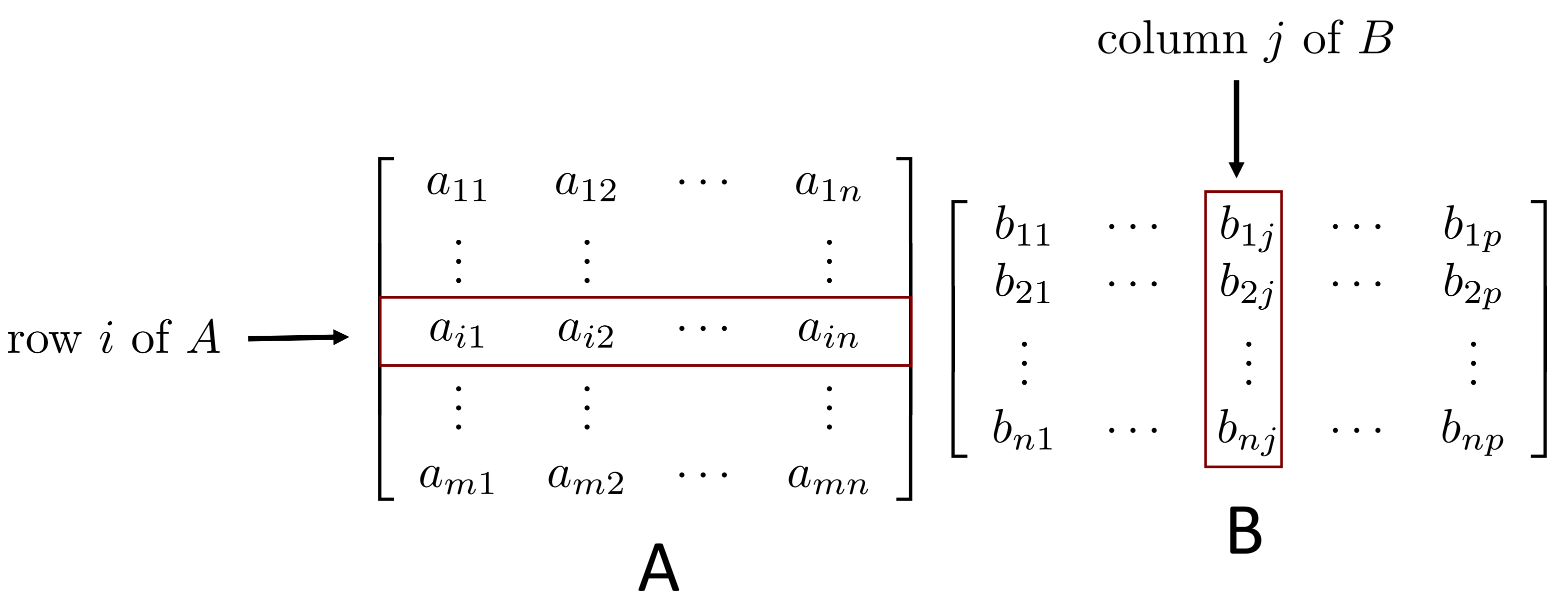
1. Inner Product

(What you have learned
in high school)

Inner Product

- Given two matrices A and B, the (*i*, *j*)-entry of AB is the inner product of row i of A and column j of B

