Orthogonality

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Reference: Chapter 7.1

Norm & Distance

- **Norm**: Norm of vector v is the length of v
  - Denoted $\|v\|$

$$\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

- **Distance**: The distance between two vectors u and v is defined by $\|v - u\|$

$$\|v\| = \sqrt{1^2 + 2^2 + 3^2}$$
$$= \sqrt{14}$$

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad u = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} \quad v - u = \begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix}$$

$$\|v - u\| = \sqrt{(-1)^2 + 5^2 + 3^2}$$
$$= \sqrt{35}$$

$$\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

2-norm

$$\|v\|_p = \left(\sum_{i=1}^{n} |v_i|^p\right)^{1/p}$$

$$\|v\|_1 = |v_1| + |v_2| + \cdots + |v_n|$$

$$\|v\|_2 = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

$$\|v\|_3 = \sqrt[3]{|v_1|^3 + |v_2|^3 + \cdots + |v_n|^3}$$

$$\|v\|_4 = \sqrt[4]{v_1^4 + v_2^4 + \cdots + v_n^4}$$

Orthogonal

https://www.youtube.com/watch?v=43BfcSkctYA

https://www.youtube.com/watch?v=EktZVposDMU

Dot Product & Orthogonal

- **Dot product**: dot product of u and v is

$$u \cdot v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

$$= \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u^T v$$

- **Orthogonal**: u and v are orthogonal if $u \cdot v = 0$

Orthogonal is actually "perpendicular"

Zero vector is orthogonal to every vector

More about Dot Product

- Let u and v be vectors, A be a matrix, and c be a scalar
- $u \cdot u = \|u\|^2$ ⇐ Connect norm and dot product
- $u \cdot u = 0$ if and only if $u = 0$
- $u \cdot v = v \cdot u$
- $u \cdot (v + w) = u \cdot v + u \cdot w$
- $(v + w) \cdot u = v \cdot u + w \cdot u$
- $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$
- $\|cu\| = |c|\|u\|$
- $Au \cdot v$

Example $||2\mathbf{u} + 3\mathbf{v}||^2 =$ ……

$$= (2u + 3v) \cdot (2u + 3v)$$

$$= 2u \cdot 2u + 3v \cdot 2u + 2u \cdot 3v + 3v \cdot 3v$$

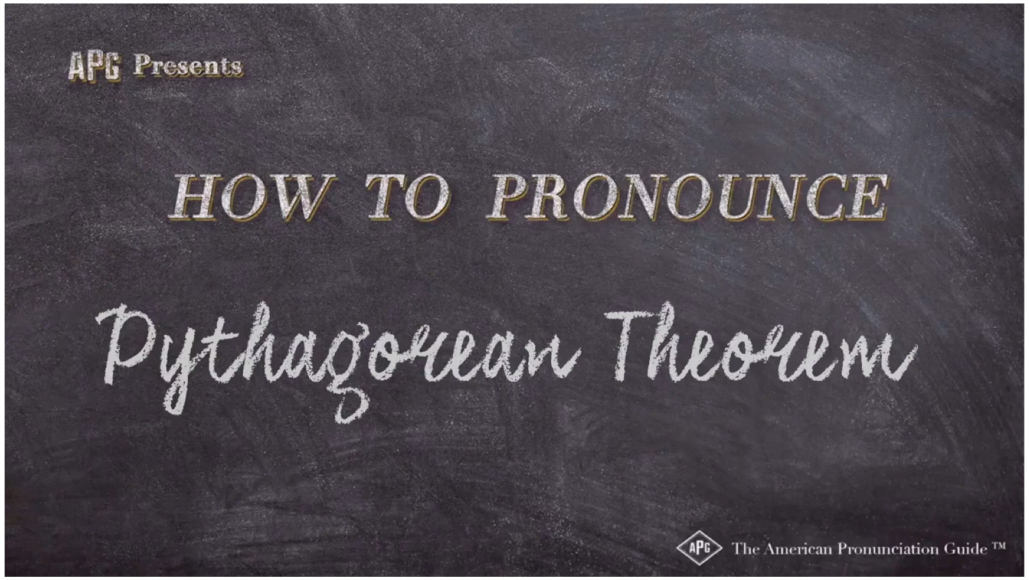
$$= 4(u \cdot u) + 6(v \cdot u) + 6(u \cdot v) + 9(v \cdot v)$$

$$\|u\|^2 \qquad 12(v \cdot u) \qquad \|v\|^2$$

$$= 4||\mathbf{u}||^2 + 12(\mathbf{v} \cdot \mathbf{u}) + 9||\mathbf{v}||^2.$$

Pythagorean Theorem

u and **v** are orthogonal ⟺ $||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$



https://www.youtube.com/watch?v=KII0C17auQU

Proof:

$$||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + \underline{2\mathbf{u}\cdot\mathbf{v}} + ||\mathbf{v}||^2$$

