Check Independence

Checking Independence

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

Linear independent or not?

A set of n vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is linear dependent

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, if there exists any $\mathbf{a}_i$ that is a linear combination of other vectors

matrix A

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, there exists scalars $x_1, x_2, \ldots, x_n$, that are **not all zero**, such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$.

vector x

$Ax = \mathbf{0}$ have non-zero solution

Checking Independence

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

Linear independent or not?

A

$Ax = \mathbf{0}$ have non-zero solution or not

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 4 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 \quad x_2 \quad x_3 \quad x_4$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 2 & 0 & 4 & 2 & 0 \\ 1 & 1 & 1 & 3 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$x_1 \quad x_2 \quad x_3 \quad x_4$

Checking Independence

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 2 & 0 & 4 & 2 & 0 \\ 1 & 1 & 1 & 3 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(columns labelled x_1, x_2, x_3, x_4; coefficient columns boxed: dependent)

$$x_1 + 2x_3 = 0$$
$$x_2 - x_3 = 0$$
$$x_4 = 0$$

$\Rightarrow$

$$x_1 = -2x_3$$
$$x_2 = x_3$$
$$x_3 \textit{ is free}$$
$$x_4 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

setting $x_3 = 1$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

Checking Independence

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

Linear independent or not?

其實這題用看的就知道答案了!

A set of n vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is linear dependent

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, if there exists any $\mathbf{a}_i$ that is a linear combination of other vectors

matrix A

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, there exists scalars $x_1, x_2, \ldots, x_n$, that are **not all zero**, such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$.

vector x

$Ax = \mathbf{0}$ have non-zero solution

Column Correspondence Theorem

pivot columns

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix}$$

linear independent

Leading entries

$$R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

linear independent

The pivot columns are linear independent.

Column Correspondence Theorem

pivot columns

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix}$$

Leading entries

$$R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathbf{a}_2 = 2\mathbf{a}_1$