$$C = AB \qquad c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$



Inner Product

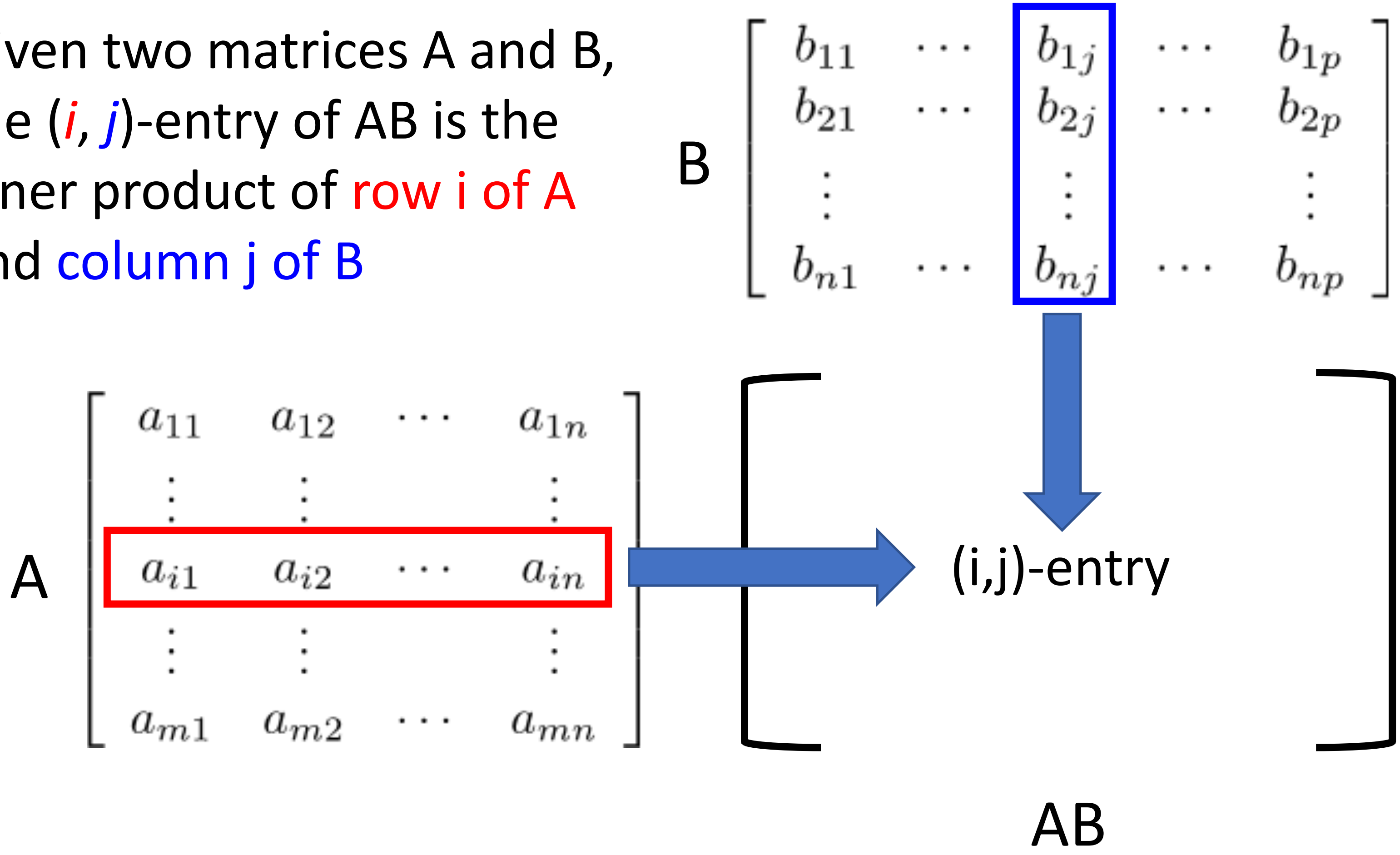
- Given two matrices A and B, the (*i*, *j*)-entry of AB is the inner product of row i of A and column j of B

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix}$$

$$C = AB = \begin{bmatrix} (-1)\times 1 + 3\times 2 & 1\times 1 + 2\times 2 \\ (-1)\times 3 + 3\times 4 & 1\times 3 + 2\times 4 \\ (-1)\times 5 + 3\times 6 & 1\times 5 + 2\times 6 \end{bmatrix}$$