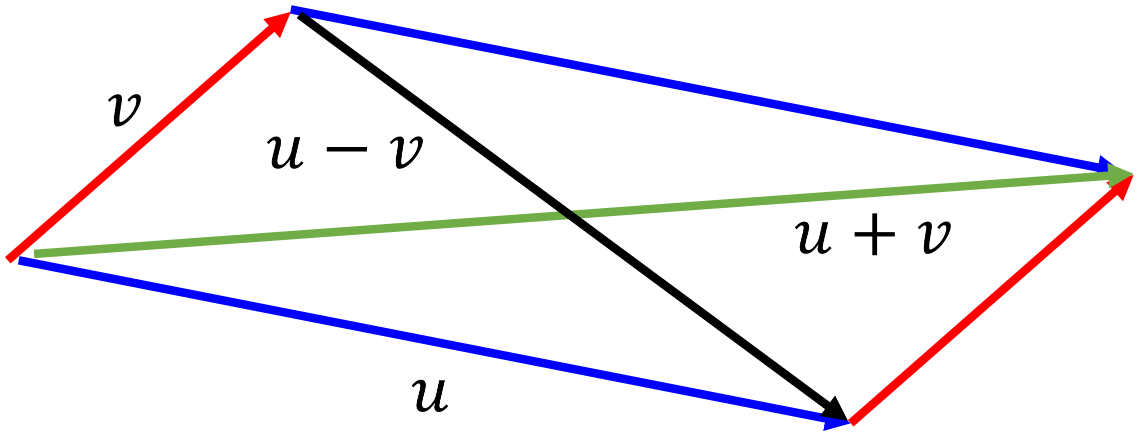
=0 if and only if u and v are orthogonal

Dot Product v.s. Geometry

The diagonals of a parallelogram are orthogonal. ⇔ The parallelogram is a rhombus (菱形).

Proof: $(u+v)\cdot(u-v)=0$

$= \|u\|^2 - \|v\|^2 \quad = 0$ ⟷ $\|\mathbf{u}\| = \|\mathbf{v}\|$



Triangle Inequality

- For any vectors u and v,

$$||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$$

<u>Proof:</u>

$$\|u + v\|^2 = \|u\|^2 + 2u \cdot v + \|v\|^2$$

$$\leq \|u\|^2 + 2|u \cdot v| + \|v\|^2$$

Cauchy-Schwarz Inequality $\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$

$$\leq (\|u\| + \|v\|)^2$$

Proof:

$$\|u+v\|^2 = \|u\|^2 + 2u \cdot v + \|v\|^2$$

$$\leq \|u\|^2 + 2|u \cdot v| + \|v\|^2$$

$$|u \cdot v| \leq \|u\| \cdot \|v\|$$

Proof: If $u = 0$ or $v = 0$ ➡ $|u \cdot v| \leq \|u\| \cdot \|v\|$

$$z \cdot z = 1$$

Otherwise

$$w = \frac{u}{\|u\|} \quad z = \frac{v}{\|v\|} \quad w \cdot w = \frac{u}{\|u\|} \cdot \frac{u}{\|u\|} = \frac{u \cdot u}{\|u\|\|u\|} = 1$$

$$0 \leq \|w \pm z\|^2 = (w \pm z) \cdot (w \pm z) = w \cdot w \pm 2(w \cdot z) + z \cdot z$$

$$= 2 \pm 2(w \cdot z)$$

$$|w \cdot z| \leq 1 \qquad \left|\frac{u}{\|u\|} \cdot \frac{v}{\|v\|}\right| = \frac{|u \cdot v|}{\|u\|\|v\|} \leq 1$$

Cauchy-Schwarz Inequality $\leq \|u\|^2 + 2\|u\| \cdot \|v\| + \|v\|^2$

$$\leq (\|u\| + \|v\|)^2$$