Orthogonal Basis

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Outline

- Orthogonal/Orthonormal Basis
- Orthogonal Decomposition Theory
- How to find Orthogonal Basis

Reference: Textbook Chapter 7.2, 7.3

Orthogonal Set

- A set of vectors is called an orthogonal set if every pair of distinct vectors in the set is orthogonal.

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} \right\}$$ An orthogonal set?

By definition, a set with only one vector is an orthogonal set.

Is orthogonal set independent?

NO

(A vector set with all zero vectors)

Independent?

- Any orthogonal set of nonzero vectors is linearly independent.

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be an orthogonal set $\mathbf{v}_i \neq \mathbf{0}$ for $i = 1, 2, \ldots, k$.

Assume $c_1, c_2, \ldots, c_k$ make $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$

$$(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_i\mathbf{v}_i + \cdots + c_k\mathbf{v}_k) \cdot \mathbf{v}_i$$

$$= \quad c_1\mathbf{v}_1 \cdot \mathbf{v}_i + c_2\mathbf{v}_2 \cdot \mathbf{v}_i + \cdots + c_i\mathbf{v}_i \cdot \mathbf{v}_i + \cdots + c_k\mathbf{v}_k \cdot \mathbf{v}_i$$

$$= \quad c_i(\mathbf{v}_i \cdot \mathbf{v}_i) \quad = \quad c_i||\mathbf{v}_i||^2$$

$||\mathbf{v}_i||^2 \neq 0$ ⇒ $c_i = 0$

⇒ $c_1 = c_2 = \cdots = c_k = 0$

Orthonormal Set

[redacted]

- A set of vectors is called an orthonormal set if it is an orthogonal set, and the norm of all the vectors is 1

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} \right\}$$

$$\frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \frac{1}{\sqrt{42}}\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$$

Is orthonormal set independent?

Yes

A vector that has norm equal to 1 is called a unit vector.

Orthogonal Basis

- A basis that is an orthogonal (orthonormal) set is called an orthogonal (orthonormal) basis

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Orthogonal basis of R^3

Orthonormal basis of R^3

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Orthogonal Basis

$$[u]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

- Let $S = \{v_1, v_2, \cdots, v_k\}$ be an orthogonal basis for a subspace W, and let u be a vector in W.

$$u = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

$$\frac{u \cdot v_1}{\|v_1\|^2} \quad \frac{u \cdot v_2}{\|v_2\|^2} \quad \frac{u \cdot v_k}{\|v_k\|^2}$$

Proof

How about orthonormal basis?

To find c_i

$$u \cdot v_i = (c_1 v_1 + c_2 v_2 + \cdots + c_i v_i + \cdots + c_k v_k) \cdot v_i$$

$$= c_1 v_1 \cdot v_i + c_2 v_2 \cdot v_i + \cdots + c_i v_i \cdot v_i + \cdots + c_k v_k \cdot v_i$$

$$= c_i (v_i \cdot v_i) = c_i \|v_i\|^2 \quad \Rightarrow \quad c_i = \frac{u \cdot v_i}{\|v_i\|^2}$$

Example

$$[u]_S = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

- Example: $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for R^3

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$$

$$\text{Let } \mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \text{ and } \mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3.$$

c_1 c_2 c_3

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Orthogonal Basis

Let $\{u_1, u_2, \cdots, u_k\}$ be a basis of a subspace W. How to transform $\{u_1, u_2, \cdots, u_k\}$ into an orthogonal basis $\{v_1, v_2, \cdots, v_k\}$?

Gram-Schmidt Process

Then $\{v_1, v_2, \cdots, v_k\}$ is an orthogonal basis for W

Example

S = {$\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$} Is a basis for subspace W

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad u_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad u_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

(independent vectors)

Then S' = {$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$} is an orthogonal basis for W .

S'' = {$\mathbf{v}_1$, $\mathbf{v}_2$, 4$\mathbf{v}_3$} is also an orthogonal basis.