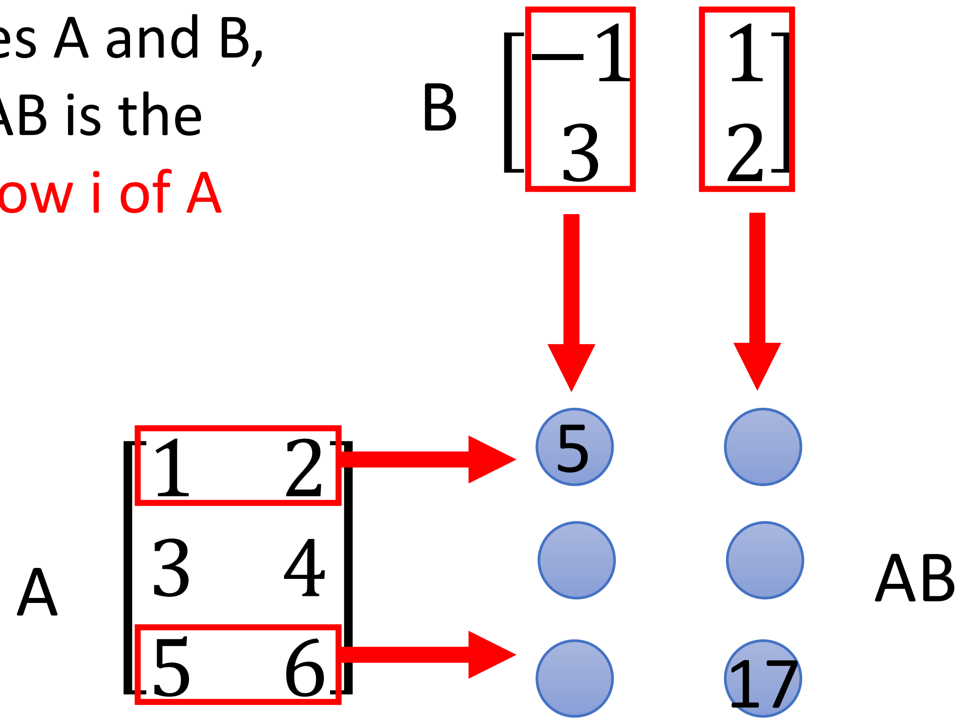
Inner Product

Given two matrices A and B, the (*i*, *j*)-entry of AB is the inner product of row i of A and column j of B



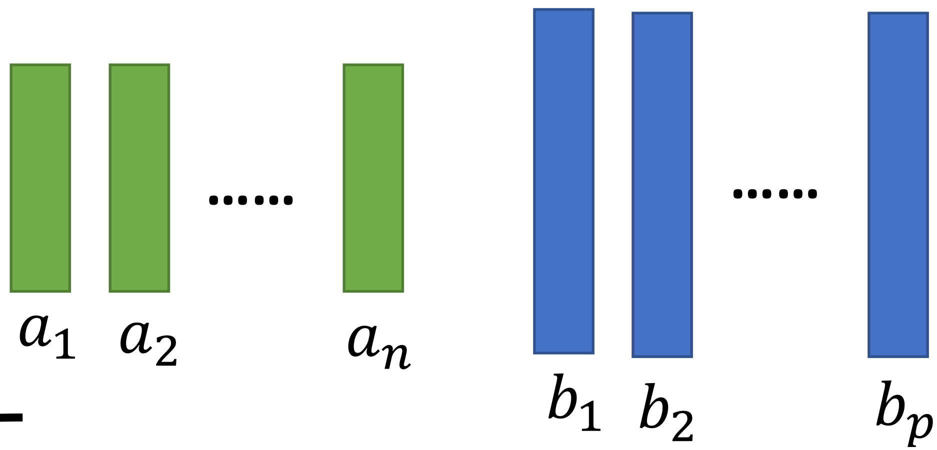
Inner Product

Given two matrices A and B, the (i, j)-entry of AB is the inner product of row i of A and column j of B



