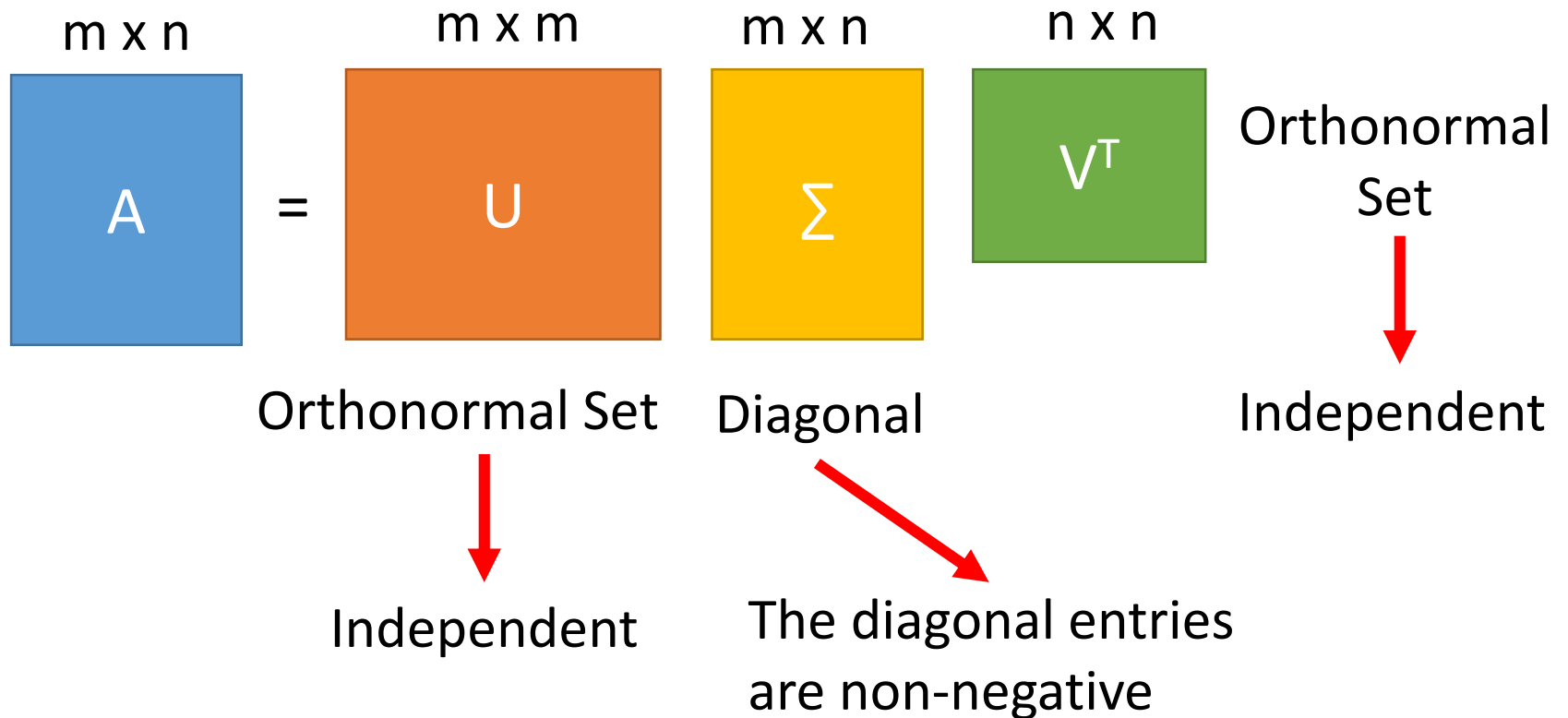# Singular Value Decomposition

Hung-yi Lee

Outline

- Diagonalization can only apply on some square matrices.