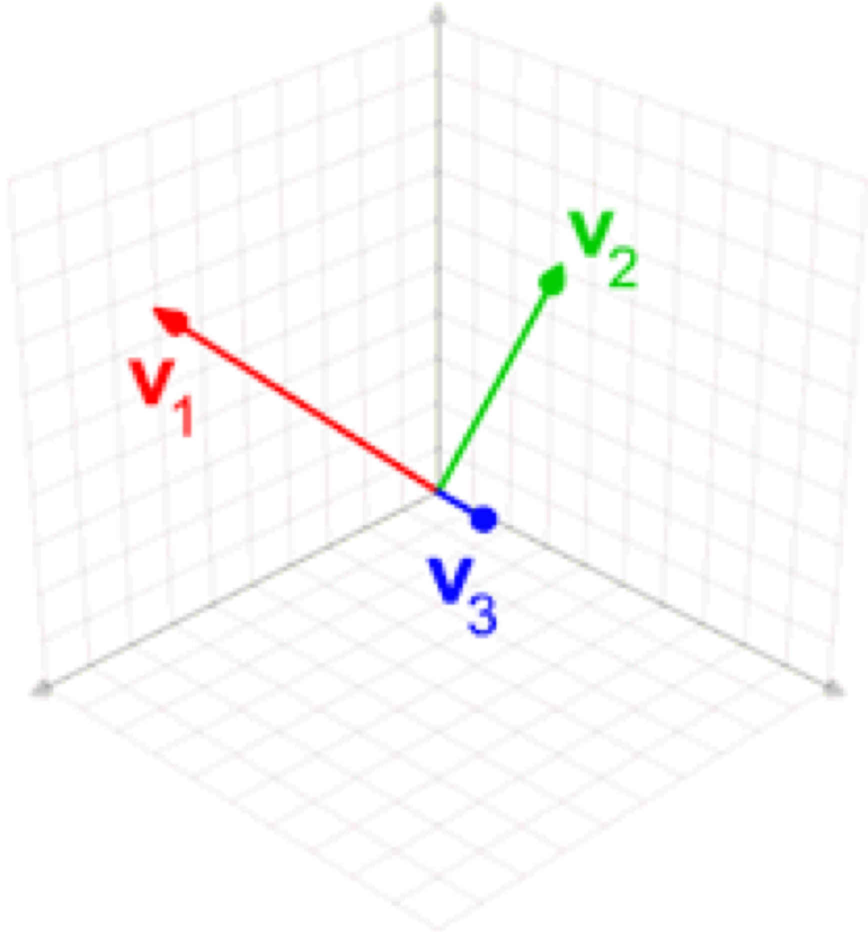
$$\mathbf{v}_1 = \mathbf{u}_1$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1$$

,

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{||\mathbf{v}_2||^2}\mathbf{v}_2$$

Visualization



https://www.youtube.com/watch?v=Ys28-Yq21B8

$$\mathbf{v}_1 = \mathbf{u}_1,$$
$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1,$$
$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{||\mathbf{v}_2||^2}\mathbf{v}_2,$$
$$\vdots$$
$$\mathbf{v}_k = \mathbf{u}_k - \frac{\mathbf{u}_k \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\mathbf{u}_k \cdot \mathbf{v}_2}{||\mathbf{v}_2||^2}\mathbf{v}_2 - \cdots - \frac{\mathbf{u}_k \cdot \mathbf{v}_{k-1}}{||\mathbf{v}_{k-1}||^2}\mathbf{v}_{k-1}$$

$\{u_1, u_2, \cdots, u_k\}$ be a basis of a subspace W

How to prove $\{v_1, v_2, \ldots, v_k\}$ is an orthogonal basis of W

$\{v_1\}$ is orthogonal

$\{v_1, v_2, \ldots, v_{n-1}\}$ is orthogonal ➔ $\{v_1, v_2, \ldots, v_{n-1}, v_n\}$ is orthogonal

$v_n \cdot v_i = 0?$ (i < n)

$$v_n \cdot v_i = u_n \cdot v_i - \frac{u_n \cdot v_i}{||v_i||^2} v_i \cdot v_i = 0$$

$$
\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1, \\
\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1, \\
\mathbf{v}_3 &= \mathbf{u}_3 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{||\mathbf{v}_2||^2}\mathbf{v}_2, \\
&\vdots \\
\mathbf{v}_k &= \mathbf{u}_k - \frac{\mathbf{u}_k \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\mathbf{u}_k \cdot \mathbf{v}_2}{||\mathbf{v}_2||^2}\mathbf{v}_2 - \cdots - \frac{\mathbf{u}_k \cdot \mathbf{v}_{k-1}}{||\mathbf{v}_{k-1}||^2}\mathbf{v}_{k-1}
\end{aligned}
$$

$\{u_1, u_2, \cdots, u_k\}$ be a basis of a subspace W

How to prove $\{v_1, v_2, \dots, v_k\}$ is an orthogonal basis of W

Independent? nonzero?

$\{v_1\}$ is non-zero set

Can v_n be zero?

$\{v_1, v_2, \dots, v_{n-1}\}$ non-zero set ➜ $\{v_1, v_2, \dots, v_{n-1}, v_n\}$ non-zero set

$Span\{v_1, \dots, v_{n-1}\}$
$= Span\{u_1, \dots, u_{n-1}\}$

If $v_n = 0$

$u_n \in Span\{v_1, \dots, v_{n-1}\}$

$u_n \in Span\{u_1, \dots, u_{n-1}\}$