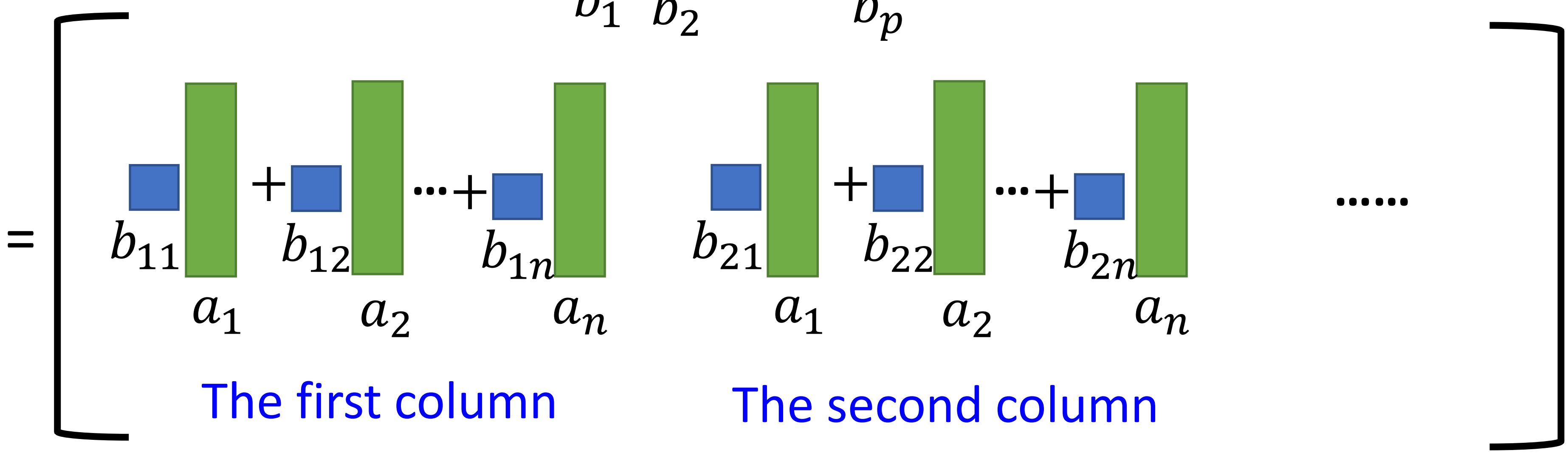
2. Combination of Columns

Combination of Columns



$$AB = A[b_1 \quad b_2 \quad \cdots \quad b_p]$$

$$= [Ab_1 \quad Ab_2 \quad \cdots \quad Ab_p]$$



Combination of Columns

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix}$$

$$AB = A\begin{bmatrix} b_1 & b_2 & \cdots & b_p \end{bmatrix}$$
$$= \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_p \end{bmatrix}$$

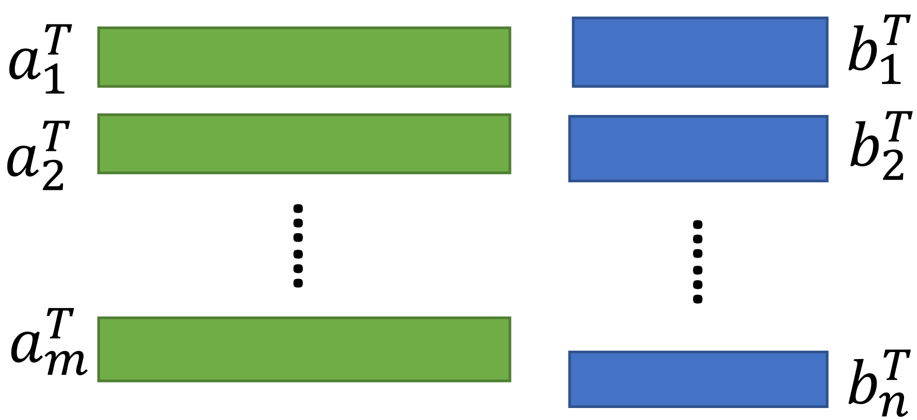
$$= \left[\; -1\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \qquad 1\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \; \right]$$

The first column

The second column

3. Combination of Rows

Combination of Rows



$$= \begin{bmatrix} a_{11}b_1^T + a_{12}b_2^T \cdots + a_{1n}b_n^T \\ a_{21}b_1^T + a_{22}b_2^T \cdots + a_{2n}b_n^T \\ \vdots \\ a_{m1}b_1^T + a_{m2}b_2^T \cdots + a_{mn}b_n^T \end{bmatrix}$$

Combination of Rows

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1[-1 \quad 1] + 2[3 \quad 2] \\ 3[-1 \quad 1] + 4[3 \quad 2] \\ 5[-1 \quad 1] + 6[3 \quad 2] \end{bmatrix}$$