$\mathbf{a}_5 = -\mathbf{a}_1 + \mathbf{a}_4$

$\mathbf{a}_6 = -5\mathbf{a}_1 - 3\mathbf{a}_3 + 2\mathbf{a}_4$

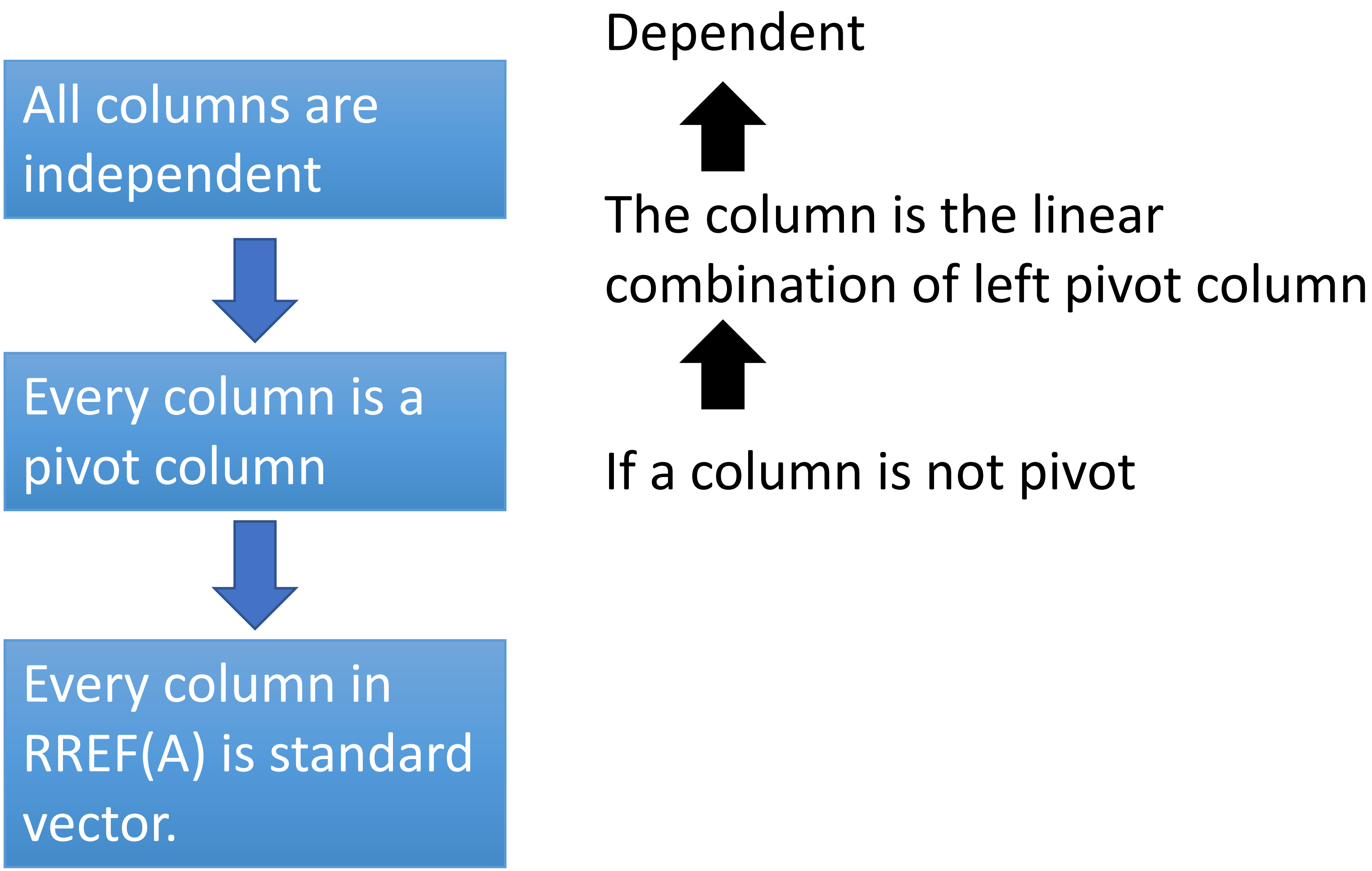
$\mathbf{r}_2 = 2\mathbf{r}_1$

$\mathbf{r}_5 = -\mathbf{r}_1 + \mathbf{r}_4$

$\mathbf{r}_6 = -5\mathbf{r}_1 - 3\mathbf{r}_3 + 2\mathbf{r}_4$

The non-pivot columns are the linear combination of the previous pivot columns.

Independent



Independent

All columns are independent

↓

Every column is a pivot column

↓

Every column in RREF(A) is standard vector.

3X3

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

Columns are linear independent

RREF ↓

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Identity matrix

Independent

All columns are independent

↓

Every column is a pivot column

↓

Every column in RREF(A) is standard vector.

4X3

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

Columns are linear independent

RREF ↓

$$\begin{bmatrix} & & \\ & & \\ & & \\ & & \end{bmatrix} \quad \begin{bmatrix} I \\ O \end{bmatrix}$$

Independent

All columns are independent

↓

Every column is a pivot column

↓

Every column in RREF(A) is standard vector.

3X4

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

Columns are linea
independent

RREF ↓

Cannot be a pivot column

$$\begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix}$$

Independent

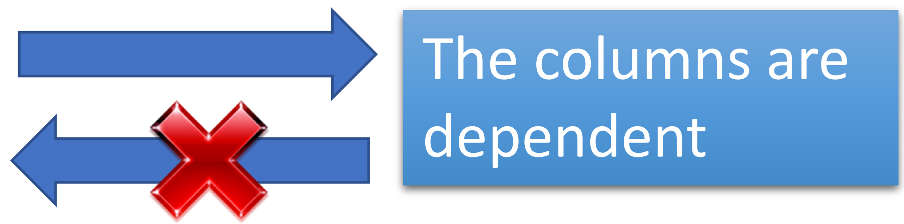
$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix}$$

dependent

因為太胖了，自己走不動

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

(矮胖型)



$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

Dependent or Independent?

dependent

More than 3 vectors in R^3 must be dependent.

More than m vectors in R^m must be dependent.