- Singular value decomposition (SVD) can apply on any matrix.
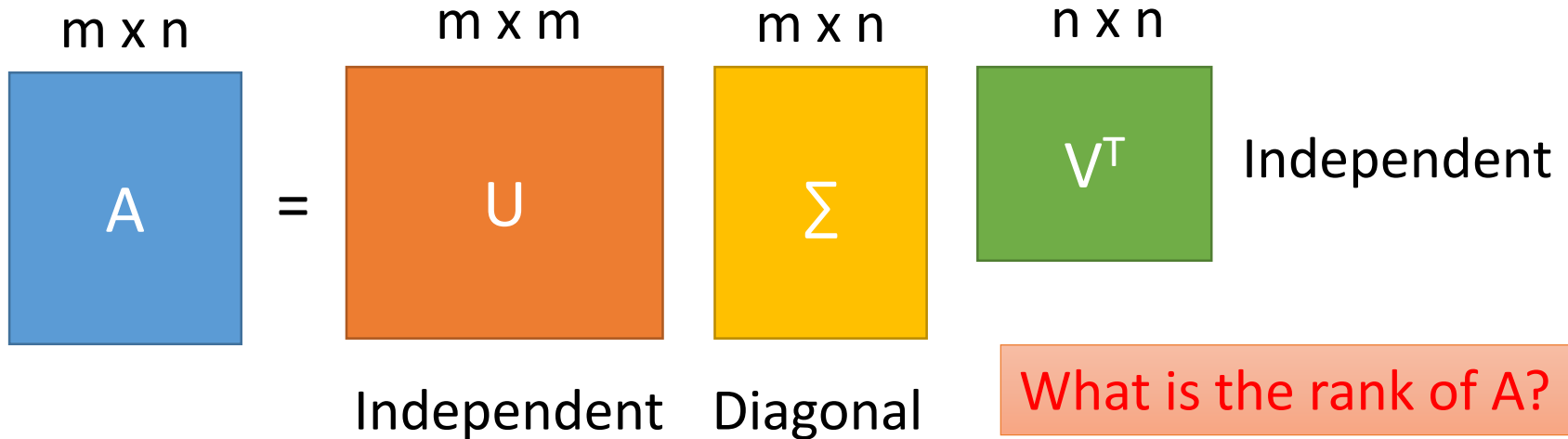
- Reference: Chapter 7.7

SVD

- Any m x n matrix A

m x n   m x m   m x n   n x n

$$A = U \Sigma V^T$$

Orthonormal Set

Orthonormal Set   Diagonal

Independent

Independent   The diagonal entries are non-negative

SVD

(We can exchange some rows and columns to achieve that)

$$\begin{bmatrix} \sigma_1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & & \sigma_k & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

- Any m x n matrix A

m x n

A =

m x m

U

Independent

m x n

Σ

Diagonal

n x n

V^T Independent

What is the rank of A?

If A is a m x n matrix, and B is a n x k matrix.

$$Rank(AB) \leq min\big(Rank(A), Rank(B)\big)$$

If B is a matrix of rank n, then $Rank(AB) = Rank(A)$

If A is a matrix of rank n, then $Rank(AB) = Rank(B)$

SVD

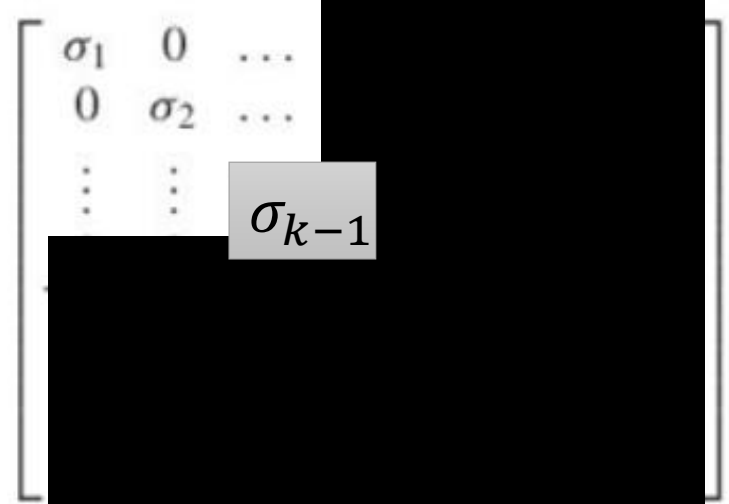$$\begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & \sigma_k \end{bmatrix}$$