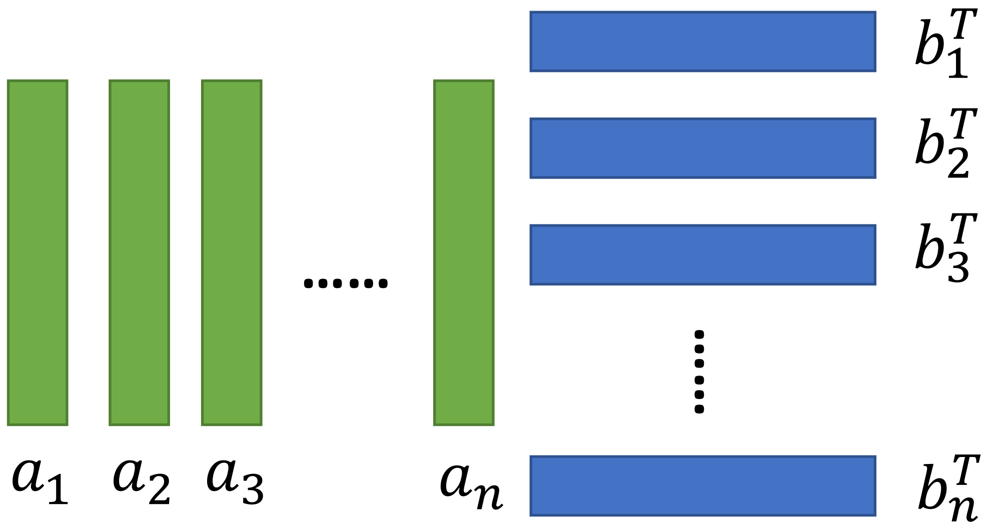
The first row

The second row

The third row

4. Summation of Matrices

Summation of Matrices



$$= a_1 b_1^T + a_2 b_2^T + \cdots + a_n b_n^T$$

matrices

Summation of Matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix} = \left[\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \begin{bmatrix} 3 & 2 \end{bmatrix} \right]$$

"1 x 2" "2 x 1"

"1 x 1"

$$= \left[\begin{bmatrix} -1 & 1 \\ -3 & 3 \\ -5 & 5 \end{bmatrix} + \begin{bmatrix} 6 & 4 \\ 12 & 8 \\ 18 & 12 \end{bmatrix} \right]$$

Rank = ? Rank = ?

Block Multiplication

Augmentation and Partition

- Augment: the augment of A and B is [A B]
- Partition:

$$A = \left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 \end{array}\right]$$

$$A = \left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 4 & 3 & 2 & 1 \end{array}\right]$$

$$A = \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 4 & 3 & 2 & 1 \end{array}\right]$$

$$A = \left[\begin{array}{c|cc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 \end{array}\right]$$

Block Multiplication

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 5 & -1 & 6 \\ 1 & 0 & 3 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \quad \end{bmatrix}$$

Multiply as the small matrices are scalar

Don’t switch the order

Block Multiplication

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 5 & -1 & 6 \\ 1 & 0 & 3 & -1 \end{bmatrix}$$