- Any m x n matrix A

m x n $\quad$ m x m $\quad$ m x n $\quad$ n x n

A = U Σ V^T Independent

Independent $\quad$ Diagonal

m x n $\quad$ m x k $\quad$ k x k $\quad$ k x n

A = U_1 Σ' V_1^T

SVD

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0$$

σ_k is deleted

$$\begin{bmatrix} \sigma_1 & 0 & \ldots & \\ 0 & \sigma_2 & \ldots & \\ \vdots & \vdots & & \\ & & & \sigma_{k-1} \end{bmatrix}$$

- Any m x n matrix A

	m x n		m x m	m x n	n x n	

A = U Σ V^T

Independent   Independent   Diagonal

What is the rank of A'?

m x n   m x (k-1)   (k-1) x (k-1)   (k-1) x n

A' = U$_1$ Σ' V$_1$^{T}
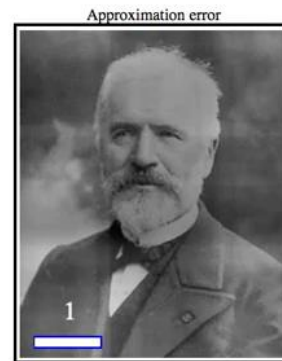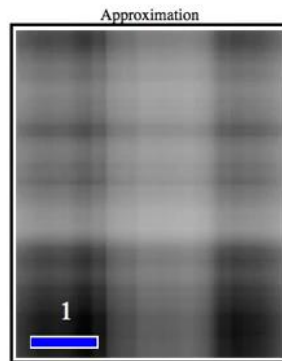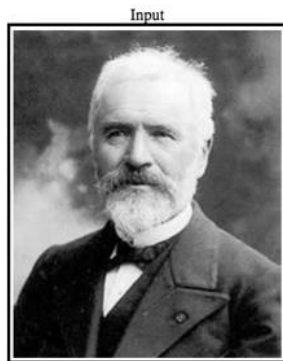
k-1
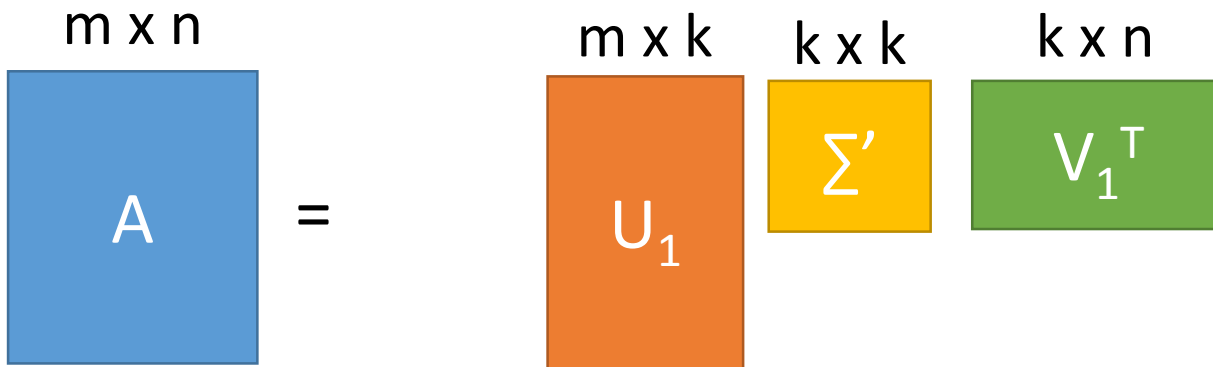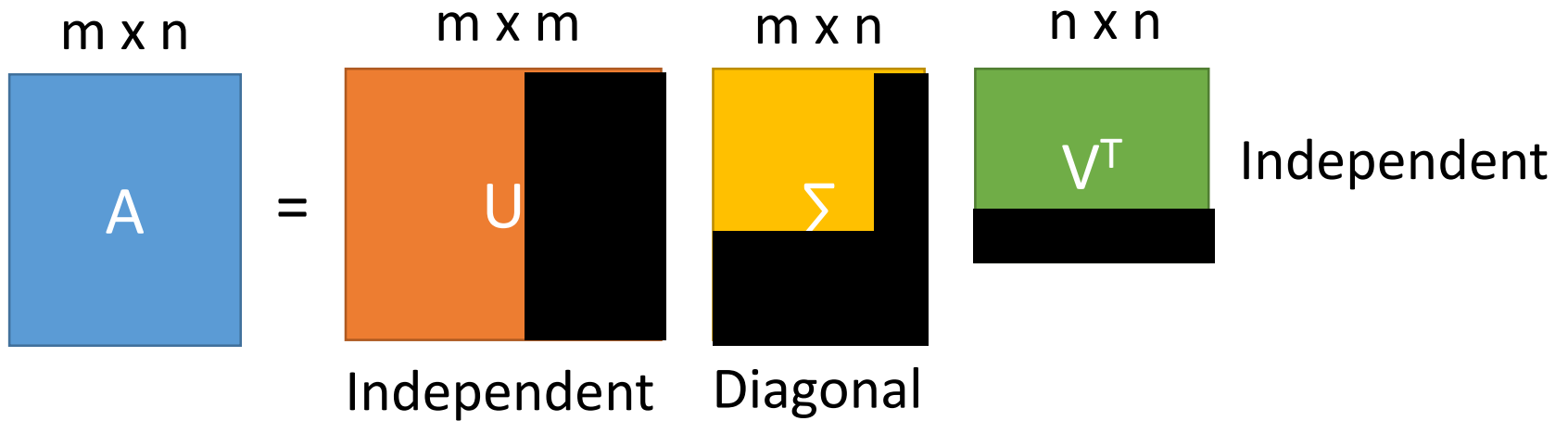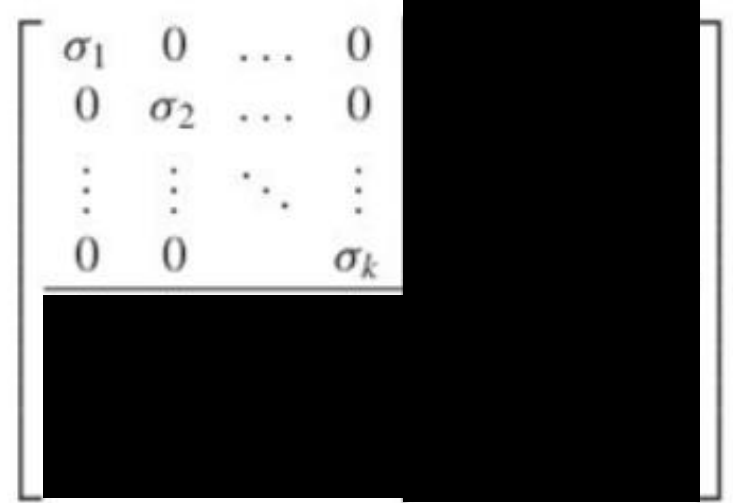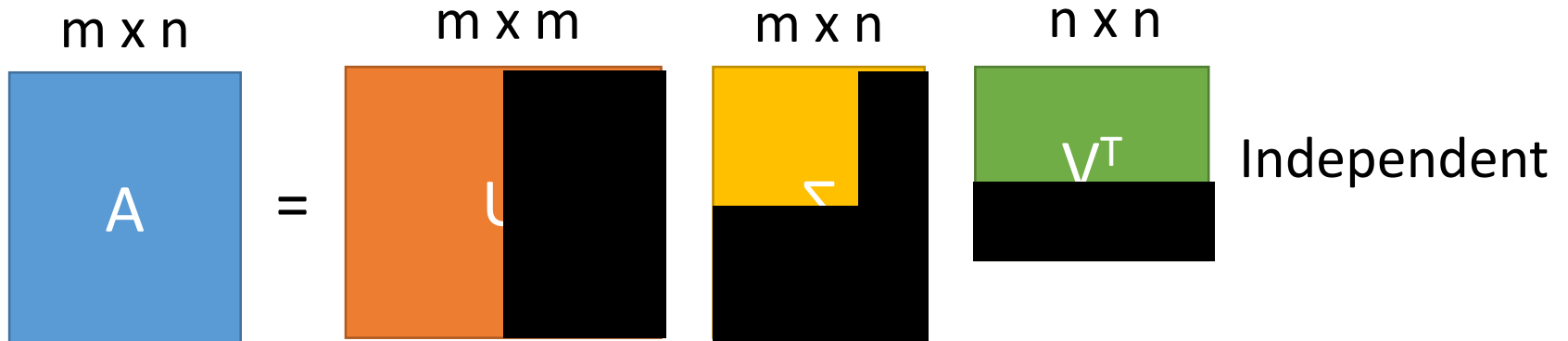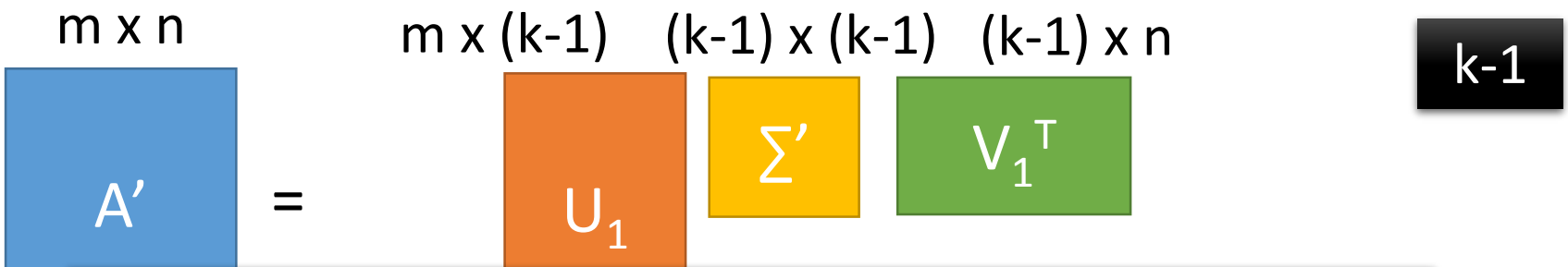
A' is the rank k-1 matrix minimizing $\|A - B\|$

Low rank approximation using the singular value decomposition

Input   Approximation   Approximation error   Spectrum of singular values

https://www.youtube.com/watch?v=pAiVb7gWUrM

https://www.youtube.com/watch?v=fKVRSbFKnEw
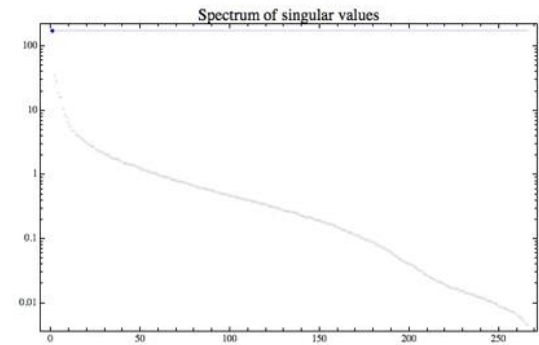
It Had To Be U

The Singular Value Decomposition
(SVD)

Thank You for Your Attention

https://www.youtube.com/watch?v=R9UoFyqJca8