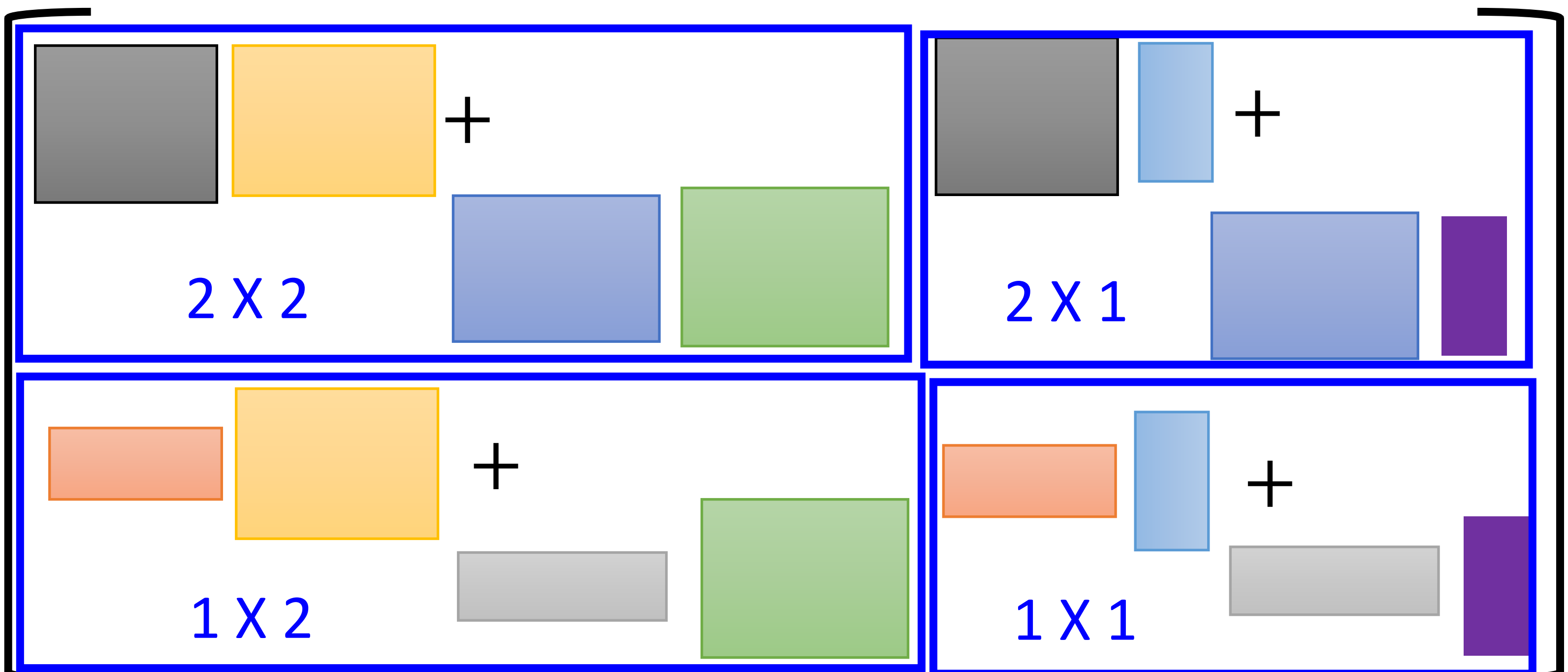
"2 x 2"

$$B = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$$

"2 x 2"

$AB =$

"2 x 2"



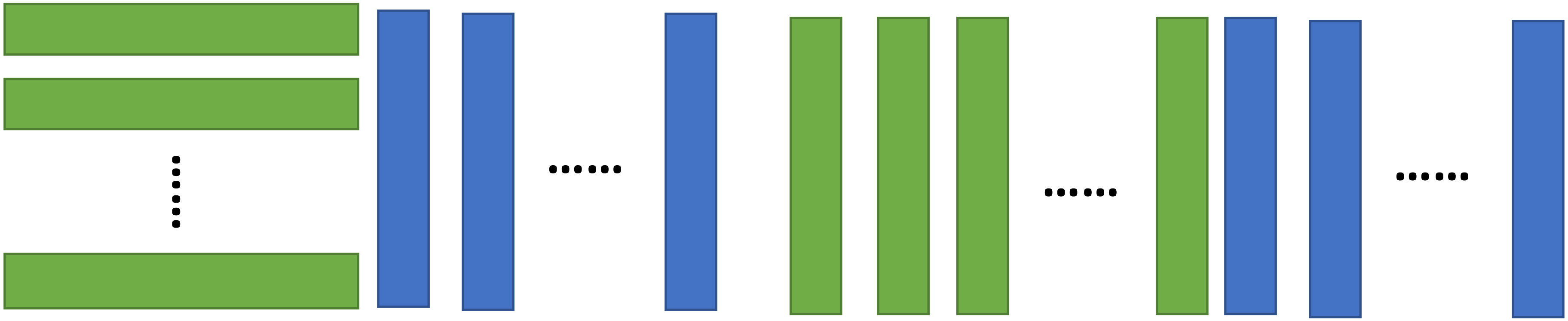
Block Multiplication

$$A = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 6 & 8 & 5 & 0 \\ -7 & 9 & 0 & 5 \end{array}\right]$$

$$A = \begin{bmatrix} I_2 & O \\ B & 5I_2 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & 8 \\ -7 & 9 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} I_2 & O \\ B & 5I_2 \end{bmatrix} \begin{bmatrix} I_2 & O \\ B & 5I_2 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$A^3 = AA^2 = \begin{bmatrix} I_2 & O \\ B & 5I_2 \end{bmatrix} \begin{bmatrix} I_2 & O \\ 6B & 25I_2 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$



Matrix Multiplication

Four aspects for multiplication